Stochastic Analysis and Control of Fluid Flows

Lecture 5

School of Mathematics – IISER - TVM

December 3 – December 20, 2012

Coupling between estimation and feedback control

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Plan of Lecture 5

1. The estimation problem for a linear dynamical system

Examples of measure operators for finite dimensional systems

Asymptotic observers

2. Coupling between estimation and control

Observability and Detectability

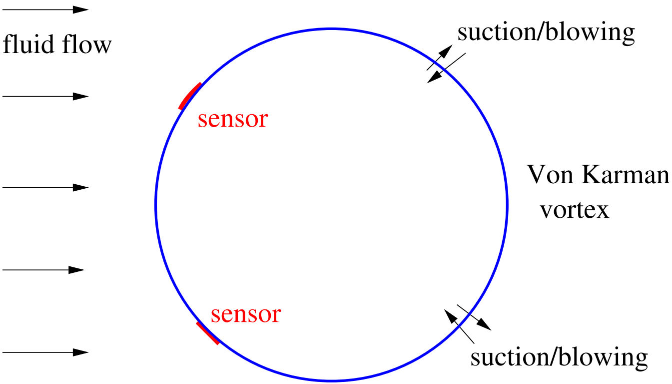
3. Detectability of finite dimensional systems

4. Detectability of infinite dimensional systems

4.1. Detectability of the heat equation for different boundary measure operators

4.2. Detectability of the Stokes equation for different boundary measure operators

The complete problem: estimation + feedback control



Stabilization by feedback with full information

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Estimation with partial information

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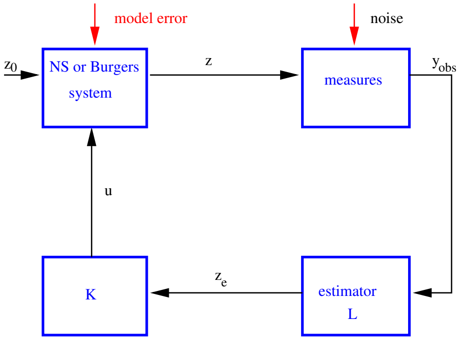
Stabilization by feedback with partial information



1. Estimation problem

We consider

$$z' = Az + f + \mu, \quad z(0) = z_0 + \mu_0.$$

The data f and z_0 are assumed to be known, while μ and η are model errors. We would like to estimate z thanks to some measurements

$$y_{obs}(t) = Hz(t) + \eta(t) \in Y_o.$$

Here, Y_o is the space of observations, $\eta(t)$ is a measure error and $y_{obs}(t)$ is the noisy observation.

Without measurements the only estimation of the state is made by solving the equation

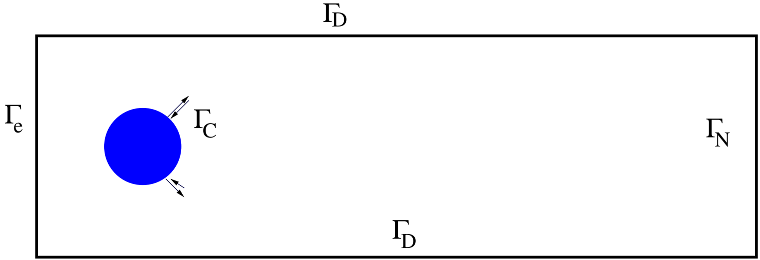
$$z_e' = Az_e + f, \quad z_e(0) = z_0.$$

The error $e = z - z_e$ is

$$e(t) = e^{tA}\mu_0 + \int_0^t e^{(t-s)A}\mu(s)\, ds.$$

The goal is to use the measure $y_{obs}(t)$ to improve the estimation of z.

The model error μ may correspond to an unknown perturbation of the inflow boundary condition



$$w = u_s + \zeta(t) \quad \text{on} \quad \Gamma_e,$$

and

$$\mu(t) = B_e\,\zeta(t).$$

Example – The simplified linearized inverted pendulum

As in lecture 2, we consider the system

$$\theta'' = \theta + u, \quad \theta(0) = \theta_0, \quad \theta'(0) = \theta_1,$$

and the noisy model

$$\theta'' = \theta + u + \mu, \quad \theta(0) = \theta_0 + \mu_0, \quad \theta'(0) = \theta_1 + \mu_1.$$

Setting $\zeta = \theta'$, we rewrite this system in the form

$$\begin{pmatrix} \theta \\ \zeta \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \zeta \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Assume that u is given by the feedback law determined in lecture 2

$$u = -2(\theta + \zeta).$$

The next step consists in using an estimation (θ_e, ζ_e) of (θ, ζ) in the feedback law.

Consider the case when $\theta_{obs} = \theta + \eta$, where η is a noise. We measure θ but the feedback law depends on θ and $\zeta = \theta'$. The naive approach consists in using θ_{obs} and θ'_{obs} in the feedback law. The noise is not necessarily differentiable and this naive approach introduces huge errors.

Another approach consists in using an asymptotic state estimation of the form

$$z_e' = Az_e + f + L(Hz_e - y_{obs}), \quad z_e(0) = z_0.$$

The term $L(Hz_e - y_{obs})$ is called a filtering gain and $L \in \mathcal{L}(Y_o, Z)$. The filtering gain is a corrector taking into account the measures.

We look for $L \in (Y_o, Z)$ such that

$$\left(e^{t(A+LH)}\right)_{t\geq 0} \quad \text{is exponentially stable on } Z.$$

When the semigroup $\left(e^{t(A+LH)}\right)_{t\geq 0}$ is exponentially stable on Z, a dynamical system of the form

$$z_e' = Az_e + f + L(Hz_e - y_{obs}), \quad z_e(0) = z_0, \quad \text{with } L \in (Y_o, Z),$$

is called a Luenberger observer.

The equation for the error $e = z - z_e$ is

$$e' = (A + LH)e + \mu - L\eta, \quad e(0) = \mu_0.$$

Theorem. Assume that

$$\|e^{t(A+LH)}\|_{\mathcal{L}(Z)} \leq Ce^{-t\omega}, \quad \text{with } \omega > 0.$$

If $e^{t\omega}\,\eta \in L^2(0,\infty;Y_o)$ and $e^{t\omega}\,\mu \in L^2(0,\infty;Z)$, then

$$\|e\|_{L^2(0,\infty;Z)} \leq Ce^{-t\omega_1}\left(\|\mu_0\|_Z + \|e^{t\omega}\,\eta\|_{L^2(0,\infty;Y_o)} + \|e^{t\omega}\,\mu\|_{L^2(0,\infty;Z)}\right),$$

with $0 < \omega_1 < \omega$.

How to determine L ?

Assume that $\mu_0 = 0$. When μ is a white Gaussian noise with mean value zero and with covariance Q_o ($Q_o = Q_o^* \geq 0$), and when η is a white Gaussian noise with mean value zero and with covariance R_o ($R_o = R_o^* > 0$), it can be shown that the best linear estimator of z, without bias, knowing y_{obs}, is the solution to the problem

$$(\mathcal{EP}) \qquad \begin{aligned} &\inf J(z, \mu, \eta), \quad (z, \mu, \eta) \text{ obeys} \\ &z' = Az + f + \mu, \quad z(0) = z_0, \\ &y_{obs}(t) = Hz(t) + \eta(t) \in Y_o, \end{aligned}$$

and

$$J(z, \mu, \eta) = \frac{1}{2}\int_0^\infty \Big(R_o^{-1}(Hz - y_{obs}), Hz - y_{obs}\Big)_{Y_o} + \frac{1}{2}\int_0^\infty (Q_o^{-1}\mu, \mu)_Z.$$

The weights R_o^{-1} and Q_o^{-1} are used to obtain the best ponderation between the model error and the measure error.

