# Lectures-ICTS, Yaffe

## 1 Introduction

The goal is to calculate non equilibrium response from first principles. We will focus on "real" systems ($\hbar \neq 0$) – quantum dynamics! These systems at sufficiently long length scales can be thought of as fluids. [ water, quantum hall fluids, plasma at the centre of the sun ] The advantage is we don't have to think about finite size effects, neither about a heat-bath. Volume is $\infty$. Also in these systems we will consider, we will have complete understanding of scattering processes. In this section we will assume that a useful description of relevant excitations in terms of *quasi-particles* will be present in these systems. [ phonons, dressed electrons, holes, spin-waves, magnons, plasmons, W - Z bosons, Higgs particle ]
Quasiparticle : Is a weakly interacting, long-lived excitation.

$$\tau >> \hbar/E \tag{1.1}$$

Same information is to say, look at the decay rate,

$$\Gamma = \hbar/\tau << E \tag{1.2}$$

And from now onwards, $\hbar = c = k_B = 1$ [ temperatures are same energies same as momenta same as frequencies] In a plot between response and energy, long lived peaks ( resonances) correspond to long-lived excitations. Also the above inequalities correspond to in terms of mean-free-path,

$$\ell_{mfp} >> \hbar/p \tag{1.3}$$

In a distance scale, far left of which is the de Broglie wavelength, and then moving right is $\ell_{mfp}$, and then moving right further we have *hydrodynamics*, where the distance scale is larger than the relevant mean free path. Even if we have a microscopic theory valid at all distance scales, we would be using an intermediate theory - kinetic theory - and then use this theory to derive the appropriate hydrodynamic coefficients.
A good quasiparticle description implies this large separation of scales, and implies the appropriate effective theory is a kinetic theory.

## 2 Kinetic Theory

We want to track the phase space distribution : $f^s(\vec{x}, \vec{p}, t)$ of the excitation and study its time evolution. $s$ here is a species index. This function at a point and time tells us the phase space density of quasiparticle excitations.

$$T^{\mu\nu}(x) = \int_p p^\mu p^\nu \sum_s f^s(p, x) \tag{2.1}$$

We introduced here four-vectors, $(x^0, \vec{x}) = x$ an $(E(\vec{p}), \vec{p}) = p$. And here, $E(\vec{p})$ is fixed by dispersion relations. The integration over $p$ means integration with the Lorentz invariant measure $\int \dfrac{d^3\vec{p}}{(2\pi)^3} \dfrac{1}{E(\vec{p})}$. We will also be interested in current (time component gives charge, spatial component gives flux of the charge),

$$J^\mu(x) = \int_p p^\mu \sum_s q_s f^s(p, x) \tag{2.2}$$

And the dynamics is given by the Boltzmann equation:

$$\left[\frac{\partial}{\partial t} + \vec{v_p} \cdot \frac{\partial}{\partial \vec{x}} - \vec{F}_{ext} \cdot \frac{\partial}{\partial \vec{p}}\right] f^a = -C^a[f] \tag{2.3}$$

For, a collision of 2-2 particles, $p$(species $a$) and $k$ (species $b$) are incoming while $p'$(species $c$) and $k'$(species $d$) are outgoing, the operator on the left will from now be called, $\mathcal{K}$. The collision term (counting the rate at which quasiparticles are entering or leaving the phase space) appearing on the right is

$$\begin{aligned} C^a[f](p, x) &= \sum_{b,c,d} \int_{k,p',k'} |\mathcal{M}^{cd}_{ab}(p, k; p', k')|^2 (2\pi)^4 \delta^4(p + k - p' - k') \\ &\times \left[f_p f_k (1 \pm f_{p'})(1 \pm f_{k'}) - f_{p'} f_{k'} (1 \pm f_p)(1 \pm f_k)\right] \end{aligned} \tag{2.4}$$

The $\pm$ is for bosons/fermions. The outer sum is over species.
At equilibrium,

$$f(p, x) = f_0(p) = \left[e^{\beta(u\cdot p - \mu)} \mp 1\right]^{-1} \tag{2.5}$$

