Stochastic Analysis and Control of Fluid Flows

Lecture 3

School of Mathematics – IISER-TVM

December 3 – December 20, 2012

Stabilization of the Navier-Stokes equations with mixed boundary conditions

Jean-Pierre Raymond – Institut Mathématiques de Toulouse

Plan of lecture 3

1. Problem and models

2. Rewriting P.D.E. as control systems

3. Stabilizability of linearized models

4. Feedback stabilization of linearized models

5. Local feedback stabilization of nonlinear models

1. Problem and models

- We consider a fluid flow governed by the N.S.E.
- Given an unstable stationary solution w_s.
- Find a Dirichlet boundary control u in feedback form

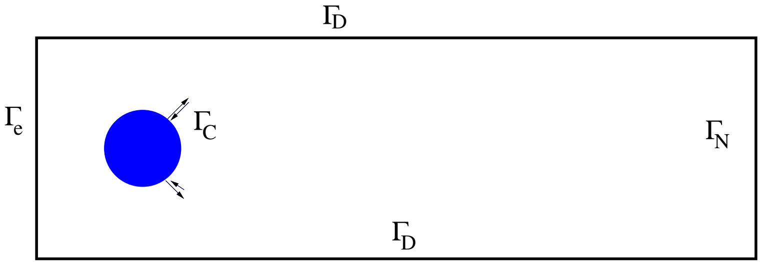
$$u(t) = K(w(t) - w_s)$$

able to stabilize $w(t) - w_s$ exponentially when $w(0) = w_s + z_0$, provided that z_0 is small enough.

For regular domain with Dirichlet B.C., see Barbu, Lasiecka, Triggiani, Fursikov, Badra, Raymond, Rowley, Sipp...

Numerical Algorithms, see Benner, Styckel, Mermann...

The case of the flow around a cylinder with an outflow boundary condition – 2D domain



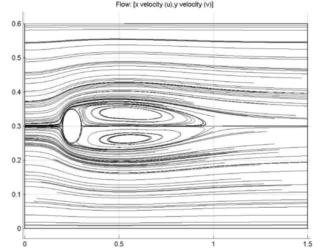
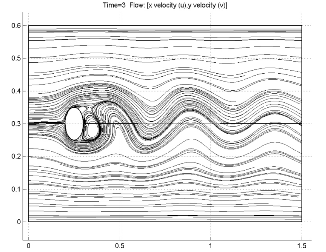
Boundary conditions

$$z = u_s \quad \text{on} \quad \Gamma_e \times (0,\infty), \qquad z = 0 \quad \text{on} \quad \Gamma_d \times (0,\infty),$$

$$z = Mu \quad \text{on} \quad \Gamma_c \times (0,\infty),$$

$$\nu \frac{\partial z}{\partial n} - pn = 0 \quad \text{or} \quad \sigma(z,p)n = 0 \quad \text{on} \quad \Gamma_N \times (0,\infty).$$

Control of the wake behind an obstacle – $Re = u_e Diam/\nu$

	$5 < Re < 50$	A fixed pair of vortices
	$50 < Re < 150$	Vortex street

The unstable stationary solution w_s of the N.S.E.

$$-\nu\Delta w_s + (w_s\cdot\nabla)w_s + \nabla p_s = 0, \quad \text{in } \Omega,$$

$$\text{div } w_s = 0 \quad \text{in } \Omega, \quad w_s = u_s \text{ on } \Gamma_e \quad + \text{ Other B.C. on } \Gamma\setminus\Gamma_e.$$

The stabilization problem

$$\text{Find } u \text{ in feedback form } \quad u(t) = K(w(t) - w_s),$$

$$\text{s.t.} \quad |w(t) - w_s|_{L^2} \longrightarrow 0 \qquad \text{as } t \longrightarrow \infty,$$

$$\frac{\partial w}{\partial t} - \nu\Delta w + (w\cdot\nabla)w + \nabla q = 0, \qquad \text{div } w = 0 \quad \text{in } Q,$$

$$w = u_s \text{ on } \Sigma_e = \Gamma_e \times (0,\infty), \quad w = Mu \text{ on } \Sigma_c = \Gamma_c \times (0,\infty),$$

$$+ \text{ Other B.C. on } \Sigma\setminus(\Sigma_e\cup\Sigma_c), \qquad w(0) = w_0 \text{ in } \Omega.$$

Set $z = w - w_s$, $p = q - p_s$. The linearized (resp. nonlinear) equation is

$$\frac{\partial z}{\partial t} - \nu\Delta z + (w_s \cdot \nabla)z + (z \cdot \nabla)w_s + (z \cdot \nabla)z + \nabla p = 0,$$

$$\text{div}\, z = 0 \quad \text{in } Q, \quad z = Mu \quad \text{on } \Sigma_c,$$

$$+ \text{ Other B.C. on } \Sigma \setminus (\Sigma_e \cup \Sigma_c), \qquad z(0) = z_0 \quad \text{in } \Omega\,.$$

with $u(t) = Kz(t)$ and

$$\text{supp}\, M \subset \Gamma_c.$$

2. Rewriting the P.D.E. as a control system

In the case of an internal control we can write the controlled Navier-Stokes system as

$$z' = Az + Bu + F(z), \quad z(0) = z_0, \quad F(0) = F'(0) = 0.$$

• $(A, D(A))$ is the Oseen operator and Bu stands for the internal control operator. The pressure is eliminated with the Leray projector Π. We are in the case when $z = \Pi z$.