When $Q_o = 0$, then $\mu = 0$. The model error is equal to zero and the best estimation (which is also the exact one) is given by

$$z' = Az + f, \quad z(0) = z_0.$$

At the opposite, if $R_o = 0$, then $\eta = 0$ and we have to use the equations $y_{obs}(t) = Hz(t)$ and $z' = Az + f + \mu$, $z(0) = z_0$ to identify μ.

Does $(\mathcal{EP})$ admit a solution ? How to characterize it ?

The existence of solutions to $(\mathcal{EP})$ is related to what is called the detectability of the pair (A, H). Let us explain why.

Let us denote by $z_{z_0,f}$ the solution to

$$z'_{z_0,f} = Az_{z_0,f} + f, \quad z_{z_0,f}(0) = z_0,$$

set $\zeta = z - z_{z_0,f}$, $\bar{y}_{obs} = y_{obs} - Hz_{z_0,f}$ and

$$I(\zeta, \mu, \eta) = \frac{1}{2}\int_0^\infty \Big(R_o^{-1}(H\zeta - \bar{y}_{obs}), H\zeta - \bar{y}_{obs}\Big)_{Y_o} + \frac{1}{2}\int_0^\infty (Q_o^{-1}\mu, \mu)_Z.$$

Problem $(\mathcal{EP})$ is transformed as follows

$$(\mathcal{NEP}) \qquad \begin{aligned} &\inf I(\zeta,\mu,\eta), \quad (\zeta,\mu,\eta) \text{ obeys} \\ &\zeta' = A\zeta + \mu, \quad \zeta(0) = 0, \\ &\bar{y}_{obs}(t) = H\zeta(t) + \eta(t) \in Y_o. \end{aligned}$$

Assume that this problem admits a unique solution and let us write *formally* the optimality system

$$\begin{aligned} &\zeta' = A\zeta - Q_o\phi, \quad \zeta(0) = 0, \\ &-\phi' = A^*\phi + H^*R_o^{-1}(H\zeta - \bar{y}_{obs}), \quad \phi(\infty) = 0. \end{aligned}$$

In this *primal formulation*, ζ is the state variable and ϕ the adjoint state. We can look for a *dual problem* in which ϕ will be the state variable and ζ the adjoint state. For that, we set $\psi = R_o^{-1/2}H\zeta$ and we rewrite the above system as follows

$$\begin{aligned} &\zeta' = A\zeta - Q_o^{1/2}Q_o^{1/2}\phi, \quad \zeta(0) = 0, \\ &-\phi' = A^*\phi + H^*R_o^{-1/2}\psi - H^*R_o^{-1}\bar{y}_{obs}, \quad \phi(\infty) = 0. \end{aligned}$$

We can verify that this system is the O.S. of the dual problem

$$(\mathcal{DP}) \qquad \begin{array}{l} \inf F(\phi,\psi), \quad (\phi,\psi) \text{ obeys} \\ -\phi' = A^*\phi + H^* R_o^{-1/2}\psi - H^* R_o^{-1}\bar{y}_{obs}, \quad \phi(\infty) = 0, \end{array}$$

where

$$F(\phi,\psi) = \frac{1}{2}\int_0^\infty \Big(Q_o\phi,\phi\Big)_Z + \frac{1}{2}\int_0^\infty \|\psi\|^2_{Y_o}.$$

For the well posedness of the state equation of $(\mathcal{DP})$, we need that (A^*, H^*) is stabilizable. This is exactly equivalent to the definition of the detectability of the pair (A, H). The Riccati equation for $(\mathcal{DP})$ is

$$P_e = P_e^* \geq 0, \quad P_e A^* + A P_e - P_e H^* R_o^{-1} H P_e + Q_o = 0.$$

This Riccati equation enable us to define a Luenberger observer by setting

$$L = -P_e H^* R_o^{-1}.$$

Then $A^* + H^* L^*$ and $A + LH$ are exponentially stable.

The simplified linearized inverted pendulum revisited

We have

$$\begin{pmatrix} \theta \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \zeta \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\theta_{obs} = \theta + \eta.$$

We assume that a, the variance of η, is positive and b, the variance of μ is also positive. We have

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad H = (1\ 0).$$

The pair (A^*, H^*) is stabilizable and the pair (A, H) is detectable. The minimization problem giving the best estimator is

$$\inf \frac{1}{2a} \int_0^\infty (\theta - \theta_{obs})^2 dt + \frac{1}{2b} \int_0^\infty \mu^2 dt,$$
$$\theta'' = \theta + u + \mu, \quad \theta(0) = \theta_0, \quad \theta'(0) = \theta_1.$$

The Riccati equation for the filtering operator is

$$P_e A^* + AP - \frac{1}{a} P_e H^* H P_e + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We obtain

$$p_{12} + p_{21} - \frac{1}{a} p_{11}^2 = 0, \quad p_{11} + p_{22} - \frac{1}{a} p_{11} p_{12} = 0,$$
$$p_{11} + p_{22} - \frac{1}{a} p_{11} p_{21} = 0, \quad p_{21} + p_{12} - \frac{1}{a} p_{21} p_{12} + b = 0.$$

Since $p_{12} = p_{21}$, we obtain

$$2p_{12} - \frac{1}{a} p_{11}^2 = 0, \quad p_{11} + p_{22} - \frac{1}{a} p_{11} p_{12} = 0,$$
$$-2a\, p_{12} + p_{12}^2 - ab = 0.$$

Which gives $p_{12} = a + \sqrt{a^2 + ab}$ and

$$p_{11} = \sqrt{2a^2 + 2a\sqrt{a^2 + ab}}, \quad p_{22} = \sqrt{2a + 2\sqrt{a^2 + ab}}\,\sqrt{a^2 + b}.$$

2. Coupling between control and estimation

The stabilization problem with full information. Consider a noisy control system

$$z' = Az + Bu + \mu, \quad z(0) = z_0 + \mu_0.$$

Assume that (A, B) is stabilizable, we can find $K \in \mathcal{L}(Z, U)$, such that $A + BK$ is exponentially stable on Z, by solving an Algebraic Riccati Equation of the form

$$P = P^* \geq 0, \quad A^*P + PA - PBR^{-1}B^*P + Q = 0,$$

with $R = R^* > 0$ and $Q = Q^* \geq 0$, and by choosing

$$K = -R^{-1}B^*P.$$

The estimation problem. Consider a noisy model and a noisy observation

$$z' = Az + f + \mu, \quad z(0) = z_0 + \mu_0, \quad y_{obs}(t) = Hz(t) + \eta(t) \in Y_o.$$

We look for an estimation z_e of z by solving an equation of the form

$$z_e' = Az_e + f + L(Hz_e - y_{obs}), \quad z_e(0) = z_0.$$

$$\|z(t) - z_e(t)\|_Z \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.$$

The filtering operator $L \in \mathcal{L}(Y_o, Z)$ may be determined by solving an A.R.E. of the form

$$P_e = P_e^* \geq 0, \quad AP_e + P_e A^* - P_e H^* R_o^{-1} H P_e + Q_o = 0,$$

and by choosing $L = -P_e H^* R_o^{-1}$.