Here, $\beta = 1/T$, $u^\alpha$ =4-velocity [in rest frame it is $(1, 0, 0, 0)$] and $\mu =$ chemical potential. And the metric convention is $u \cdot p = -u^0 p^0 + \vec{u} \cdot \vec{p}$. For this distribution the LHS of the Boltzmann equation is zero if $F_{ext} = 0$. [ Detailed balance is assumed in the structure of $\mathcal{M}^{cd}_{ab}$ ] ( So this is a equilibrium distribution ). But equilibrium is boring, so we want to understand small departures from equilibrium. So lets look at the following distribution function ( and then we will look at the linearized Boltzmann equation )

$$f(p,x) = f_0(p,x) + f_0(1 \pm f_0)\Phi(p,x) \tag{2.6}$$

now the first term is same as $f_0(p)$ with, $\beta = \beta(x)$, $u = u(x)$ and $\mu = \mu(x)$. So just the first term will not satisfy the Boltzmann equation (non-vanishing derivatives) , hence the second term. $\Phi$ denotes the departure from equilibrium. Now putting into the Boltzmann equation and keeping only the small terms ( small is derivatives of $\beta$, $u$ and $\mu$, and $\Phi$ ) The RHS is now,

$$C[f_0](\text{which is zero}) - \mathcal{C}\Phi$$

where $\mathcal{C}$ is the linearized collision operator. The first order in small quantities in the LHS will come just from spacetime derivatives acting on the first term,

$$\mathcal{K}f_0 \sim \mathcal{O}(\nabla\beta, \nabla u, \nabla\mu, F_{ext}) = S$$

So we have the linear equations,

$$S = -\mathcal{C}\Phi \qquad or, \quad \Phi = -\mathcal{C}^{-1}S \tag{2.7}$$

For the stress-energy tensor we now have,

$$T^{\mu\nu} = T^{\mu\nu}_{eq} + \int_p p^\mu p^\nu f_0(p,x)(1 \pm f_0(p,x))\Phi(p,x) \tag{2.8}$$

Consider, for example, an incompressible, irrotational flow for which,

$$\nabla\beta = \nabla\mu = F_{ext} = 0 \tag{2.9}$$

But there is shear in the flow (no curl in the flow ) which contributes now to the source, thus

$$\Phi = (\partial_i u_j + \partial_j u_i - 2/3\delta_{ij}\nabla\cdot\vec{u})\chi_{ij}$$

Thus now from equation(2.8) we have,

$$\Delta T_{ij} = -\eta(\partial_i u_j + \partial_j u_i - 2/3\delta_{ij}\nabla\cdot\vec{u}) \tag{2.10}$$

Where $\eta$ is the shear viscosity.

$$\eta = \frac{2}{15}(I_{ij}, \chi_{ij}) = \frac{2}{15}(I_{ij}, \mathcal{C}^{-1}I_{ij}) = \max_{\chi_{ij}} \frac{2}{15}\left[(\chi_{ij}, I_{ij}) - \frac{1}{2}(\chi_{ij}, \mathcal{C}\chi_{iJ})\right] \tag{2.11}$$

here,

$$S = I_{ij}(\partial_i u_j + \partial_j u_i - 2/3\delta_{ij}\nabla\cdot\vec{u})$$

and the inner product,

$$(f,g) = \int_p f\cdot g$$

Thus we see that because there is complicated dynamics on many different scales, using appropriate effective theory (kinetic theory, in this case) vastly simplifies the analysis. For details and a continuation of this story about calculation of transport coefficients in weakly-coupled theories, see [1].

## 2.1 Hot QCD

We choose to be at $T >> T_c \approx \Lambda_{QCD} \sim 160 MeV$. Interaction strength is $g^2(T)$ which decreases as $T$ increases ( asymptotic freedom). For high temperatures, the quarks and gluons are weakly interacting. Thus quarks and gluons are good quasiparticles. This is a highly relativistic plasma:

$$E(\vec{p}) \approx |\vec{p}|$$

and typically $E(\vec{p}) \approx T$ some flavors of quarks have $m << T$ in this case we ignore the mass. in case $m >> T$ we ignore the quark altogether. ( their number is small ). The number density $\sim T^3$ and energy density $\sim T^4$.