With non homogeneous Dirichlet B.C., we obtain a system of the form

$$\Pi z' = A\Pi z + Bu + F(\Pi z + (I - \Pi)z), \quad z(0) = z_0, \quad F(0) = F'(0) = 0,$$

$$(I - \Pi)z = (I - \Pi)DMu.$$

The Helmholtz decomposition in the case of mixed D/N boundary conditions

$$V^0_{n,\Gamma_D}(\Omega) = \Big\{ z \in L^2(\Omega;\mathbb{R}^d) \mid \ \mathrm{div}\, z = 0,\ z \cdot n = 0 \text{ on } \Gamma_D \Big\},$$

$$L^2(\Omega;\mathbb{R}^d) = V^0_{n,\Gamma_D}(\Omega) \oplus \mathrm{grad}\, H^1_{\Gamma_N}(\Omega),$$

$$\mathrm{grad}\, H^1_{\Gamma_N}(\Omega) = \{ p \in H^1(\Omega) \mid p|_{\Gamma_N} = 0 \}.$$

$$\Pi \ : \ L^2(\Omega;\mathbb{R}^d) \longmapsto V^0_{n,\Gamma_D}(\Omega).$$

To define the Stokes operator, we need

$$V^1_{\Gamma_D}(\Omega) = \Big\{ z \in H^1(\Omega;\mathbb{R}^d) \cap V^0_{n,\Gamma_D}(\Omega) \mid z = 0 \text{ on } \Gamma_D \Big\},$$

$$V^1_{\Gamma_D}(\Omega) \hookrightarrow V^0_{n,\Gamma_D}(\Omega) \hookrightarrow V^{-1}_{\Gamma_D}(\Omega) = (V^1_{\Gamma_D}(\Omega))'.$$

The Helmholtz projector Π

$$\Pi f = f - \nabla p - \nabla q,$$

$$\Delta p = \operatorname{div} f \in H^{-1}(\Omega), \quad p \in H_0^1(\Omega),$$

$$\Delta q = 0, \quad \frac{\partial q}{\partial n} = (f - \nabla p) \cdot n \quad \text{on } \Gamma_D, \quad q = 0 \quad \text{on } \Gamma_N.$$

The Stokes operator $(A_0, D(A_0))$ in the case of Mixed D/N B.C. with a junction between the Dirichlet and the Neumann condition

$$D(A_0) = \Big\{ z \in V^1_{\Gamma_D}(\Omega) \mid$$

$$\exists p \in L^2(\Omega) \text{ s. t. } \mathrm{div}\sigma(z,p) \in L^2(\Omega;\mathbb{R}^d)$$

$$\text{and } \sigma(z,p)n = 0 \quad \text{on} \quad \Gamma_N \Big\},$$

$$A_0 z = \Pi \mathrm{div}\sigma(z,p) \quad (\text{does not depend on } p).$$

The Oseen operator $(A, D(A))$ is defined by

$$D(A) = D(A_0) \quad \text{and} \quad Az = A_0 z + \Pi\left((w_s\cdot\nabla)z + (z\cdot\nabla)w_s\right).$$

In the 3D case with a right angle junction, we have

$$D(A_0) \subset H^{3/2+\varepsilon}(\Omega; \mathbb{R}^d) \quad \text{for some} \quad \varepsilon > 0.$$

(See Maz'ya and Rossmann, 2007.)

Theorem. The operator $(A, D(A))$ is the infinitesimal generator of an analytic semigroup on $V^0_{n,\Gamma_D}(\Omega)$. Its resolvent is compact.

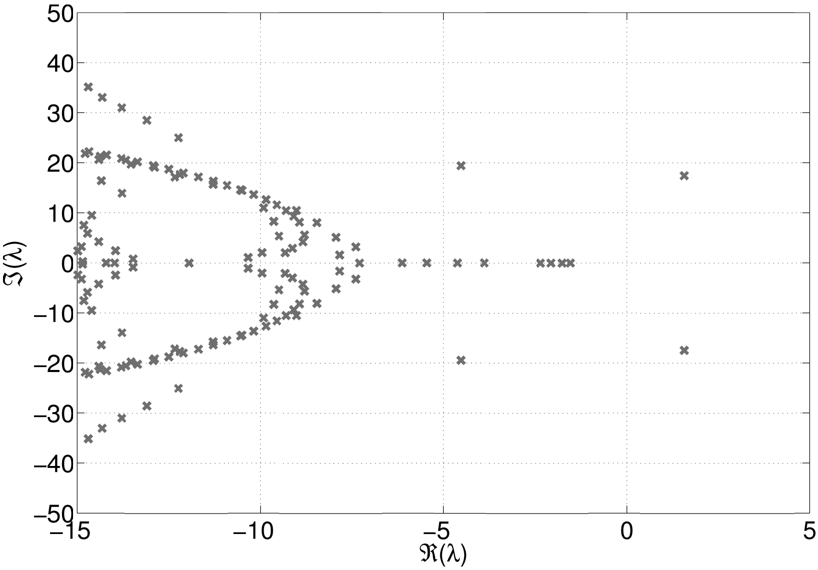
Proof.

$$((\lambda_0 I - A)z, z) \geq \frac{1}{2}\|z\|^2_{V^1_{\Gamma_D}(\Omega)} \quad \forall z \in D(A),$$

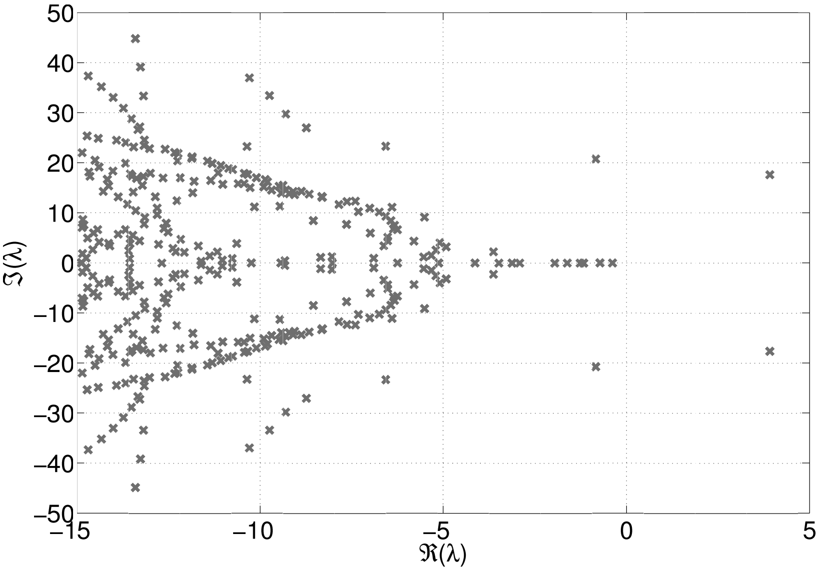
with $\lambda_0 > 0$ big enough.