The operator $Q_o = Q_o^* \geq 0$ and $R_o = R_o^* > 0$ are used to balance the model error and the measurement error.

When we compare the Riccati equation for P_e and the Riccati equation used to define the control law

$$P = P^* \geq 0, \quad A^* P + PA - PBR^{-1} B^* P + Q = 0,$$

we notice that the roles of A and A^* are interchanged.

Next we use the filtering equation to determine the control by solving

$$z_e' = Az_e + BKz_e + L(Hz_e - y_{obs}), \quad z_e(0) = z_0.$$

After that we prove that the original system with the feedback coming from the estimator

$$z' = Az + BKz_e + \mu, \quad z(0) = z_0 + \mu_0,$$

is stable. Indeed the system satisfied by $(z, z_e)^T$ is

$$\begin{pmatrix} z \\ z_e \end{pmatrix}' = \begin{pmatrix} A & BK \\ -LH & A + BK + LH \end{pmatrix} \begin{pmatrix} z \\ z_e \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix}.$$

Theorem. If $(e^{t(A+BK)})_{t\geq 0}$ is exponentially stable and if $(e^{t(A+LH)})_{t\geq 0}$ is exponentially stable, then the semigroup generated by

$$\mathcal{A} = \begin{bmatrix} A & BK \\ -LH & A + BK + LH \end{bmatrix}$$

is also exponentially stable on $Z \times Z$.

Proof. If $e = z - z_e$, we have

$$\begin{pmatrix} z \\ e \end{pmatrix}' = \begin{pmatrix} A + BK & -BK \\ 0 & A + LH \end{pmatrix} \begin{pmatrix} z \\ e \end{pmatrix} + \begin{pmatrix} \mu \\ L\eta \end{pmatrix}.$$

Observability.

An initial condition $z_0 \in Z$ is unobservable for the pair (A, H) when

$$H\, e^{tA}\, z_0 = 0 \quad \text{for all } t \geq 0.$$

For finite dimensional systems, an initial condition $z_0 \in Z$ is unobservable for the pair (A, H) if and only if it is not reachable for the pair (A^*, H^*). Indeed, if $z_0 \in Z$ is unobservable, then

$$0 = \int_0^T \big(y(t), H\, e^{tA}\, z_0\big)_{Y_o}\, dt = \int_0^T \Big(e^{tA^*}\, H^* y(t), z_0\Big)_Z\, dt$$

for all $y \in L^2(0, T; Y_o)$. The converse is obvious.

A system (A, H) of finite dimension is observable when the set of unobservable state is reduced to $\{0\}$.

The pair (A, H) is observable iff the pair (A^*, H^*) is controllable.

Detectability. The dual notion of stabilizability is the notion of detectability. We say that the pair (A, H) is detectable iff there exists $L \in (Y_o, Z)$ such that

$$\left(e^{t(A+LH)}\right)_{t\geq 0} \quad \text{is exponentially stable on } Z.$$

The semigroup generated by $A + LH$ is exponentially stable on Z iff the semigroup generated by $A^* + H^* L^*$ is exponentially stable on $Z^* \equiv Z$.

This means that the pair (A, H) is detectable iff the pair (A^*, H^*) is stabilizable. One way to find L^* such that $A^* + H^* L^*$ is exponentially stable consists in using a stabilizing feedback control by solving the Riccati equation

$$P_e = P_e^* \geq 0, \quad P_e A^* + A P_e - P_e H^* H P_e + I = 0.$$

Taking into account the knowledge of covariance noises, we can solve a Riccati equation of the form

$$P_e = P_e^* \geq 0, \quad P_e A^* + A P_e - P_e H^* R_o^{-1} H P_e + Q_o = 0,$$

where $R_o \in \mathcal{L}(Y_o)$ and $Q_o \in \mathcal{L}(Z)$ are two symmetric and semidefinite positive operators (the covariance operators of the noises).

3. Detectability of finite dimensional systems

Example 1

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + f + \mu, \quad z(0) = z_0 + \mu_0.$$

Let us take

$$H_1 z = z_1 + z_2.$$

The state $z_0 = (1, -1)^T$ is not observable. But if $\lambda < 0$, then (A, H_1) is detectable. If $\lambda > 0$, the pair (A, H_1) is not detectable.

If $H_2 z = (z_1 + z_2, z_1 - z_2)$, then (A, H_2) is detectable. (We have a full 'noisy' information.)

Example 2

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + f + \mu, \quad z(0) = z_0 + \mu_0.$$

Let us take

$$H_1 z = z_1 + z_2.$$

If $\lambda_1 \neq \lambda_2$, the pair (A, H_1) is detectable and observable.

Conditions for detectability

• The pair (A, H) is detectable iff there exists $\alpha > 0$ such that

$$\int_0^\infty \left\| H\,\pi_u\, e^{-tA_u} z \right\|_{Y_o}^2 \, dt \geq \alpha \|z\|_Z^2 .$$

• The best constant $\alpha > 0$ can be taken as an evaluation of *a degree of detectability*.

• The pair (A, H) is detectable iff, for each 'unstable' eigenvalue λ_j of A, the corresponding family of eigenvectors (eigenfunctions) $(e_j^k)_{1\leq k\leq \ell_j}$ is such that the family

$$(He_j^1, He_j^2, \cdots, He_j^{\ell_j})$$

is linearly independent.

Example 3

$$z' = Az + f + \mu, \quad z(0) = z_0 + \mu_0, \quad z(t) \in \mathbb{R}^N, \quad A = \mathrm{diag}(\lambda_1, \cdots, \lambda_N),$$

$$Hz(t) = \sum_{i=1}^N z_i(t) \in \mathbb{R} \quad \text{and} \quad y_{obs}(t) = Hz(t) + \eta(t).$$

We are going to see that if the eigenvalues are two by two distinct, then the measure Hz is enough to estimate z.

Assume that all the eigenvalues are unstable and of multiplicity equal to 1. We have

$$H = (1, \cdots, 1).$$

The eigenvectors are the basis vectors e_i, $1 \leq i \leq N$. We have

$$He_i = 1.$$

Thus the detectability condition is trivially satisfied. However, we have

$$e^{-tA_u^*} H^* H e^{-tA_u} = \mathrm{diag}(e^{-\lambda_1 t}, \cdots, e^{-\lambda_N t})\, H^* H\, \mathrm{diag}(e^{-\lambda_1 t}, \cdots, e^{-\lambda_N t}),$$

$$H^* H = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix},$$

and

$$\int_0^\infty e^{-tA_u^*} H^* H e^{-tA_u} dt =$$

$$= \begin{pmatrix} \frac{1}{2\lambda_1} & \cdots & \frac{1}{\lambda_1+\lambda_j} & \cdots & \frac{1}{\lambda_1+\lambda_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\lambda_i+\lambda_1} & \cdots & \frac{1}{2\lambda_i} & \cdots & \frac{1}{\lambda_i+\lambda_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\lambda_N+\lambda_1} & \cdots & \frac{1}{\lambda_N+\lambda_j} & \cdots & \frac{1}{2\lambda_N} \end{pmatrix}.$$

In the case when $N = 2$,

$$\det \int_0^\infty e^{-tA_u^*} H^* H e^{-tA_u} dt = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2.$$

Thus if two eigenvalues are very close, the pair (A, H) is 'weakly' detectable.

