Stochastic Analysis and Control of Fluid Flows

Lecture 2

School of Mathematics – IISER-TVM

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Representation and boundary stabilizability of Infinite Dimensional Systems

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Plan of lecture 2

1. Infinite dimensional linear systems with Homogeneous Boundary Conditions

The Duhamel formula with Homogeneous Boundary Conditions (Heat, Lin. Burgers, Stokes, Lin. N.S.E.)

2. The Duhamel formula with Nonhomogeneous Boundary Conditions (Heat, Lin. Burgers, Stokes, Lin. N.S.E.)

3. Boundary stabilizability of infinite dimensional linear systems

4. Feedback stabilization of infinite dimensional linear systems

1. Infinite Dimensional Systems – The Duhamel formula

1.1. The Duhamel formula for the heat equation with homogeneous B.C.

$$\begin{aligned}
&\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q = \Omega \times (0,\infty),\\
&z = 0 \quad \text{in } \Sigma = \Gamma \times (0,\infty),\\
&z(0) = z_0.
\end{aligned}$$

The spectrum of Δ with D.B.C. is constituted of isolated eigenvalues of finite multiplicity

$$\begin{aligned}
&(\lambda_j)_{1\le j\le\infty}, \quad \lambda_j \in \mathbb{R}, \quad E(\lambda_j) = \mathrm{Ker}(\Delta - \lambda_j I) \subset H_0^1(\Omega),\\
&L^2(\Omega) = \oplus_{j=1}^{\infty} E(\lambda_j), \quad \mathrm{dim} E(\lambda_j) = \ell_j.
\end{aligned}$$

Let

$$\{e_j^k,\ 1 \le k \le \ell_j\},$$

be an orthonormal basis of $E(\lambda_j)$, constituted of eigenfunctions of Δ in $H_0^1(\Omega)$.

Then

$$\{e_j^k \mid j \in \mathbb{N}^*,\ 1 \leq k \leq \ell_j\}$$

is a Hilbertian basis in $L^2(\Omega)$ and

$$e^{tA} z_0 = \sum_{j=1}^{\infty} \sum_{k=1}^{\ell_j} e^{\lambda_j t} \left(z_0, e_j^k\right)_{L^2(\Omega)} e_j^k.$$

The projection of z_0 onto $E(\lambda_{j_0})$ along $\oplus_{j=1,\, j\neq j_0}^{\infty} E(\lambda_j)$ is

$$\pi_{E(\lambda_{j_0})} z_0 = \sum_{k=1}^{\ell_{j_0}} \left(z_0, e_{j_0}^k\right)_{L^2(\Omega)} e_{j_0}^k.$$

Example. In the 1D case, $\Omega =]0, L[$, $\ell_j = 1$,

$$\lambda_j = -\frac{j^2 \pi^2}{L^2}, \quad e_j(x) = \frac{\sqrt{2}}{\sqrt{L}} \sin\left(\sqrt{-\lambda_j} x\right) = \frac{\sqrt{2}}{\sqrt{L}} \sin\left(\frac{j\pi x}{L}\right),$$

$$e^{tA} z_0(x) = \sum_{j=1}^{\infty} e^{\lambda_j t} \left(\int_{\Omega} z_0(\xi) e_j(\xi) d\xi\right) e_j(x).$$

1.2. The Duhamel formula for the linearized Burgers equation

$$\frac{\partial z}{\partial t} - \Delta z + 2\,\partial_i w_s\, z + 2\,\partial_i z\, w_s = 0,$$

$$z = 0 \quad \text{on } \Sigma, \quad z(0) = z_0 \quad \text{in } \Omega.$$

The spectrum of A, with $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, $Az = \Delta z - 2\,\partial_i w_s\, z - 2\,\partial_i z\, w_s$, is constituted of isolated eigenvalues of finite multiplicity