One could have found shear viscosity alternatively from,

$$\eta = \lim_{\omega\to 0}\langle T_{xy}T_{xy}\rangle$$

But the above method is really messy. Shear viscosity tells us how efficient it is to transport momentum density from one part to another. The longer the mean free path is the larger the shear viscosity is. Dimensional analysis gives,

$$\eta \sim \text{energy-density} \times \ell$$

But,

$$\ell \sim \frac{1}{n\sigma v}$$

where $\sigma$ is the scattering cross-section, and $\sigma \sim |\mathcal{M}|^2 \sim g^4$. Thus,

$$\eta \sim \frac{T^3}{g^4(T)}\left(c_1 + \mathcal{O}(g) + \cdots\right) \tag{2.12}$$

And getting this result will be very complicated. There is a whole series of inverse logarithms,

$$\eta = \frac{2}{15}\frac{T^3}{g^4(T)}\left(1067/\ln(\frac{3T}{m_D}) + 2.4/\ln(\frac{3T}{m_D})^3 - 0.1/\ln(\frac{3T}{m_D})^4 + \mathcal{O}(g)\right)$$

Just doing self-energy corrections is not sufficient, since while calculating Feynman diagrams we damage the physics when we integrate over all spacetime, assuming that the quasi-particles live forever, which they do not. The weakly coupled theory does not make things easy since many different scales have been involved. There is important physics at the scales, $T^{-1}$, $(gT)^{-1}$, $(g^2T)^{-1}$, $(g^4T)^{-1}$, $\cdots$.

# 3 Strongly correlated systems

There are various examples of such a system. It includes, high $T_C$ superconductors, unitary atomic gases and quark-gluon plasma at temperatures, $T_C < T \leq 4T_C$ (experimentally relevant) For any of these systems the quasiparticle description does not work ( no simple weak-coupling description, hence no effective kinetic theory )
For some systems one can do calculations using $AdS/CFT$ duality.

## 3.1 AdS/CFT

Consider the simplest case : A $SU(N_c)$ theory with the largest amount of symmetry you can possibly have : Maximally supersymmetric Yang-Mills theory. ( $\mathcal{N} = 4$ SYM )
This is same at Type IIB String theory on $AdS_5 \times S^5$. We will focus on a particular limit, when the field theory is at strong coupling and $N_c \to \infty$, $\lambda(= g^2 N_c) >> 1$. At this limit, string theory reduces to classical (super)gravity. The whole point of this, is that this is relating strongly coupled QFT to classical gravity (in 10 dimensions) [1]

[1] But for our discussions, nothing interesting will be going on in the $S^5$

| QCD | $\mathcal{N}=4$ SYM |
|---|---|
| adjoint gluons | adjoint gluons |
| fundamental quarks | adjoint fermions(4) + scalars(6) |
| asymptotic freedom | Conformal (scale invariant) |
| T=0 | |
| confinement, hadrons, S-matrix | none |
| $T > T_C$ | |
| non-abelian plasma | ✓ |
| Debye screening | ✓ |
| Corr length $\xi < \infty$ | ✓ |
| Hydrodynamics | ✓ |

On the $AdS^5$ side now, we will focus on the Poincare patch where the metric is:

$$ds^2 = \frac{r^2}{L_{AdS}^2}(-dt^2 + d\vec{x}^2) + \frac{L_{AdS}^2}{r^2}dr^2 \tag{3.1}$$

At $r \to \infty$ the spacetime boundary $\sim$ Minkowski space. This picture also gives a geometric representation of the renormalization group. At $r = 0$ the geometry is pinching off, and corresponds to the $IR$ scale of the theory, while the boundary corresponds to the $UV$. The dictionary is:

| $AdS^5 \times S_5$ | $\mathcal{N}=4$ SYM |
|---|---|
| $L_{AdS}$ | none |
| $G_{N(5)}$ | $\frac{\pi}{2}L^3/N_c^2$ |
| (string length) $\ell_s$ | $\lambda^{-1/4}L$ |
| $(L/\ell_s)^4$ | $\lambda$ |
| (string coupling) $g_s$ | $g_{YM}^2/4\pi$ |