Example 4 – The linearized inverted pendulum

We measure θ and x. We have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{g(M+m)}{M\ell} & 0 \end{pmatrix} \quad \text{and} \quad H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The controllability matrix of (A^*, H_1^*) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{mg}{M} & \frac{g(M+m)}{M\ell} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{mg}{M} & \frac{g(M+m)}{M\ell} \end{pmatrix}.$$

Thus (A^*, H_1^*) is stabilizable and (A, H_1) is detectable.

If we choose the measure operator

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix},$$

the controllability matrix of (A^*, H_2^*) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & -\frac{mg}{M} \end{pmatrix}.$$

Thus (A^*, H_2^*) is also stabilizable and (A, H_2) is detectable.

4. Detectability of infinite dimensional systems

4.1. The heat equation

$$\Omega = (0,a)\times(0,b), \quad \Gamma_o = \{0\}\times(0,b), \quad \Gamma_c = \{a\}\times(0,b)$$

$$Q = \Omega\times(0,\infty), \quad \Sigma_c = \Gamma_c\times(0,\infty), \quad \Sigma_o = \Gamma_o\times(0,\infty),$$

$\mathcal{O} = (a_1,a_2)\times(b_1,b_2), \quad \chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$,

m is a truncation function on Γ, m_o is a truncation function in Ω,

$$\frac{\partial z}{\partial t} - \Delta z = f + \mu \quad \text{in } Q,$$

$$z = m\,u \quad \text{on } \Sigma_c, \quad z = 0 \quad \text{on } \Sigma_o, \quad z(0) = z_0 + \mu_0 \quad \text{in } \Omega\,.$$

Example of bounded (in $L^2(\Omega)$) measure operators

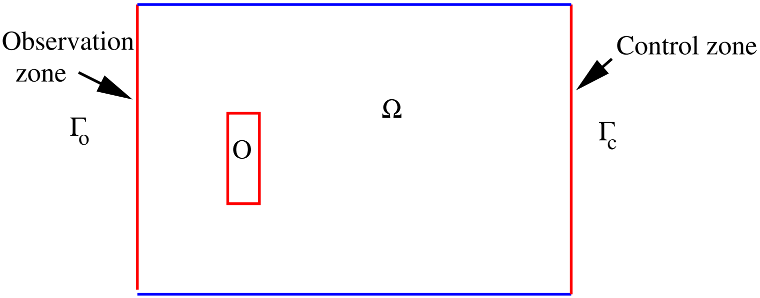
$$H_1 z(t) = z(t)\,\chi_{\mathcal{O}}, \quad H_2\, z(t) = m_o\, z(t).$$

Examples of boundary measure operators (unbounded operators in $L^2(\Omega)$)

$$H_3 z = \frac{\partial z}{\partial n} \quad \text{on } \Sigma_o, \quad H_4 z = \int_{\Gamma_o} \frac{\partial z}{\partial n},$$

$$H_5 z = \left(\int_{\Gamma_1} \frac{\partial z}{\partial n}, \cdots, \int_{\Gamma_{N_o}} \frac{\partial z}{\partial n} \right),$$

where $\Gamma_1, \cdots, \Gamma_{N_o}$ are N_o intervals of Γ_o.



Detectability condition for the heat operator.

The heat operator in $L^2(\Omega)$ is $A = \Delta$ with domain $H^2(\Omega) \cap H^1_0(\Omega)$ (if Ω is regular). Let

$$(e_j^1, e_j^2, \cdots, e_j^{\ell_j})$$

be a basis of $\mathrm{Ker}(A - \lambda_j)$.

• The pair $(A + \omega I, H_1)$ is detectable. We must verify that, for all *unstable* eigenvalue λ_j,

$$(H_1 e_j^1, H_1 e_j^2, \cdots, H_1 e_j^{\ell_j}) = \left(e_j^1\, \chi_{\mathcal{O}}, \cdots, e_j^{\ell_j}\, \chi_{\mathcal{O}} \right)$$

is lin. independent in $L^2(\mathcal{O})$.

It is a consequence of the Holmgren Theorem.

• The pair $(A + \omega I, H_3)$ is detectable. Indeed, for all unstable eigenvalues λ_j,

$$(H_3 e_j^1, H_3 e_j^2, \cdots, H_3 e_j^{\ell_j}) = \left(\frac{\partial e_j^1}{\partial n}, \cdots, \frac{\partial e_j^{\ell_j}}{\partial n} \right)$$

is lin. independent in $L^2(\Gamma_o)$.

It is a consequence of a unique continuation property.

- Detectability of the pair $(A + \omega I, H_4)$.

$$\text{Is} \quad (H_4 e_j^1, H_4 e_j^2, \cdots, H_4 e_j^{\ell_j}) = \left(\int_{\Gamma_o} \frac{\partial e_j^1}{\partial n}, \cdots, \int_{\Gamma_o} \frac{\partial e_j^{\ell_j}}{\partial n} \right)$$

lin. independent in $\mathbb{R}$ for all j such that $\lambda_j$is unstable ?

$(A + \omega I, H_4)$ is detectable if the unstable eigenvalues are simple (that is $\ell_j = 1$) and if the corresponding eigenvector e_j^1 is such that $\int_{\Gamma_o} \frac{\partial e_j^1}{\partial n} \neq 0$.

- Detectability of the pair $(A + \omega I, H_5)$.

We can choose $\Gamma_1, \cdots, \Gamma_{N_o}$ so that

$(H_5 e_j^1, H_5 e_j^2, \cdots, H_5 e_j^{\ell_j})$ is lin. independent in $\mathbb{R}^N$,

$$\text{with} \quad H_5 e_j^k = \left(\int_{\Gamma_1} \frac{\partial e_j^k}{\partial n}, \cdots, \int_{\Gamma_{N_o}} \frac{\partial e_j^k}{\partial n} \right).$$

Thus, it is possible to choose $\Gamma_1, \cdots, \Gamma_{N_o}$ so that $(A + \omega I, H_5)$ is detectable.

Calculation of H_1^*, H_2^*, H_3^*, H_4^*, H_5^*.

$$H_1 z = z \quad \text{on } \mathcal{O}, \quad H_1^* \phi = \phi\, \chi_{\mathcal{O}},\ \chi_{\mathcal{O}} \text{ is the characteristic function of } \mathcal{O},$$

$$H_2 z = m_0\, z, \quad H_2^* \phi = m_0\, \phi,$$

$$H_3 z = \frac{\partial z}{\partial n} \quad \text{on } \Gamma_o, \quad H_3^* \phi = (-A) D\, (\phi\, \chi_{\Gamma_o})\,,$$

$$H_4 z = \int_{\Gamma_o} \frac{\partial z}{\partial n}, \quad H_4^* r = (-A) D\, (r\, \chi_{\Gamma_o}) = r\, (-A) D\, (\chi_{\Gamma_o})\,,\ r \in \mathbb{R},$$

χ_{Γ_o} is the characteristic function of Γ_o,

$$H_5 z = \left(\int_{\Gamma_1} \frac{\partial z}{\partial n}, \cdots, \int_{\Gamma_{N_o}} \frac{\partial z}{\partial n} \right),$$

$$H_5^* r = H_5^* (r_1, \cdots, r_{N_o}) = \big(r_1\, (-A) D\, (\chi_{\Gamma_1})\,, \cdots, r_{N_o}\, (-A) D\, \big(\chi_{\Gamma_{N_o}}\big) \big)\,.$$

4.2. The linearized Burgers equation

The same measure operators can be considered for the linearized Burgers equation

$$\frac{\partial z}{\partial t} - \Delta z + 2\,\partial_i w_s\, z + 2\,\partial_i z\, w_s = \mu,$$

$$z = m\,u \quad \text{on } \Sigma, \quad z(0) = z_0 + \mu_0 \quad \text{in } \Omega\,.$$

The operator A in $L^2(\Omega)$ is $Az = \Delta z - 2\,\partial_i w_s\, z - 2\,\partial_i z\, w_s$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ (if Ω is regular).