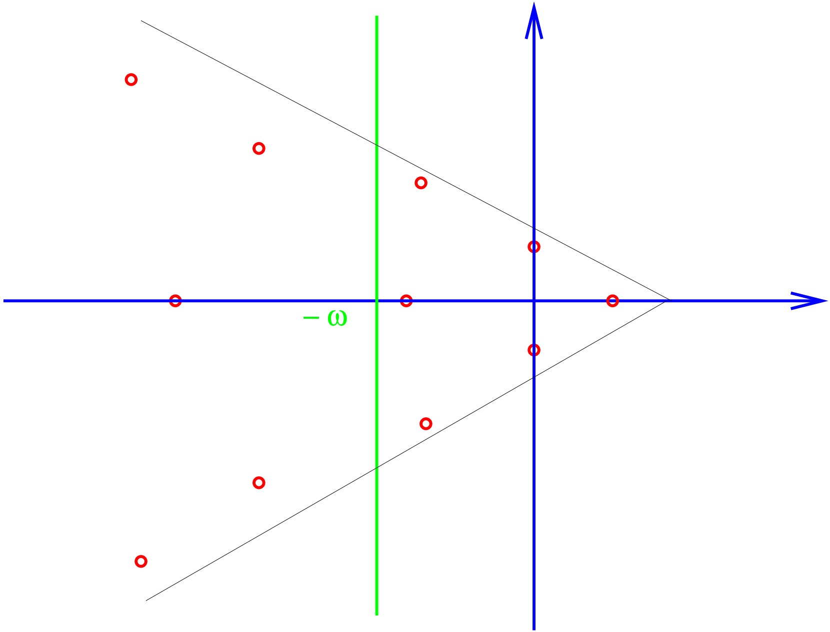
$$(\lambda_j)_{1 \le j \le \infty}, \quad \lambda_j \in \mathbb{C},$$

$$L^2_{\mathbb{C}}(\Omega) = \oplus_{j=1}^{\infty} G_{\mathbb{C}}(\lambda_j), \quad \dim G_{\mathbb{C}}(\lambda_j) = N(\lambda_j),$$

if λ_j is an eigenvalue $\overline{\lambda_j}$ is an eigenvalue too,

if $z_0 \in G_{\mathbb{C}}(\lambda_j)$, then $z_{z_0}(t) \in G_{\mathbb{C}}(\lambda_j)$.

The spectrum of A^* the adjoint of A, with $A^* z = \Delta z + 2\,\partial_i z\, w_s$, is identical to the spectrum of A.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Since we are interested in real valued solutions, it is interesting to introduce the real generalized eigenspaces

$$\text{If } Im(\lambda_j) = 0, \quad G_{\mathbb{R}}(\lambda_j) = \text{vec}(\text{Re}\, G_{\mathbb{C}}(\lambda_j)).$$

$$\text{If } Im(\lambda_j) \neq 0, \quad G_{\mathbb{R}}(\lambda_j) = \text{vec}(\text{Re}\, G_{\mathbb{C}}(\lambda_j), \text{Im}\, G_{\mathbb{C}}(\lambda_j)).$$

Let $\pi_{G_{\mathbb{R}}(\lambda_{j_0})}$ be the projection onto $G_{\mathbb{R}}(\lambda_{j_0})$ along $\oplus_{j=1,\, j\neq j_0}^{\infty} G_{\mathbb{R}}(\lambda_j)$. The system satisfied by $\pi_{G_{\mathbb{R}}(\lambda_j)} z_{z_0}$ is a finite dimensional system.

To define this projection, we introduce a basis of $G_{\mathbb{R}}(\lambda_{j_0})$

$$\left\{ e_{j_0}^k \mid k = 1, \cdots, N(\lambda_{j_0}) \right\},$$

and a basis of $G_{\mathbb{R}}^*(\bar{\lambda}_{j_0}) = G_{\mathbb{R}}^*(\lambda_{j_0})$ (the generalized eigenspace for A^* associated with $\bar{\lambda}_{j_0}$)

$$\left\{ \phi_{j_0}^k \mid k = 1, \cdots, N(\lambda_{j_0}) \right\}.$$

We can choose these bases in order that they satisfy the following bi-orthogonality property

$$\left(e_{j_1}^{k_1}, \phi_{j_2}^{k_2} \right)_{L^2(\Omega)} = \delta_{k_1}^{k_2}\, \delta_{j_1}^{j_2}.$$