Consequence. The spectrum of A is contained in a sector. The eigenvalues are isolated, pairwise conjugate when they are not real, and of finite multiplicity.

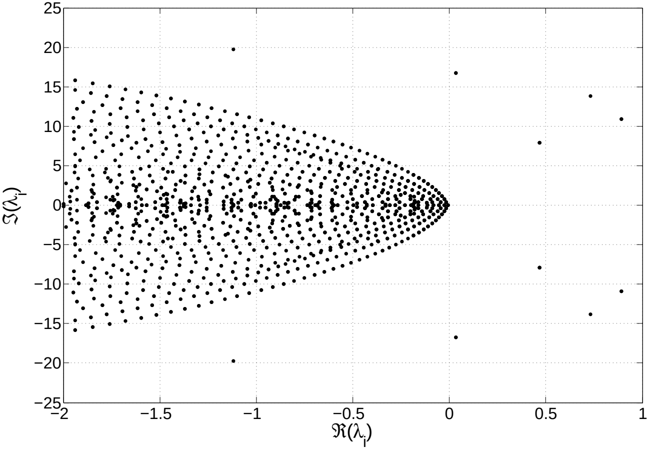
Spectrum of A. $Re = u_e\, Diam/\nu = 80$ (Cylinder)



Spectrum of A with $Re = 200$



Spectrum of *A*. $Re = u_e \times h_{cavity}/\nu = 7500$ (Cavity)



Rewriting the Oseen and N.S. equations as a control system. For the Stokes equation, see lecture 2. We look for (z, p) in the form

$$z = y + w \quad \text{and} \quad p = q + \rho,$$

where (w, ρ) is a lifting of the B.C. $z = Mu$ on Γ_c. We define $DMu(t) = w(t)$ by

$$\lambda_0 w(t) - \nu\Delta w(t) + (w_s \cdot \nabla)w(t) + (w(t) \cdot \nabla)w_s + \nabla\rho(t) = 0,$$

$$\operatorname{div} w(t) = 0, \quad w(t) = Mu(t) \quad \text{on } \Gamma_D, \quad \sigma(w(t), \rho(t))n = 0 \quad \text{on} \quad \Gamma_N.$$

The equation for y is:

$$\frac{\partial y}{\partial t} = \nu\Delta y - (w_s \cdot \nabla)y - (y \cdot \nabla)w_s - \nabla q - w' + \lambda_0 w, \quad \operatorname{div} y = 0,$$

$$y = 0 \quad \text{on } \Sigma_D, \quad \sigma(y, q)n = 0 \quad \text{on } \Sigma_N, \quad y(0) = z_0 - w(0).$$

Evolution equation satisfied by y:

$$y'(t) = Ay - \Pi w'(t) + \lambda_0 \Pi w(t), \qquad y(0) = \Pi(z_0 - w(0)).$$

With the Oseen semigroup we obtain

$$y(t) = e^{tA}(z_0 - w(0)) - \int_0^t e^{(t-\tau)A}\left(\Pi w'(\tau) - \lambda_0 \Pi w(\tau)\right) d\tau.$$

Integrating by parts

$$y(t) = e^{tA} z_0 + \int_0^t (\lambda_0 I - A) e^{(t-\tau)A}\, \Pi w(\tau)\, d\tau - \Pi w(t).$$

Therefore

$$\Pi z(t) = y(t) + \Pi w(t) = e^{tA} z_0 + \int_0^t (\lambda_0 I - A) e^{(t-\tau)A} \Pi DMu(\tau)\, d\tau.$$

This means that

$$\Pi z' = A\Pi z + (\lambda_0 I - A)\Pi DMu, \qquad \Pi z(0) = z_0.$$

(we have to extend the semigroup to $(\mathcal{D}(A^*))'$)

What is the equation satisfied by $(I - \Pi)z$?

$$(I - \Pi)z(t) = (I - \Pi)w(t) = (I - \Pi)DMu(t).$$

The system satisfied by z is finally :

$$\Pi z' = A\Pi z + (\lambda_0 I - A)\Pi DMu, \qquad \Pi z(0) = z_0,$$

$$(I - \Pi)z = (I - \Pi)DMu = (I - \Pi)D(Mu \cdot n\, n).$$

3. Stabilizability of the linearized N.S.E.

i. Null controllability results. (Fernandez-Cara, Guerrero, Imanuvilov, Puel 04, Immanuvilov and Fursikov, 96–01) (Carleman inequality)

ii. Linear independence of the generalized eigenfunctions of A^* restricted to the control zone, associated to the unstable eigenvalues, implies the stabilizability of the L.N.S.E.. (Fursikov 01, 04, Barbu-Triggiani 04), $\partial\Omega$ is regular and B.C. are of Dirichlet type.

iii. To stabilize the L.N.S.E. up to a decay rate $-\omega$, it is sufficient to stabilize the finite dimensional systems obtained by projecting the L.N.S.E. onto the unstable subspace.

The stabilizability of the projected system is equivalent to **the linear independence of the images by B^* of the bases** of eigenfunctions associated to each unstable eigenvalues. (Fattorini, Triggiani, Badra-Takahashi 10, JPR 11.)