If H is a measure operator, the detectability of (A, H) is equivalent to the linear independence of

$$(He_j^1, He_j^2, \cdots, He_j^{\ell_j})$$

for all *unstable* eigenvalue λ_j, where (as before)

$$(e_j^1, e_j^2, \cdots, e_j^{\ell_j})$$

is a basis of $\mathrm{Ker}(A - \lambda_j I)$.

4.3. The Stokes equation

Let us consider the Stokes equation with mixed boundary conditions

$$\frac{\partial z}{\partial t} - \nu\Delta z + \nabla p = \mu, \quad \operatorname{div} z = 0 \quad \text{in } Q,$$

$$z = m\,u \quad \text{on } \Sigma_c, \quad z = 0 \quad \text{on } \Sigma_D, \quad \sigma(z,p)n = 0 \quad \text{on } \Sigma_N,$$

$$z(0) = z_0 + \mu_0 \quad \text{in } \Omega.$$

Some boundary measure operators

$$H_1(z(t),p(t)) = \sigma(z(t),p(t))n|_{\Gamma_1}, \quad H_2(z(t),p(t)) = p(t)|_{\Gamma_1},$$

$$H_3(z(t),p(t)) = \int_{\Gamma_1} \sigma(z(t),p(t))n, \; H_4(z(t),p(t)) = \left(\int_{\Gamma_1} p(t), \dots, \int_{\Gamma_{N_o}} p(t)\right),$$

where $\Gamma_1, \cdots \Gamma_{N_o}$ are N_o intervals of Γ_o and $\Gamma_o \subset \Gamma_D$.

The pressure $p(t)$ is the solution to

$$\Delta p(t) = 0 \quad \text{in } \Omega, \quad p(t) = \nu(\nabla z + (\nabla z)^T)n \cdot n \quad \text{on } \Gamma_N,$$

$$\frac{\partial p(t)}{\partial n} = \nu \Delta z \cdot n - \frac{\partial z}{\partial t} \cdot n \quad \text{on } \Gamma \setminus \Gamma_N.$$

The same type of measure operator can be considered for the L.N.S.E.

Let us set

$$p(t) = N\left(\nu \Delta z(t) \cdot n - \frac{\partial z}{\partial t} \cdot n\right)$$

and

$$p(t)|_{\Gamma_1} = N_1\left(\nu \Delta z(t) \cdot n - \frac{\partial z}{\partial t} \cdot n\right).$$

Let us recall that

$$z(t) = \Pi z(t) + (I - \Pi)z(t) = \Pi z(t) + (I - \Pi)Du(t),$$

and

$$\frac{\partial \Pi z}{\partial t}|_{\Gamma_0} \cdot n = 0.$$

Thus, we have

$$p(t)|_{\Gamma_0} = H\Pi z(t) + H(I - \Pi)z = H_z \Pi z(t) + H_u\, u(t) + H_{u'}\, u'(t).$$

We have to give a precise meaning to the operator H_z, H_u and H_v.

For the measure corresponding to H_2, according to the equation satisfied by z, we set

$$H_z \Pi z(t) = \rho|_{\Gamma_0},$$

where ρ is the solution to

$$\Delta\rho(t) = 0 \quad \text{in } \Omega, \quad \rho(t) = \nu(\nabla \Pi z + (\nabla \Pi z)^T)n \cdot n \quad \text{on } \Gamma_N,$$

$$\frac{\partial \rho(t)}{\partial n} = \nu \Delta \Pi z \cdot n \quad \text{on } \Gamma \setminus \Gamma_N.$$

We set

$$H_u u(t) = \theta|_{\Gamma_0},$$

where θ is the solution to

$$\begin{aligned} \Delta\theta(t) = 0 \quad &\text{in } \Omega, \\ \theta(t) = \nu(\nabla(I-\Pi)Dmu + (\nabla(I-\Pi)Dmu)^T)n \cdot n \quad &\text{on } \Gamma_N, \\ \frac{\partial\theta}{\partial n}(t) = \nu\Delta(I-\Pi)Dmu \cdot n \quad &\text{on } \Gamma \setminus \Gamma_N. \end{aligned}$$

Finally, we set

$$H_v v(t) = \vartheta|_{\Gamma_0},$$

where ϑ is the solution to

$$\Delta\vartheta(t) = 0 \quad \text{in } \Omega, \quad \vartheta(t) = 0 \quad \text{on } \Gamma_N,$$

$$\frac{\partial\vartheta(t)}{\partial n} = -(I-\Pi)Dmv \cdot n \quad \text{on } \Gamma \setminus \Gamma_N.$$

To estimate z, we have to assume that u' is well defined. In that case the estimator is of the form

$$\begin{aligned}
&\Pi z_e' = A\Pi z_e + Bu + L(H_z\Pi z_e + H_u\, u(t) + H_{u'}\, u'(t) - y_{obs}),\\
&z_e(0) = z_0,\\
&(I-\Pi)z_e(t) = (I-\Pi)Du(t),
\end{aligned}$$

where $L = -P_e H_z^* R_o^{-1}$ and P_e is the solution of

$$P_e = P_e^* \geq 0, \quad P_e A^* + AP_e - P_e H_z^*\, R_o^{-1}\, H_z P_e + Q_o = 0.$$

To determine the control

$$u(t) = -B^* P \Pi z_e,$$

we have to solve the equation

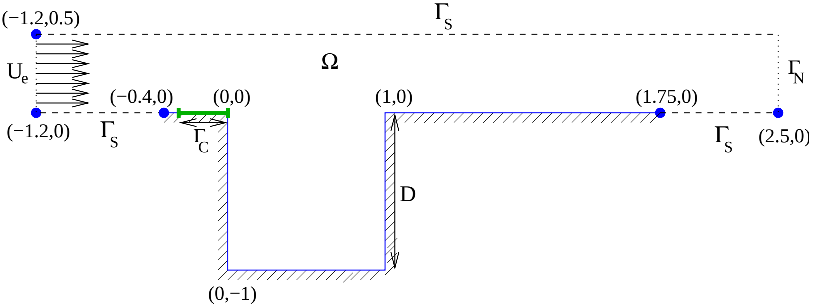
$$\Pi z_e' = A \Pi z_e + Bu + L(H_z \Pi z_e - H_u B^* P \Pi z_e - H_{u'} B^* P \Pi z_e' - y_{obs}),$$
$$z_e(0) = z_0.$$

This equation is not necessarily well posed because the time derivative z_e' appears on both sides of the equation.

Thus if we want to couple the estimator and the control law, we have to consider another system in which u will play the role of a new state variable and u' is the new control variable. It will be called 'extended system'.

The same boundary measure operators can be considered for the L.N.S.E.