We are looking at the limit $N_c \to \infty$ which means that the string theory is weakly coupled. What about observable?

| $AdS^5 \times S_5$ | $\mathcal{N}=4$ SYM |
|---|---|
| Fields $\phi$ | Operators $\mathcal{O}$ |
| $\phi(r \to \infty)$ | source, $J$ coupled to $\mathcal{O}$ |
| $g^{\mu\nu}$ | $T^{\mu\nu}$ |
| $A_\mu^a$ | $J_\mu^a$ |

If we have some correlation function in the field theory:

$$\langle \mathcal{O}\mathcal{O}\cdots\rangle = \frac{\delta}{\delta J}\frac{\delta}{\delta J}\cdots W[J] \tag{3.2}$$

Here due to the dictionary $W[J]$ is same as the gravity action, and $J$ is the boundary value of the bulk field. This dictionary only allows us to talk about the gauge invariant quantities.

| $\mathcal{N}=4$ SYM | $AdS^5 \times S_5$ |
|---|---|
| $T=0$ | $AdS_5$ |
| $T>0$ (eq. plasma) | $AdS_5$-Schwarzchild |
| entropy density $s=\frac{S}{V}=\frac{\pi^2}{2}N_c^2T^3$ | $\frac{A/4G}{V}$ |

A property about the entropy density : $s(\lambda=\infty)=3/4s(\lambda=0)$. In the above for the black hole the metric is

$$ds_{BH}^2=\frac{r^2}{L^2}(-f(r)dt^2+d\vec{x}^2)+\frac{L^2}{r^2}\frac{dr^2}{f(f)}$$

where

$$f(r)=1-(\frac{r_h}{r})^4$$

here horizon radius, $r_h=\pi T L^2$

When calculating non-equilbrium response,

**1**. Long time/ long distance = dissipative hydrodynamics

shear $\eta=\langle T_{xy}T_{xy}\rangle=?$

bulk $\zeta=\langle T_M^M T_\nu^\mu\rangle=0$(since the trace is zero)

**2**. Quasi normal modes, $\sum_n e^{-i\omega_n t}f_k$

**3**. Projectile drag : we can shoot a heavy quark into the medium, with some momentum, then it will follow :

$$0=\frac{dp}{dt}+\mu p$$

And now we can ask what is the friction?

### 3.1.1 Example : $\eta$

Starting with 5 dimensional gravity :

$$S_{5D}=\frac{N_c^2}{8\pi^2L^3}\int d^5x\ \sqrt{g}(R-2\Lambda+\cdots)$$

The solution to this is:

$$g_{MN} = g^0_{MN} + h_{MN} \tag{3.3}$$

Here $g^0_{MN}$ is either the $AdS$ or $AdS/BH$ metric. There will be quadratic corrections to the action ( lets try to extract out shear viscosity $\eta$) Thus we perturb with $h^1_2 = \phi$. The resulting quadratic piece of the action is :

$$S_{quad} = \frac{N_c^2}{8\pi^2 L^3}\int d^4x dr\ \sqrt{g_0}(-\frac{1}{2}g_0^{MN}\partial_M\phi\partial_N\phi) \tag{3.4}$$

And the equation of motion from the variation is :

$$\partial_M(\sqrt{g_0}g_0^{MN}\partial_N\phi) = 0 \tag{3.5}$$

We will need to keep track of the non-trivial dependence in the $r$ direction, hence we take a 4D fourier transform of the Minkowski space. Using $z = L^2/r$ we have,

$$\phi(p,z) = h_p(z)\phi_0(p) \tag{3.6}$$

where,

$$p = (\omega, \vec{q})$$

$h_p$ is the mode function and $\phi_0$ is the amplitude. We would like to plug back the solution on the gravitation action because the gravitational action viewed as a function of the boundary values of the fields is the generating functional of the field theory. The contribution is just from the surface term and it will involve only 1 derivative, since we are doing integration by parts.