Theorem. Assume that the semigroup generated by $(A, D(A))$ is analytic on Y, the resolvent of A is compact, $(\lambda_0 I - A)^{\alpha-1} B \in \mathcal{L}(U, Y)$, and the spectrum of A obeys

$$\ldots < \mathrm{Re}\lambda_{N_u+1} < -\omega < \mathrm{Re}\lambda_{N_u} \leq \mathrm{Re}\lambda_{N_u-1} \leq \ldots \leq \mathrm{Re}\lambda_1 .$$

For $1 \leq j \leq N_u$, let $(\phi_j^k)_{1\leq k\leq \ell_j}$ be a basis of $\mathrm{Ker}(A^* - \lambda_j I)$.

The pair $(A + \omega I, B)$ is stabilizable iff, for all $1 \leq j \leq N_u$, the family

$$(B^* \phi_j^k)_{1\leq k\leq \ell_j}$$

is linearly independent.

Proof of the stabilizability.

$$A^*\phi = \lambda\,\phi \quad \text{and} \quad B^*\phi = M\Big(\sigma(\phi,\psi)n + w_s\cdot n\,\phi\Big) = 0,$$

implies that $\phi = 0$.

We can invoque the unique continuation results by Fabre-Lebeau.

If

$$\lambda\phi - \nu\Delta\phi - (w_s\cdot\nabla)\phi + (\nabla w_s)^T\phi + \nabla\psi = 0,$$

$$\text{div}\,\phi = 0 \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \Gamma_D,$$

and

$$\sigma(\phi,\psi)n + w_s\cdot n\,\phi = 0 \quad \text{on} \quad \Gamma_C \subset \Gamma_D,$$

then

$$\phi = 0 \quad \text{and} \quad \psi = 0.$$

2.2. Stabilizability – Feedback controls of minimal norm for the infinite dimensional system

The spectrum of A obeys

$$\ldots < \mathrm{Re}\lambda_{N_u+1} < -\omega < \mathrm{Re}\lambda_{N_u} \le \mathrm{Re}\lambda_{N_u-1} \le \ldots \le \mathrm{Re}\lambda_1 .$$

We can decompose Z and $Z^* \equiv Z$ as follows

$$Z = Z_{\omega,s} \oplus Z_{\omega,u}, \quad Z_{\omega,u} = \oplus_{j=1}^{N_u} G_{\mathbb{R}}(\lambda_j), \quad Z_{\omega,s} = \oplus_{j=N_u+1}^{\infty} G_{\mathbb{R}}(\lambda_j),$$

$$Z^* = Z^*_{\omega,s} \oplus Z^*_{\omega,u}, \quad Z^*_{\omega,u} = \oplus_{j=1}^{N_u} G^*_{\mathbb{R}}(\lambda_j), \quad Z^*_{\omega,s} = \oplus_{j=N_u+1}^{\infty} G^*_{\mathbb{R}}(\lambda_j).$$

$G_{\mathbb{R}}(\lambda_j)$ is the real generalized eigenspace for A.

$G^*_{\mathbb{R}}(\lambda_j)$ is the real generalized eigenspace for A^*.

Let $\pi_{\omega,u}$ the projection onto $Z_{\omega,u}$ along $Z_{\omega,s}$ and set $\pi_{\omega,s} = I - \pi_{\omega,u}$.

Similarly let $\pi^*_{\omega,u}$ the projection onto $Z^*_{\omega,u}$ along $Z^*_{\omega,s}$ and set $\pi^*_{\omega,s} = (I - \pi_{\omega,u})^*$.

There exist bases $(e_1, \cdots, e_K)$ of $Z_{\omega,u}$ and of $(\xi_1, \cdots, \xi_K)$ of $Z^*_{\omega,u}$ s. t.

$$\pi_{\omega,u} f = \sum_{i=1}^{K} (f, \xi_i) e_i \quad \text{and} \quad \pi^*_{\omega,u} f = \sum_{i=1}^{K} (f, e_i) \xi_i, \quad \forall f \in Z,$$

$$(e_i, \xi_j) = \delta_{i,j} \quad \text{for all } 1 \le i \le K,\ 1 \le j \le K,$$

where $\delta_{i,j}$ is the Kroenecker symbol. Thanks to these formula we can extend the operators $\pi_{\omega,u}$ and $\pi^*_{\omega,u}$ to $L^2(\Omega; \mathbb{R}^2)$ by setting

$$\pi_{\omega,u} f = \sum_{i=1}^{K} (f, \xi_i) e_i \quad \text{and} \quad \pi^*_{\omega,u} f = \sum_{i=1}^{K} (f, e_i) \xi_i, \quad \forall f \in L^2(\Omega; \mathbb{R}^2).$$

By using this extension, we notice that

$$\pi_{\omega,u} f = \pi_{\omega,u} \Pi f \quad \forall f \in L^2(\Omega; \mathbb{R}^2).$$

We can also extend $\pi_{\omega,u}$ to $(D(A^*))'$ by setting

$$\pi_{\omega,u} f = \sum_{i=1}^{K} \langle f, \xi_i \rangle e_i \quad \forall f \in (D(A^*))',$$

$$\pi_{\omega,u} A \Pi z = \sum_{i=1}^{K} \left(\Pi z, A^* \xi_i \right) e_i \quad \text{and} \quad \pi_{\omega,u} B u = \sum_{i=1}^{K} \left(u, B^* \xi_i \right) e_i.$$

The pair $(A + \omega I, B)$ satisfies the FCC in Z with controls in U when

$$\forall z_0 \in Z,\ \exists u \in L^2(0, \infty, U),\ \text{such that the solution to}$$

$$z' = (A + \omega I)z + Bu, \quad z(0) = z_0 \text{ obeys } \int_0^\infty |z_{z_0,u}|_Z^2 dt < \infty.$$