Detectability of the flow in an open cavity by boundary pressure measurement



The boundary measure operators are now

$$H_1(z(t),p(t)) = \sigma(z(t),p(t))n|_{\Gamma_1}, \quad H_2(z(t),p(t)) = p(t)|_{\Gamma_1},$$

$$H_3(z(t),p(t)) = \int_{\Gamma_1} \sigma(z(t),p(t))n, \; H_4(z(t),p(t)) = \Big(\int_{\Gamma_1} p(t), \ldots, \int_{\Gamma_{N_o}} p(t) \Big),$$

where $\Gamma_1, \cdots \Gamma_{N_o}$ are N_o intervals in Γ_o and $\Gamma_o \subset \Gamma_D$.

- Detectability of $(A+\omega I, H_1)$

We have to show that if e is an eigenfunction of A and p_e the associated pressure, and if

$$H_1(e,p) = \sigma(e,p_e)n|_{\Gamma_1} = 0 \quad \text{with e.g.} \qquad \Gamma_1 = (1, 1.1),$$

then $e \equiv 0$. This is a consequence of the unique continuation property for the Oseen operator (Fabre and Lebeau, 96).

Then (A, H_1) is detectable. Indeed for each eigenfunction e, the vector $\sigma(e,p_e)n|_{\Gamma_1}$ is non zero.

- Detectability of $(A+\omega I, H_2)$

We have to show that if λ_j is an unstable eigenvalue, e_j belongs to $\mathrm{Ker}(A-\lambda_j)$, p_j is the pressure associated with e_j and if

$$H_2 e_j = p_j|_{\Gamma_1} = 0 \quad \text{with e.g.} \qquad \Gamma_1 = (1, 1.1),$$

then $(e_j, p_j) \equiv 0$.

There is no mathematical result for that, but it can be checked numerically.

- Detectability of $(A + \omega I, H_3)$ and $(A + \omega I, H_4)$

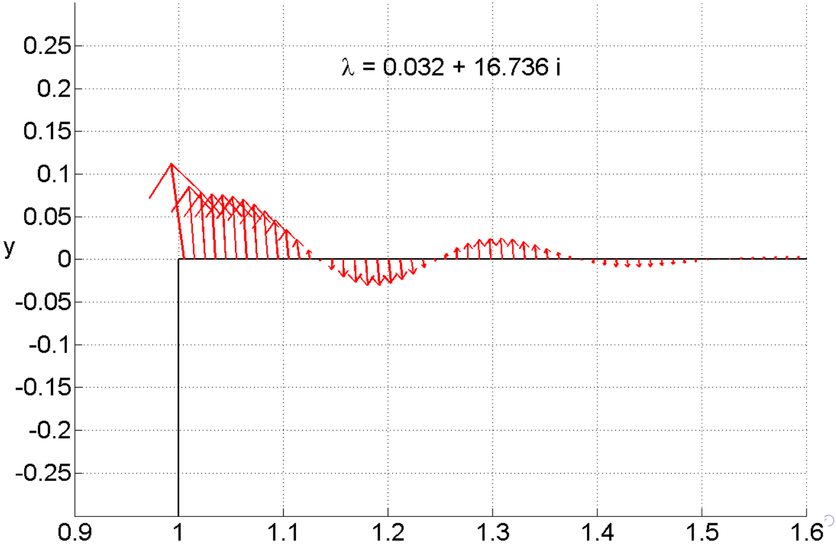
If the unstable eigenvalues are simple and if

$$\int_{\Gamma_1} p_j \quad \text{with} \quad \Gamma_1 = (1, 1.1),$$

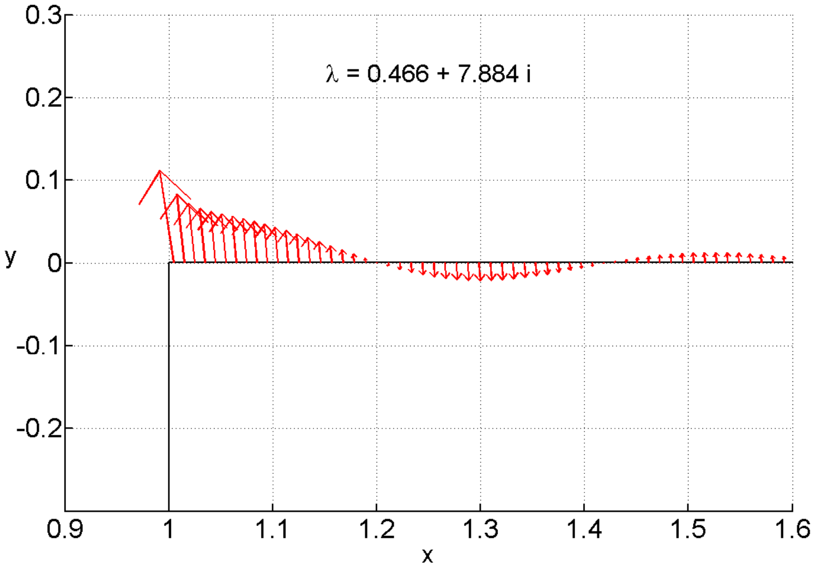
is non zero for the eigenfunctions e_j corresponding to the unstable eigenvalues, then $(A + \omega I, H_3)$ and $(A + \omega I, H_4)$ are detectable.

Numerical verification of the detectability condition for the L.N.S.E.

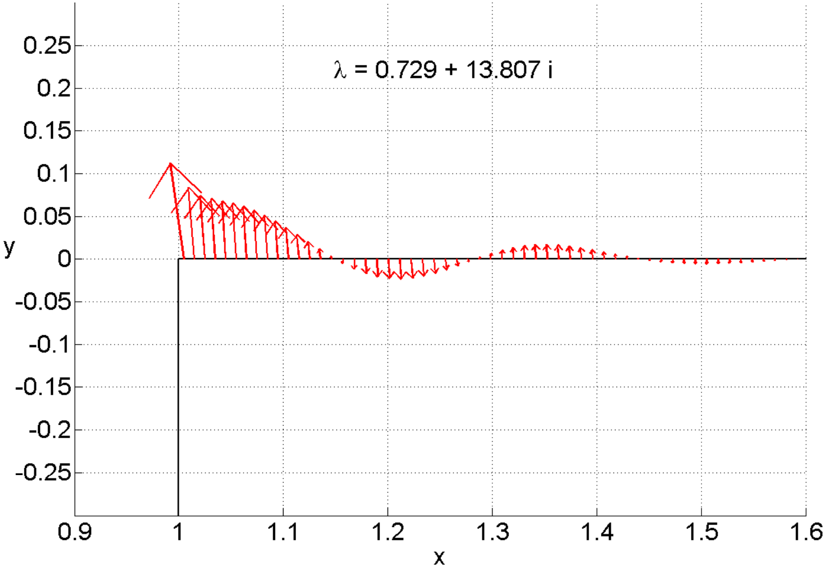
Stress tensor at the boundary for A – Cavity with $Re = 7500$