$$\begin{aligned} S &= \frac{N_c^2}{16\pi^2 L^3}\int \frac{d^4p}{(2\pi)^4}\ \frac{1}{z^3}\phi(-p,z)\partial_z\phi(p,z)|_{z\to 0} \\ &= \int \frac{d^4p}{(2\pi)^4}\ \phi_0(-p)\mathcal{F}(p,z)\phi_0(p)|_{z\to 0} \end{aligned} \tag{3.7}$$

where,

$$\mathcal{F}(p,z) = \frac{N_c^2}{16\pi^2 L^3}\frac{1}{z^3}h_{-p}(z)\partial_z h_p(z)$$

This thus gives,

$$\frac{\delta^2 S}{\delta\phi_0(p)\delta\phi_0(-p)} \sim \langle T_{xy}T_{xy}\rangle_{ret}(p) = -2\lim_{z\to 0}\mathcal{F}(p,z) \tag{3.8}$$

Lets now do the $AdS/BH$ choosing $2\pi T = 1$. Also we will use $u = r_h^2/r^2$. In these coordinates, $f(u) = 1 - u^2$. Now the mode equation is:

$$h_p'' - \frac{1+u^2}{uf}h_p' + \frac{\omega^2 - q^2}{uf^2}fh_p = 0 \tag{3.9}$$

For the purpose of extracting shear viscosity it is sufficient to consider, $\omega$ and $|q|$ both smaller than the temperature (i.e, 1). And now we will find,

$$h_p(u) = (1-u^2)^{-i\omega/2} + \mathcal{O}(\omega^2, q^2) \tag{3.10}$$

This satisfies, $h_p(0) = 1$, and more importantly $h_p(u)$ looks like and incoming wave at the horizon. So perturbations don't have stuff coming out of the horizon. Plugging this back into $\mathcal{F}$ with the change of variables from $u$ to $z$, we get

$$\mathcal{F}(p,z) \sim i\omega \frac{N_c^2 \pi^2 T^4}{8} + \mathcal{O}(\omega^4) \tag{3.11}$$

Thus reading off,

$$\langle T_{xy} T_{xy} \rangle_{ret}(p) = -i\omega \frac{N_c^2 \pi^2 T^4}{4} \tag{3.12}$$

And finally, the imaginary part of this correlation function gives the spectral density and the slope of the spectral density appears in the Kubo relation. Basically we take a derivative[2] with $\omega$ :

$$\eta = \frac{\pi}{8} N_c^2 T^3 \tag{3.13}$$

This is consistent with $\eta/s = 1/(4\pi)$. With a wide class of holographic calculations we have the same answer.[3] Qualitatively,

$$\eta \sim \text{energy-density} \times \text{mean free time}$$

And,

$$\text{energy density} \sim \text{entropy density} \times \text{typical energy of excitation}$$

So, qualitatively,

$$\frac{\eta}{s} \sim (\text{typical excitation energy}) \times (\text{mean free time})$$

and the latter product, by the uncertainly principle, is bounded below by $\mathcal{O}(\hbar)$. It is liquid helium specifically near the $\lambda$ point which has near-minimal $\eta/s$. The ratio increases as one goes to either higher, or lower temperatures. Experimentally this ratio is within an order of magnitude for liquid helium. A nice reference for this subsection is Son and Starinets [2].

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[2] $\eta = -\frac{1}{2\pi T} \partial_\omega Im(\langle T_{xy} T_{xy} \rangle_{ret})$

[3] Suppose there's a fluid, we can ask, can we ignore viscosity ( ideal ) or is it a real fluid with some viscosity. And thus we have to compare it to something, that's why this quantity $\eta/s$ is important

### 3.1.2 Quasi Normal Modes

Now suppose we have a box of a hot plasma and we shake it little bit and watch it evolve. Expectation is that the perturbations fall of exponentially with some relaxation time, $\sum_i c_i \exp(-i\omega_n t)$. One can ask, what is the set of these relaxation frequencies?
Here we have QNMs (Quasi Normal Modes) fluctuations in $T_{xy}$ without any driving, thus we want to look for $\phi(p,z) = \phi_0(p)h_p(z)$ but now with $h_p(0) = 0$ (no source). We still of course want $h_p(z \to 1)$ to be incoming to the horizon. This is a boundary value problem, and there will be solutions for certain $\omega$s ( complex allowable ). The results are some discrete set of frequencies but the wave vector can be anything we want, e.g, we can choose values of $q$ and get the discrete set of frequencies (which now have negative imaginary parts, for the perturbations to relax). There is always a gap, which is given by the lowest QNM. As $q$ is reduced to zero, the gap reduces to zero as well, signifying that in the infinite wavelength limit the damping totally vanishes. For more details on this, see Kovtun and Starinets [3].