In that case, we have

$$\pi_{G_{\mathbb{R}}(\lambda_{j_0})} f = \sum_{k=1}^{N(\lambda_{j_0})} \left(f, \phi_{j_0}^k\right)_{L^2(\Omega)} e_{j_0}^k.$$

and

$$\pi_{G_{\mathbb{R}}^*(\lambda_{j_0})} f = \sum_{k=1}^{N(\lambda_{j_0})} \left(f, e_{j_0}^k\right)_{L^2(\Omega)} \phi_{j_0}^k.$$

With the Jordan decomposition of the finite dimensional projected systems, we can determine the semigroup of the Linearized Burgers equation

$$e^{tA} z_0 = \sum_{j=1}^{\infty} \sum_{k=1}^{N(\lambda_{j_0})} e^{\lambda_j t} (z_0, \phi_{j_0}^k)_{L^2(\Omega)} p_k(t) e_{j_0}^k,$$

where p_k is a polynomial.

1.3. The Duhamel formula for the Stokes equation with Homogeneous Dirichlet B.C.

The Helmholtz decomposition

$$V_n^0(\Omega) = \Big\{ z \in L^2(\Omega; \mathbb{R}^d) \mid \text{ div } z = 0,\ z \cdot n = 0 \text{ on } \Gamma \Big\},$$

$$L^2(\Omega; \mathbb{R}^d) = V_n^0(\Omega) \oplus \text{grad}\, H^1(\Omega),$$

$$\Pi\ :\ L^2(\Omega; \mathbb{R}^d) \longmapsto V_n^0(\Omega).$$

$$\Pi z = z - \nabla p - \nabla q,$$

$$\Delta p = \mathrm{div}\, z \in H^{-1}(\Omega), \quad p \in H_0^1(\Omega),$$

$$\Delta q = 0, \quad \frac{\partial q}{\partial n} = (z - \nabla p) \cdot n \quad \text{on } \Gamma.$$

We set

$$V_0^1(\Omega) = H_0^1(\Omega; \mathbb{R}^d) \cap V_n^0(\Omega).$$

The Stokes operator with Dirichlet B.C.

$$A_0 = \Pi\Delta \quad \text{with} \quad \mathcal{D}(A_0) = H^2(\Omega;\mathbb{R}^d) \cap V_0^1(\Omega).$$

The spectrum of A_0 is real. We have a formula similar to the one for the heat equation

$$e^{tA}z_0 = \sum_{j=1}^{\infty}\sum_{k=1}^{\ell_j} e^{\lambda_j t}(z_0, e_j^k)_{L^2(\Omega)} e_j^k.$$

The Oseen operator with Dirichlet B.C.

$$\mathcal{D}(A) = H^2(\Omega;\mathbb{R}^d) \cap \cap V_0^1(\Omega),$$

$$Az = \nu\Pi\Delta z - \Pi((w_s\cdot\nabla)z) - \Pi((z\cdot\nabla)w_s),$$

$$\mathcal{D}(A^*) = H^2(\Omega;\mathbb{R}^d) \cap V_0^1(\Omega),$$

$$A^*\phi = \nu\Pi\Delta\phi + \Pi((w_s\cdot\nabla)\phi) - \Pi((\nabla w_s)^T\phi).$$

The spectrum of A is symmetric with respect to the real axis. The representation of the semigroup is similar to that of the Linearized Burgers equation.

The Helmholtz decomposition

$$V^0_{n,\Gamma_D}(\Omega) = \Big\{ z \in L^2(\Omega;\mathbb{R}^d) \mid \ \text{div}\, z = 0,\ z \cdot n = 0 \text{ on } \Gamma_D \Big\},$$

$$L^2(\Omega;\mathbb{R}^d) = V^0_{n,\Gamma_D}(\Omega) \oplus \text{grad}\, H^1_{\Gamma_N}(\Omega),$$

$$\Pi \ : \ L^2(\Omega;\mathbb{R}^d) \longmapsto V^0_{n,\Gamma_D}(\Omega).$$

$$\Pi z = z - \nabla p - \nabla q,$$

$$\Delta p = \mathrm{div}\, z \in H^{-1}(\Omega), \quad p \in H^1_0(\Omega),$$

$$\Delta q = 0, \quad \frac{\partial q}{\partial n} = (z - \nabla p) \cdot n \quad \text{on } \Gamma_D, \quad q = 0 \quad \text{on } \Gamma_N.$$