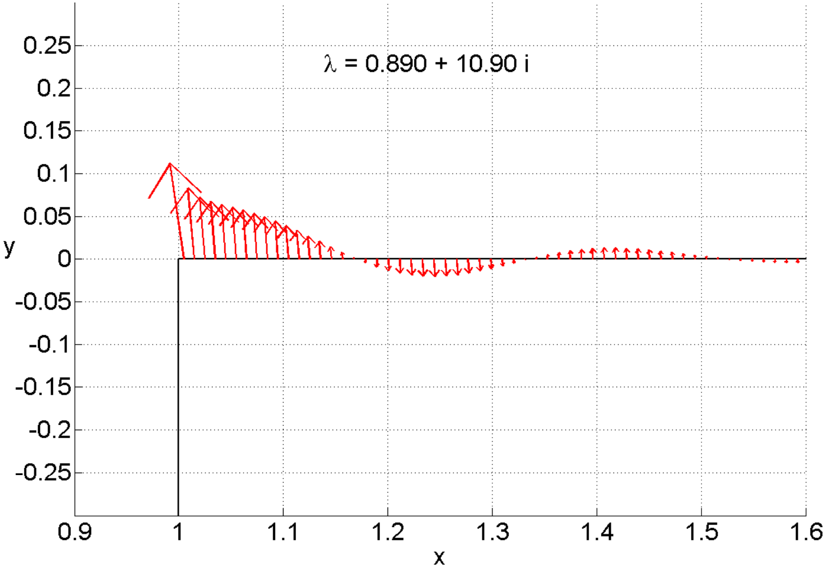
Stress tensor at the boundary for A – Cavity with $Re = 7500$



Stress tensor at the boundary for A – Cavity with $Re = 7500$



Stress tensor at the boundary for A – Cavity with $Re = 7500$



One way to improve the efficiency of the estimator is to choose H so that the restriction of H to Z_u, that is $H\,\pi_u$, satisfies

$$\pi_u^*\, H^*\, H\,\pi_u > 0.$$

5. An extended system

If we look for a control u belonging to $H^1(0,\infty;U)$, we can consider u as an additional state variable, we can add the equation $u' = \Lambda_u u + v$, with Λ_u diagonal (for simplicity) with real coefficients, and we can choose v as the new control variable

$$\tilde{z}' = \begin{pmatrix} \Pi z \\ u \end{pmatrix}' = \begin{pmatrix} A & B \\ 0 & \Lambda_u \end{pmatrix} \begin{pmatrix} \Pi z \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ I_u \end{pmatrix} v, \quad \begin{pmatrix} z(0) \\ u(0) \end{pmatrix} = \begin{pmatrix} z_0 \\ u_0 \end{pmatrix}.$$

We set

$$\widetilde{A} = \begin{pmatrix} A & B \\ 0 & \Lambda_u \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} 0 \\ I_u \end{pmatrix}.$$

We assume that U is of finite dimension. The operator $\widetilde{A}$, with

$$D(\widetilde{A}) = \{(z,u) \in V^0_{n,\Gamma_D}(\Omega) \times U \mid Az + Bu \in V^0_{n,\Gamma_D}(\Omega)\},$$

is the generator of an analytic semigroup on $V^0_{n,\Gamma_D}(\Omega) \times U$. And $\widetilde{B} \in \mathcal{L}(U, Z \times U)$ with $Z = V^0_{n,\Gamma_D}(\Omega)$.

Thus, we have replaced the unbounded operator B by $\widetilde{B} \in \mathcal{L}(U, \widetilde{Z})$, with $\widetilde{Z} = V^0_{n,\Gamma_D}(\Omega) \times U$.

5.1. Stabilizability of the extended system

The adjoints of $\widetilde{A}$ and $\widetilde{B}$ are

$$\widetilde{A}^* = \begin{pmatrix} A^* & 0 \\ B^* & \Lambda_u \end{pmatrix}, \quad \widetilde{B}^* = (0 \quad I_u),$$

with

$$D(\widetilde{A}^*) = \{(\phi, \gamma) \in V^0_{n,\Gamma_D}(\Omega) \times U \mid A^*\phi \in V^0_{n,\Gamma_D}(\Omega),\ B^*\phi \in U\}.$$

Now, we study the stabilizability of the extended system

$$\widetilde{z}' = \widetilde{A}\widetilde{z} + \widetilde{B}v, \quad \widetilde{z}(0) = \widetilde{z}_0.$$

Theorem

We assume that $\sigma(A) \cap \sigma(\Lambda_u) = \emptyset$ and $-\omega \notin \sigma(A)$. The system $(A + \omega I_z, B)$ is stabilizable by a control $u \in L^2(0, \infty; U)$ iff the extended system $(\widetilde{A} + \omega I_{\widetilde{z}}, \widetilde{B})$ is stabilizable by a control $v \in L^2(0, \infty; U)$.

Consequence. The system $(A + \omega I_z, B)$ is stabilizable by a control $u \in L^2(0, \infty; U)$ iff it is stabilizable by a control $u \in H^1(0, \infty; U)$.

Idea of the proof. Consider $(\phi, \gamma) \in D(A^*) \times U$, an eigenfunction of $\widetilde{A}^*$

$$A^*\phi = (\lambda - \omega)\,\phi, \quad B^*\phi + \Lambda_u\gamma = (\lambda - \omega)\,\gamma.$$

Thus ϕ is an eigenfunction of A^*, $\lambda - \omega \in \sigma(A^*)$, $\lambda - \omega \notin \sigma(\Lambda_u)$, and $\gamma = (\Lambda_u - (\lambda - \omega)I_u)^{-1}B^*\phi$. In particular the eigenvalues of A^* and $\widetilde{A}^*$ are the same ones.

The pair $(A + \omega I_z, B)$ is stabilizable iff, for all unstable eigenvalue λ_j, the family

$$(B^*\phi_j^k)_{1\leq k\leq \ell_j}$$

is linearly independent.

The pair $(\widetilde{A} + \omega I_{\tilde{z}}, \widetilde{B})$ is stabilizable iff, for all unstable eigenvalue λ_j, the family

$$(\widetilde{B}^*\phi_j^k)_{1\leq k\leq \ell_j} = (\gamma_j^k)_{1\leq k\leq \ell_j} = ((\Lambda_u - (\lambda_j - \omega)I_u)^{-1}B^*\phi_j^k)_{1\leq k\leq \ell_j}$$

is linearly independent. The equivalence is obvious.

A Riccati equation for finding a feedback control law for the extended system.

The Riccati equation for the extended system associated with the cost functional

$$J(z,u,v) = \int_0^\infty \|Cz\|_Y^2 + \int_0^\infty \|R^{1/2}u\|_U^2 + \int_0^\infty \|v\|_U^2,$$

is

$$\widetilde{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \in \mathcal{L}(Z \times U), \quad \widetilde{P} = \widetilde{P}^* \geq 0,$$

$$\widetilde{P}\widetilde{A}_\omega + \widetilde{A}_\omega^*\widetilde{P} - \widetilde{P}\widetilde{B}\widetilde{B}^*\widetilde{P} + \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} = 0, \quad Q = C^*C.$$

This is equivalent to

$$P_{11}A_\omega + A_\omega^*P_{11} - P_{12}P_{21} + Q = 0, \quad P_{11}B + A_\omega^*P_{12} - P_{12}P_{22} = 0,$$

$$P_{21}A_\omega + B^*P_{11} - P_{22}P_{21} = 0, \quad B^*P_{12} + P_{21}B - P_{22}^2 + R = 0.$$