The following conditions are equivalent

- The pair $(A + \omega I, B)$ satisfies the FCC in Z with controls in U.
- The pair $(A + \omega I, B)$ satisfies the FCC in Z with controls in U_0 with $U_0 = \text{vect} \cup_{j=1}^{N_u} (\mathrm{Re} B^* E^*(\lambda_j) \cup \mathrm{Im} B^* E^*(\lambda_j))$. $E^*(\lambda_j) = \text{Ker}(A^* - \lambda_j I)$.
- The pair $(A + \omega I, B)$ is stabilizable by feedback in Z with controls in U_0.
- The pair $(A_{\omega,u} + \omega I_{\omega,u}, B_{\omega,u}) = (\pi_{\omega,u}(A + \omega I), \pi_{\omega,u} B)$ satisfies the FCC in $Z_{\omega,u}$ with controls in U_0
- For all $1 \leq j \leq N_u$, $\mathrm{Ker}(\lambda_j I - A^*) \cap \mathrm{Ker}(B^*) = \{0\}$.

- The extended Gramian

$$W^{\infty}_{-A_{\omega,u},B_{\omega,u}} = \int_0^{\infty} e^{-tA_{\omega,u}} B_{\omega,u} B^*_{\omega,u} e^{-tA^*_{\omega,u}} \, dt$$

is invertible.

The operator

$$P_{\omega,u} = (W^{\infty}_{-A_{\omega,u},B_{\omega,u}})^{-1} \in \mathcal{L}(Z_{\omega,u}, Z^*_{\omega,u}), \quad P_{\omega,u} = P^*_{\omega,u} \geq 0,$$

provides a stabilizing feedback for $(A_{\omega,u}, B_{\omega,u})$

$$A_{\omega,u} - B_{\omega,u} B^*_{\omega,u} P_{\omega,u} \quad \text{is exponentially stable on } Z_{\omega,u}.$$

The operator $P_{\omega,u}$ satisfies the following Algebraic Bernoulli equation (a degenerate Algebraic Riccati equation)

$$
\begin{aligned}
& P_{\omega,u} \in \mathcal{L}(Z_{\omega,u}, Z^*_{\omega,u}), \quad P_{\omega,u} = P^*_{\omega,u} \geq 0, \\
& P_{\omega,u} A_{\omega,u} + A^*_{\omega,u} P_{\omega,u} - P_{\omega,u} B_{\omega,u} B^*_{\omega,u} P_{\omega,u} \; = \; 0, \\
& P_{\omega,u} \quad \text{is invertible.}
\end{aligned}
$$

This equation is equivalent to

$$
(P_{\omega,u} A_{\omega,u} y, z)_Z + (A^*_{\omega,u} P_{\omega,u} y, z)_Z - (B^*_{\omega,u} P_{\omega,u} y, B^*_{\omega,u} P_{\omega,u} z)_U \; = \; 0,
$$

for all $y \in Z_{\omega,u}$ and all $z \in Z_{\omega,u}$. To determine $P_{\omega,u}$ it is sufficient to determine the image of a basis of $Z_{\omega,u}$ by $P_{\omega,u}$ because $Z_{\omega,u}$ is of finite dimension. Thus this equation can be written as a matrix equation.

We use this choice of stabilizing control for finding the best control location, but other choice of feedback are possible.

The operator $P = \pi^*_{\omega,u} P_{\omega,u} \pi_{\omega,u} \in \mathcal{L}(Z)$ provides a stabilizing feedback for $(A + \omega I, B)$

$$A + \omega I - BB^* P \quad \text{is exponentially stable on } Z.$$

And P is the unique solution to the A.R.E.

$$P \in \mathcal{L}(Z), \quad P = P^* \geq 0, \quad P(A + \omega I) + (A^* + \omega I)P - PBB^* P = 0.$$

$$A + \omega I - BB^* P \quad \text{is exponentially stable on } Z.$$

4. Feedback stabilization of the linearized and nonlinear systems

$$\Pi z' = A\Pi z + Bu, \qquad \Pi z(0) = \Pi z_0 = z_0.$$

We look for a feedback by solving the optimal control problem

$$\text{Minimize } \; J(z,u) = \frac{1}{2}\int_0^\infty \|C\,\Pi z\|_Y^2 + \frac{1}{2}\int_0^\infty \|u\|_{L^2(\Gamma_C)}^2$$

$$\Pi z' = A\Pi z + Bu\,, \qquad \Pi z(0) = z_0 = \Pi z_0\,.$$

where $C \in \mathcal{L}(Z, Y)$. The value function of this problem is

$$z_0 \longmapsto J(\Pi z_{z_0,u_{z_0}}, u_{z_0}) = \tfrac{1}{2}\,(Pz_0, z_0)_{L^2}$$

and

$$u_{z_0}(t) = -B^* P\,\Pi z_{z_0}(t) = K\,\Pi z_{z_0}(t),$$

where P is the solution to the A.R.E.

$$P \in \mathcal{L}(Z, Z'),\; P = P^* \geq 0, \quad PA + A^*P - PBB^*P + C^*C = 0.$$

To find the control u, we have to solve

$$\Pi z' = (A - BB^* P)\Pi z + F(\Pi z + (I - \Pi)z), \qquad \Pi z(0) = \Pi z_0 = z_0,$$
$$(I - \Pi)z = -(I - \Pi)DMB^* P\,\Pi z,$$

We shall say that $K = -B^* P$ also stabilizes the nonlinear system if the solution to the closed loop nonlinear system obeys

$$\|z(t)\| \leq C(\|z_0\|)\, e^{-\delta t}, \quad \delta > 0.$$

How to choose C so that K also stabilizes the nonlinear system ?