We set

$$V^1_{\Gamma_D}(\Omega) = H^1_{\Gamma_D}(\Omega;\mathbb{R}^2) \cap V^0_n(\Omega),$$
$$H^1_{\Gamma_D}(\Omega;\mathbb{R}^2) = \{ z \in H^1(\Omega;\mathbb{R}^2) \mid z = 0 \text{ on } \Gamma_D \}.$$

Characterization of Stokes operator $(A_0, D(A_0))$ in the case of Mixed D/N B.C. with a right angle junction

$$\mathcal{D}(A_0) = \Big\{ z \in V^1_{\Gamma_D}(\Omega) \mid$$

$$\exists p \in L^2(\Omega) \text{ s. t. } \mathrm{div}\sigma(z,p) \in L^2(\Omega;\mathbb{R}^d)$$

$$\text{and } \sigma(z,p)\, n = 0 \quad \text{on} \quad \Gamma_N \Big\},$$

$$A_0 z = \Pi \mathrm{div}\sigma(z,p) \quad (\text{does not depend on } p).$$

In the case of mixed B.C.

$$A_0 \neq \nu \Pi \Delta z.$$

Indeed, the pressure p in the equation

$$-\Delta z + \nabla p = f,\ \operatorname{div} z = 0 \quad \text{in } \Omega, \qquad z = 0 \ \text{ on } \Gamma_D$$
$$\sigma(z,p)n = 0 \text{ on } \Gamma_N,$$

does not obey

$$\Pi\, \nabla p = 0 \quad \text{because} \quad p|_{\Gamma_N} \neq 0,\ p|_{\Gamma_N} = \sigma(z,p)n \cdot n.$$

In the 3D case with a right angle junction, we have

$$\mathcal{D}(A_0) \subset H^{3/2+\varepsilon}(\Omega; \mathbb{R}^d) \quad \text{for some} \quad \varepsilon > 0.$$

(See Maz'ya and Rossmann, 2007.)

The Oseen operator $(A, D(A))$ is defined by

$$\mathcal{D}(A) = \mathcal{D}(A_0) \quad \text{and} \quad Az = A_0 z + \Pi\,((w_s \cdot \nabla)z + (z \cdot \nabla)w_s)\,.$$

Theorem. The spectrum of A is symmetric with respect to the real axis. The representation of the semigroup is similar to that of the Linearized Burgers equation.
If w_s is regular enough and if $\operatorname{div} w_s = 0$, we can verify that there exists $\lambda_0 > 0$ in the resolvent set of A satisfying

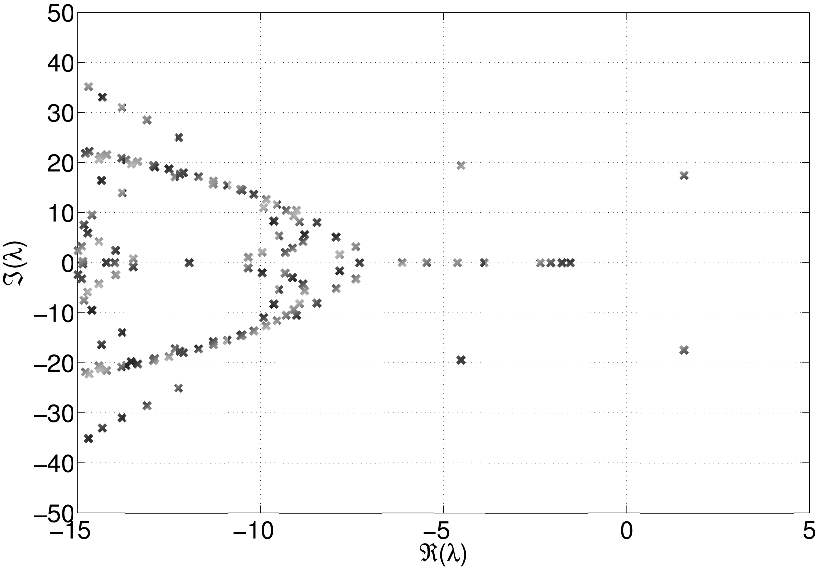
$$\big((\lambda_0 I - A)z, z\big)_{L^2(\Omega)} \geq \frac{\nu}{2}\|z\|^2_{H^1(\Omega)} \qquad \text{for all } z \in \mathcal{D}(A),$$