5.2. Detectability of the extended system

The measure operator is modified as follows

$$
\begin{aligned}
& H\left(z(t),p(t)\right) = H_z\Pi z(t) + H_u u(t) + H_{u'}\Lambda_u u(t) + H_{u'} v(t) \\
& = H_z\Pi z(t) + \widetilde{H}_u u(t) + H_v v(t) = H_{\tilde{z}}\widetilde{z}(t) + \widetilde{H}_u u(t) + H_v v(t).
\end{aligned}
$$

Assume that $\sigma(\Lambda_u) = \{\alpha_1, \cdots, \alpha_{N_c}\}$ and

$$\Lambda_u g_k = \alpha_k g_k, \quad \text{for } 1 \le k \le N_c.$$

For each $k \in \{1, \cdots, N_c\}$, we consider that following equation

$$
(\mathcal{EP}) \qquad
\begin{aligned}
& (w_s \cdot \nabla)e_k + (e_k \cdot \nabla)w_s - \mathrm{div}\sigma(e_k, p_k) = \alpha_k\, e_k \quad \text{in } \Omega, \\
& e_k = 0 \quad \text{on } \Gamma_d, \quad e_k = M g_k \quad \text{on } \Gamma_c, \\
& (e_k, p_k) \text{ obeys the O. B. C.,}
\end{aligned}
$$

and we assume that α_k does not belong to $\sigma(A)$. Thus the above equation admits a unique solution (e_k, p_k).

Theorem. Assume that $\sigma(A) \cap \sigma(\Lambda_u) = \emptyset$, the pair $(A + \omega I, H_{i,z})$ is detectable in Z, $\Lambda_u \in \mathcal{L}(U)$ is diagonalizable with a real spectrum, the eigenvalues of Λ_u are simple. Assume in addition that, for all $k \in \{1, \cdots, N_c\}$, the solution (e_k, p_k) to equation $(\mathcal{EP})$ obeys $H_z P e_k + \widetilde{H}_u g_k \neq 0$. Then the pair $(\widetilde{A} + \omega I_{\tilde{z}}, H_{\tilde{z}})$ is detectable in $\widetilde{Z} = Z \times U$.

To estimate the solution to

$$\widetilde{z}' = \widetilde{A}_\omega \widetilde{z} + \widetilde{B} v + \widetilde{\mu}, \quad \widetilde{z}(0) = \widetilde{z}_0 + \widetilde{\mu}_0,$$

based on the observation

$$y_{obs}(t) = H(z(t), p(t)) + \eta(t) = H_z \Pi z(t) + H_u u(t) + H_v v(t) + \eta(t),$$

we choose a filtering gain of the form

$$\widetilde{L} = -\widetilde{P}_e H_{\tilde{z}}^* \widetilde{R}_0^{-1}.$$

Later on v will be written in the form

$$v = -\widetilde{B}^* \widetilde{P} (\Pi z_e, u_e)^T = -P_{21} \Pi z_e - P_{22} u.$$

6. Coupling estimation and control for Linearized Navier-Stokes equations

The coupled system

$$
\begin{aligned}
&\Pi z_e' = A_\omega \Pi z_e + B u_e + \widetilde{L}_z(H_z \Pi z_e + H_u\, u_e(t) + H_v\, v(t) - y_{obs}),\\
&u_e' = (\Lambda_u + \omega I_u) u_e + \widetilde{L}_z(H_z \Pi z_e + H_u\, u_e(t) + H_v\,(P_{21}\Pi z_e + P_{22} u_e) - y_{obs})\\
&\qquad\qquad -P_{21}\Pi z_e - P_{22} u_e,\\
&\Pi z_e(0) = \Pi z_0, \quad u_e(0) = u_0,\\
&(I-\Pi) z_e(t) = (I-\Pi) D\, u_e(t), \quad (I-\Pi) z(t) = (I-\Pi) D\, u(t)\\
&\Pi z' = A_\omega \Pi z + B u + \mu,\\
&u' = (\Lambda_u + \omega I_u) u - P_{21}\Pi z_e - P_{22} u_e, \quad u(0) = u_0,\\
&\Pi z(0) = \Pi z_0 + \mu_0, \quad u(0) = u_0.
\end{aligned}
$$

is well posed.

Let us denote by $\widetilde{K}$ the feedback used to stabilize the pair $(\widetilde{A} + \omega I_{\tilde{z}}, \widetilde{B})$ and $\widetilde{L}$ the filtering operator stabilizing the pair $(\widetilde{A} + \omega I_{\tilde{z}}, \widetilde{B})$.

The decoupling principle for the extended system can be expressed has follows.

Theorem. If $(e^{t(\widetilde{A}+\omega I_{\tilde{z}}+\widetilde{B}\widetilde{K})})_{t\geq 0}$ is exponentially stable on $Z \times U \times U$ and if $(e^{t(\widetilde{A}+\omega I_{\tilde{z}}+\widetilde{L}\widetilde{H})})_{t\geq 0}$ is exponentially stable on $Z \times U \times U$, then the semigroup generated by

$$\mathcal{A}_\omega = \begin{bmatrix} \widetilde{A} + \omega I_{\tilde{z}} & \widetilde{B}\widetilde{K} \\ -\widetilde{L}\widetilde{H} & \widetilde{A} + \omega I_{\tilde{z}} + \widetilde{B}\widetilde{K} + \widetilde{L}\widetilde{H} \end{bmatrix}$$

is also exponentially stable on $(Z \times U \times U) \times (Z \times U \times U)$.

7. Local feedback stabilization of the Navier-Stokes equations with partial information

We can define the extended system for the Navier-Stokes equations as follows

$$
\begin{aligned}
&\Pi z_e' = A_\omega \Pi z_e + B u_e + \widetilde{L}_z (H_z \Pi z_e + H_u\, u_e(t) + H_v\, v(t) - y_{obs}),\\
&u_e' = (\Lambda_u + \omega I_u) u_e + \widetilde{L}_z (H_z \Pi z_e + H_u\, u_e(t) - H_v\,(P_{21}\Pi z_e + P_{22} u_e) - y_{obs})\\
&\qquad\qquad\qquad\qquad - P_{21}\Pi z_e - P_{22} u_e,\\
&\Pi z_e(0) = \Pi z_0, \quad u_e(0) = u_0,\\
&(I - \Pi) z_e(t) = (I - \Pi) D\, u_e(t), \quad (I - \Pi) z(t) = (I - \Pi) D\, u(t)\\
&\Pi z' = A_\omega \Pi z + B u + \mathcal{F}(\Pi z + (I - \Pi z)) + \mu,\\
&u' = \Lambda_u u - P_{21}\Pi z_e - P_{22} u_e, \quad u(0) = u_0,\\
&\Pi z(0) = \Pi z_0 + \mu_0, \quad u(0) = u_0,
\end{aligned}
$$

where $\mathcal{F}(\Pi z + (I - \Pi z))$ stands for the nonlinear term of the Navier-Stokes equation.

The measure depends non linearly on z

Let us notice that for the nonlinear equation, the measure Hz is now decomposed as follows

$$Hz(t) = H_z \Pi z + H_u u + H_v u'(t) + H_{\mathcal{F}} \mathcal{F}(\Pi z + (I - \Pi z)).$$

Indeed, when we solve the equation for the pressure, the nonlinear term appears as a source term. Thus $Hz(t)$ depends nonlinearly on $z(t)$.

Local feedback stabilization with partial information

As in the case of feedback with full information, we can prove that the previous nonlinear system admis a solution provided that the data are small enough.

Next step

We have to find an estimator of finite dimension.