### 3.1.3 Projectile drag

We inject into a box of hot plasma a very heavy quark with momentum $p$, then we expect,

$$\frac{d\vec{p}}{dt} = \vec{F_{drag}}(\vec{p}) \tag{3.14}$$

For the reasonable assumption, $\vec{F_{drag}}(0) = 0$ then, only dependence is

$$\vec{F_{drag}}(\vec{p}) = -\mu\vec{p} + \cdots \tag{3.15}$$

The above form need only hold for small momentum On the gravity side, we have one open string with one end stuck on the boundary [4]. Here there are two possibilities, firstly, the other end can get lost behind the horizon (deconfined quark, free to wander around), secondly the other end can also end on the boundary (confined quark antiquark pair)
**String dynamics**

$$S = -T_0 \int \; Area \tag{3.16}$$

where,

$$T_0 = \sqrt{\lambda}\frac{1}{2\pi L^2}$$

[4] historical origin of string theory before QCD, confinement with quark and antiquark on both ends of a string

. Putting a $\sigma - \tau$ patch on the world sheet, the string coordinate $X^\mu = X^\mu(\sigma, \tau)$ Thus,

$$S = -T_0 \int d\sigma d\tau \ \sqrt{-\det(\gamma)} \tag{3.17}$$

where $\gamma$ is the induced metric on the world sheet. And

$$-\det\gamma = (\dot{X} \cdot X')^2 - (X')^2(\dot{X})^2 \tag{3.18}$$

We now choose the static gauge, $\tau = t = X^0$ and $\sigma = r$, also we will pay attention to a single spatial direction. So we assume,

$$X^\mu = (t, x(r,t), 0, 0, r) \tag{3.19}$$

Plugging into the Nambu-Goto action we obtain,

$$\gamma = \begin{pmatrix} -h + r^2(\dot{x})^2 & r^2\dot{x}x' \\ r^2\dot{x}x' & h^{-1} + r^2(x')^2 \end{pmatrix} \tag{3.20}$$

And the equation of motion is,

$$\partial_r(hr^2x'/\sqrt{-\gamma}) = \frac{r^2}{h}\partial_t(\frac{\dot{x}}{\sqrt{-\gamma}}) \tag{3.21}$$

We look for solutions of the form,

$$x(r,t) = x_0 + vt \pm vF(r)$$

and a small calculation gives,

$$F(r) = \frac{1}{2r_h}[\frac{\pi}{2} - \tan^{-1}\frac{r}{r_h} - \coth^{-1}\frac{r}{r_h}] \tag{3.22}$$

We choose the sign which corresponds to energy flowing from the boundary through the string to the horizon. If we look at the energy flux then we find,

$$\frac{dp}{dt} = \frac{1}{v}\frac{dE}{dt} = \frac{dE}{dx} = -\mu\frac{mv}{\sqrt{1-v^2}} = -\mu p \tag{3.23}$$

This gives,

$$\mu = \frac{\pi}{2}\sqrt{\lambda}\frac{T^2}{m} \tag{3.24}$$

For more about this, see [4].
The examples we talked about were small departures from the equilibrium, but we can also ask what else can be done with this holographic set-up.

## References


[1] http://arxiv.org/abs/hep-ph/0010177.pdf,
http://arxiv.org/abs/hep-ph/0209353.pdf,
http://arxiv.org/abs/hep-ph/0302165.pdf

[2] http://arxiv.org/pdf/0704.0240.pdf

[3] http://arxiv.org/pdf/hep-th/0506184.pdf

[4] http://arxiv.org/pdf/hep-th/0605158.pdf