If we look for $z \in \mathcal{S}$, we have to identify the space $\mathcal{F}$, where

$$\begin{array}{ccc} \mathcal{S} & \longmapsto & \mathcal{F} \\ z & \longmapsto & F(z). \end{array}$$

with

$$\|F(z)\|_{\mathcal{F}} \leq C\,\|z\|_{\mathcal{S}}.$$

We have to verify that the solution z to the closed loop nonhomogeneous linear system

$$\Pi z' = (A - BB^* P)\Pi z + f, \qquad \Pi z(0) = \Pi z_0 = z_0,$$
$$(I - \Pi)z = -(I - \Pi)DMB^* P\,\Pi z,$$

obeys

$$\|z\|_{\mathcal{S}} \leq C\,(\|z_0\|_{Z_0} + \|f\|_{\mathcal{F}}).$$

Strategy 1. High gain functional. Choose C 'strong enough' (C may be unbounded) so that the value function of the control problem is a Lyapunov function of the closed loop nonlinear system.

• Tangential control. $N = 2, 3$, $C = (-A)^{3/4+\varepsilon}$. Barbu, Lasiecka, Triggiani 06. The operator P is unbounded and does not satisfy a standard Riccati equation.

Strategy 2. Low gain functional. Choose C 'weak enough' so that

$$\Pi \in \mathcal{L}(Y, D(A)) \quad \text{a smoothing operator.}$$

Tangential and normal controls. $N = 2$, $C = I$ (R. 06, SICON).
$N = 2, 3$, $C = (-A)^{-1/2}$ (R. 07, JMPA).
$N = 2$, C is of finite rank, C is a projector onto a finite dimensional space (R.-Thevenet 09, DCSD). $N = 2, 3$, $C = 0$, Kesavan, R. 09., R. 11.

When $C = \pi_{\omega,u}$, the projection onto the finite dimensional unstable subspace for $A + \omega I$, we have

$$\text{Minimize} \quad J(z,u) = \frac{1}{2}\int_0^\infty \|\pi_{\omega,u} z\|_{L^2}^2 + \frac{1}{2}\int_0^\infty \|u\|_{L^2(\Gamma_C)}^2$$

$$\Pi z' = (A + \omega I)\Pi z + Bu\,, \qquad \Pi z(0) = z_0\,.$$

Only the equation

$$\pi_{\omega,u} z' = \pi_{\omega,u}(A + \omega I)\pi_{\omega,u} z + \pi_{\omega,u} Bu\,, \qquad \pi_{\omega,u} z(0) = \pi_{\omega,u} z_0,$$

is taken into account.

When $C = 0$, the problem is

$$\text{Minimize} \quad \frac{1}{2}\int_0^\infty \|u\|_{L^2(\Gamma_C)}^2$$

$$\Pi z' = (A + \omega I)\Pi z + Bu\,, \qquad \Pi z(0) = z_0,$$

with the constraint $z \in L^2(0, \infty; Z)$.

Advantages of this new approach

• If we take C as the projector onto the unstable subspace of the dynamical system and we choose controls of finite dimension $\longrightarrow$ The corresponding Riccati equation is of finite dimension.

Numerical viewpoint

• The discretization is needed to compute the unstable eigenvalues and the corresponding eigenfunctions.

• Error estimates on the optimal control depend only on error estimates on the unstable eigenvalues and the corresponding eigenfunctions.

• The Riccati equation being of small dimension, its solution can be calculated accurately.

5. Feedback stabilization of the N.S.E.

The issues

- Regularity of functions belonging to $D(A)$ and to $D(A^*)$
- Regularity of $B^* P \Pi z$
- Regularity of solutions to the closed loop linear and nonlinear systems (by a fixed point method)

We start with

$$\frac{\partial z}{\partial t} + (w_s \cdot \nabla) z + (z \cdot \nabla) w_s - \omega z - \operatorname{div} \sigma(z,p) = -e^{\omega t}(z \cdot \nabla) z = f,$$

$$\operatorname{div} z = 0 \quad \text{in} \quad Q_\infty = \Omega \times (0,\infty),$$

$$z = Mu \quad \text{on} \quad \Sigma_D^\infty,$$

$$\sigma(z,p)n = 0 \quad \text{on} \quad \Sigma_N^\infty = \Gamma_N \times (0,\infty),$$

$$z(0) = z_0 \quad \text{on} \quad \Omega.$$

We rewrite the equation as

$$z'_{\omega,u} = A_{\omega,u} z_{\omega,u} + \mathcal{B}_{\omega,u} u + \pi_{\omega,u} f, \quad z_{\omega,u}(0) = \pi_{\omega,u} z_0,$$

$$z'_{\omega,s} = A_{\omega,s} z_{\omega,s} + \mathcal{B}_{\omega,s} u + \pi_{\omega,s} f, \quad z_{\omega,s}(0) = \pi_{\omega,s} z_0,$$

$$z_{\omega,u} = \pi_{\omega,u} z, \quad z_{\omega,s} = \pi_{\omega,s} z, \quad \Pi z = z_{\omega,u} + z_{\omega,s},$$

$$(I - \Pi) z = -(I - \Pi) D\, M\, u.$$

We choose the feedback control law

$$u(t) = -\mathcal{B}^*_{\omega,u}\mathcal{P}_{\omega,u}z_{\omega,u}(t)$$

for the equation satisfied by $z_{\omega,u}$. The control is of finite dimension. The full system is

$$z'_{\omega,u} = (A_{\omega,u} - \mathcal{B}_{\omega,u}\mathcal{B}^*_{\omega,u}\mathcal{P}_{\omega,u})z_{\omega,u} + \pi_{\omega,u}f, \quad z_{\omega,u}(0) = \pi_{\omega,u}z_0,$$

$$z'_{\omega,s} = A_{\omega,s}z_{\omega,s} - \mathcal{B}_{\omega,s}\mathcal{B}^*_{\omega,u}\mathcal{P}_{\omega,u}z_{\omega,u} + \pi_{\omega,s}f, \quad z_{\omega,s}(0) = \pi_{\omega,s}z_0,$$

$$z_{\omega,u} = \pi_{\omega,u}z, \quad z_{\omega,s} = \pi_{\omega,s}z, \quad \Pi z = z_{\omega,u} + z_{\omega,s},$$

$$(I - \Pi)z = -(I - \Pi)\mathcal{B}^*_{\omega,u}\mathcal{P}_{\omega,u}z_{\omega,u}.$$

We can follow the same approach when only D.B.C. are involved in the system

The closed loop nonlinear system – Dirichlet B.C.

To study the closed loop nonlinear system we have to study closed loop nonhomogeneous linear system and use a fixed point argument.

$$\frac{\partial z}{\partial t} - \nu \Delta z - \omega z + (w_s \cdot \nabla) z + (z \cdot \nabla) w_s + \nabla p = -(z \cdot \nabla) z,$$

$$\operatorname{div} z = 0 \quad \text{in } Q, \quad z = -\mathcal{B}^*_{\omega,u} \mathcal{P}_{\omega,u} \pi_{\omega,u} z \quad \text{on } \Sigma_D = \Sigma,$$

$$z(0) = z_0 \quad \text{in } \Omega,$$

In that case, we can choose

$$\mathcal{S} = L^2(0,\infty; V^1_0(\Omega)) \cap L^\infty(0,\infty; V^0_n(\Omega)), \quad Z_0 = V^0_n(\Omega),$$

$$\mathcal{F} = L^2(0,\infty; V^{-1}(\Omega)), \quad e^{-\omega t}(z \cdot \nabla) z \in L^2(0,\infty; V^{-1}(\Omega)).$$

Another choice is

$$\mathcal{S} = L^2(0,\infty; V_0^{1+\varepsilon}(\Omega)) \cap L^\infty(0,\infty; V_n^\varepsilon(\Omega)), \quad Z_0 = V_n^\varepsilon(\Omega),$$

with $\quad 0 < \varepsilon < 1/2,$

$$\mathcal{F} = L^2(0,\infty; V^{-1+\varepsilon}(\Omega)), \quad e^{-\omega t}(z\cdot\nabla)z \in L^2(0,\infty; V^{-1+\varepsilon}(\Omega)).$$

Theorem. Let ε belong to $[0,1/2)$. There exists $\mu_0 > 0$ and a nondecreasing function η from $\mathbb{R}^+$ into itself, such that if $\mu \in (0,\mu_0)$ and $\|z_0\|_{V_n^\varepsilon(\Omega)} \le \eta(\mu)$, then the nonlinear closed loop system admits a unique solution in the set

$$D_\mu = \Big\{ z \mid \|e^{\omega t} z\|_{L^2(0,\infty;V_0^{1+\varepsilon}(\Omega))\cap L^\infty(0,\infty;V_n^\varepsilon(\Omega))} \le \mu \Big\}.$$

In particular

$$\|z(t)\|_{V^\varepsilon(\Omega)} \le C(\mu) e^{-\omega t}.$$

The closed loop nonlinear system – Mixed B.C.

For the cylinder, we have

$$\frac{\partial z}{\partial t} - \nu\Delta z - \omega z + (w_s\cdot\nabla)z + (z\cdot\nabla)w_s + \nabla p = -e^{-\omega t}(z\cdot\nabla)z,$$

$$\mathrm{div}\, z = 0 \quad \text{in } Q, \quad z = -\mathcal{B}^*_{\omega,u}\mathcal{P}_{\omega,u}\pi_{\omega,u}z \quad \text{on } \Sigma_D,$$

$$\sigma(z,p)n = 0 \quad \text{on } \Sigma_N,$$

$$z(0) = z_0 \quad \text{in } \Omega\,,$$

We cannot take

$$\mathcal{S} = L^2(0,\infty; V^1_{\Gamma_D}(\Omega)) \cap L^\infty(0,\infty; V^0_{n,\Gamma_D}), \quad Z_0 = V^0_{n,\Gamma_D},$$

$$\mathcal{F} = L^2(0,\infty; V^{-1}_{\Gamma_D}(\Omega)), \quad e^{-\omega t}(z\cdot\nabla)z \notin L^2(0,\infty; V^{-1}_{\Gamma_D}(\Omega)).$$

Indeed

$$\left\langle (z \cdot \nabla) z \, \phi \right\rangle = -\int_{\Omega} (z \cdot \nabla)\phi \, z + \int_{\Gamma_N} (z \cdot n)\,(z \cdot \phi)$$

and

$$\phi \longmapsto \left\langle (z \cdot \nabla) z \,, \phi \right\rangle$$

cannot be identified with an element in $V_{\Gamma_D}^{-1}(\Omega)$.

The domain of the Oseen operator obeys

$$\mathcal{D}(A) \subset H^{3/2+\varepsilon_0}(\Omega;\mathbb{R}^2) \quad \text{for some } \varepsilon_0 \in (0,1/2).$$

The main tools

$$\mathcal{S} = L^2(0,\infty;\mathcal{D}((\lambda_0 I - A)^{1/2+\varepsilon/2})) \cap H^{1/2+\varepsilon/2}(0,\infty;V^0_{n,\Gamma_d}(\Omega)) \\ +H^1(0,\infty;H^{3/2+\varepsilon_0}(\Omega;\mathbb{R}^2)),$$

$$Z_0 = V^{\varepsilon}_{n,\Gamma_d}(\Omega),$$

$$\mathcal{F} = L^2(0,\infty;H^{-1+\varepsilon}_{\Gamma_d}(\Omega)).$$

Needed results

$$\mathcal{D}((\lambda_0 I - A)^{1/2}) = V^1_{\Gamma_d}(\Omega), \quad \mathcal{D}((\lambda_0 I - A^*)^{1/2}) = V^1_{\Gamma_d}(\Omega)$$

$$\text{and} \quad (\mathcal{D}((\lambda_0 I - A^*)^{1/2}))' = V^{-1}_{\Gamma_d}(\Omega),$$

$$\mathcal{D}((\lambda_0 I - A)^{1/2+\varepsilon/2}) \subset V^{1+\eta(\varepsilon,\varepsilon_0)}_{\Gamma_d}(\Omega) = V^1_{\Gamma_d}(\Omega) \cap H^{1+\eta(\varepsilon,\varepsilon_0)}(\Omega;\mathbb{R}^2).$$

with $\eta(\varepsilon,\varepsilon_0) = \frac{\varepsilon}{2} + \varepsilon\varepsilon_0$.

Theorem. Let ε belong to $(0,1/2)$. There exists $\mu_0 > 0$ and a nondecreasing function η from $\mathbb{R}^+$ into itself, such that if $\mu \in (0,\mu_0)$ and $\|z_0\|_{V^\varepsilon_{n,\Gamma_D}(\Omega)}$, then the nonlinear closed loop system admits a unique solution in the set

$$D_\mu = \Big\{ z \mid$$

$$\|e^{\omega t} z\|_{L^2(0,\infty;\mathcal{D}((\lambda_0 I - A)^{1/2+\varepsilon/2}))\cap H^{1/2+\varepsilon/2}(0,\infty;V^0_{n,\Gamma_d}(\Omega))+H^1(0,\infty;H^{3/2+\varepsilon_0}(\Omega;\mathbb{R}^2))} \le \mu \Big\}.$$

In particular

$$\|z(t)\|_{V^\varepsilon(\Omega)} \le C(\mu) e^{-\omega t}.$$

Conclusion

• Local feedback stabilization of the N.S.E. in 2D with a Dirichlet B.C. in the case of mixed Dirichlet/Neumann B.C. The 3D case is under investigation.

• The boundaries Γ_D and Γ_N, at the junction, make a right angle.

• The control is of finite dimension.

• The Riccati equation used to calculate the feedback is of finite dimension.

References

M. Badra, Local stabilization of the Navier-Stokes equations with a feedback controller localized in an open subset of the domain, Numer. Funct. Anal. Optim., Vol. **28** (2007), 559–589.

M. Badra, Lyapunov function and local feedback boundary stabilization of the Navier-Stokes equations, SIAM J. Control Optim, 43 (2009), 1797–1830.

M. Badra, T. Takahashi, Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers. Application to the Navier-Stokes system, submitted to SIAM J. Control.

V. Barbu, Feedback stabilization of the Navier-Stokes equations, ESAIM COCV, Vol. 9 (2003), 197–206.

V. Barbu, I. Lasiecka, R. Triggiani, Tangential boundary stabilization of Navier-Stokes equations, Memoirs of the A.M.S., 2006, number 852.

V. Barbu, R. Triggiani, Internal stabilization of Navier-Stokes equations with finite-dimensional controllers, Indiana University Mathematics Journal, Vol. 53 (2004), 1443–1494.

V. Barbu, G. Wang, Feedback stabilization of semilinear heat equations. Abstr. Appl. Anal. 2003, 697–714.

V. Barbu, G. Wang, Feddback stabilization of periodic solutions to nonlinear parabolic-like evolution systems, Ind. Univ. Math. Journal, 54 (2005), 1521–1546.

C. Fabre, G. Lebeau, Prolongement unique des solutions de l'équation de Stokes, Comm. P. D. E., 21 (1996), 573–596.

C. Fabre, G. Lebeau, Régularité et unicité pour le problème de Stokes, C. P. D. E., 27 (2002), 437–475.

E. Fernandez-Cara, S. Guerrero, O. Yu. Imanuvilov, J.-P. Puel, Local exact controllability of the Navier-Stokes system. J. Math. Pures Appl. 83 (2004), 1501–1542.

A. V. Fursikov, Stabilizability of two-dimensional Navier-Stokes equations with help of a boundary feedback control, J. Math. Fluid Mech., 3 (2001), 259–301.

A. V. Fursikov, Stabilizability of a quasi-linear parabolic equation by means of a boundary control with feedback, Sbornik Mathematics, 192:4 (2001), 593–639.

A. V. Fursikov, Stabilization for the 3D Navier-Stokes system by feedback boundary control, Discrete and Cont. Dyn. Systems, 10 (2004), 289–314.

S. Kesavan, J.-P. Raymond, On a degenerate Riccati equation, Control and Cybernetics, 38 (2009), 1393–1410.

V. Maz'ya, J. Rossmann, L_p estimates of solutions to mixed boundary value problems for the Stokes system in polyhedral domains, Math. Nachr. 280 (2007), 751–793.

J.-P. Raymond, Boundary feedback stabilization of the two dimensional Navier-Stokes equations, SIAM J. Control and Optim., Vol. 45 (2006), 790–828.

J.-P. Raymond, L. Thevenet, Boundary feedback stabilization of the two dimensional Navier-Stokes equations with finite dimensional controllers, DCDS-A, 27 (2010).

J.-P. Raymond, Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions, Annales de l'IHP, An. non lin., Vol. 6 (2007), 921–951.

J.-P. Raymond, Feedback boundary stabilization of the three dimensional incompressible Navier-Stokes equations, J. Math. Pures Appl., 87 (2007), 627–669.

J.-P. Raymond, Stabilizability of infinite dimensional systems by finite dimesional controls, Control and Cybernetics (submitted).

R. Triggiani, Unique continuation of boundary over-determined Stokes and Oseen eigenproblems, Discrete and Cont. Dyn. Systems, Series S, 2 (2009), 647–677.