and

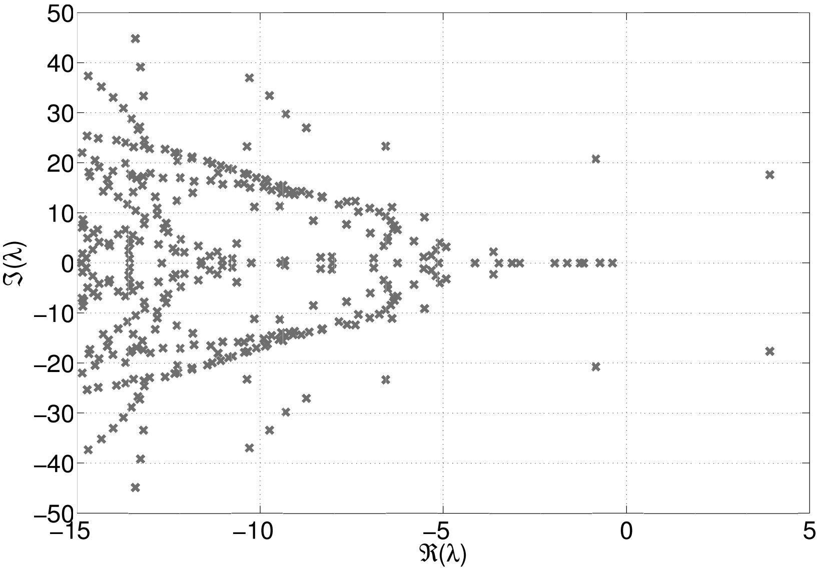
$$\big((\lambda_0 I - A^*)\phi, \phi\big)_{L^2(\Omega)} \geq \frac{\nu}{2}\|\phi\|^2_{H^1(\Omega)} \qquad \text{for all } \phi \in \mathcal{D}(A^*).$$

Consequence. The spectrum of A is contained in a sector. The eigenvalues are isolated, pairwise conjugate when they are not real, and of finite multiplicity.

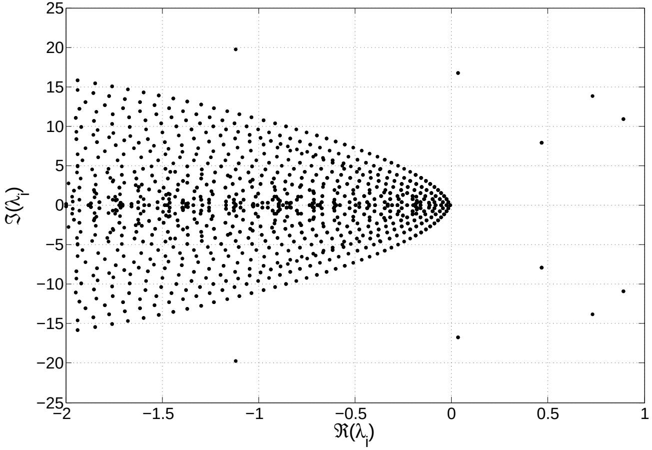
Spectrum of A. $Re = u_s \, Diam \; of \; disc/\nu = 80$ (Cylinder)



Spectrum of A with $Re = 200$



Spectrum of A. $Re = u_s \times h_{cavity}/\nu = 7500$ (Cavity)



2. The Duhamel formula in the case of non homogeneous Dirichlet B.C.

2.1. The Heat equation with non homogeneous B.C.

$$\frac{\partial z}{\partial t} - \Delta z = 0,$$

$$z = u \quad \text{on } \Sigma, \quad z(0) = z_0 \quad \text{in } \Omega\,.$$

It is of the form

$$z' = Az + Bu, \qquad z(0) = z_0.$$

But $B \notin \mathcal{L}(U, Z)$, and $B \in \mathcal{L}(U, (\mathcal{D}(A^*))')$.

Recall that $\mathcal{D}(A) = \mathcal{D}(A^*) = H^2(\Omega) \cap H_0^1(\Omega)$, $A = A^* = \Delta$.

The Dirichlet operator for the Laplace equation

$Du(t) = w(t)$ is the solution to

$$-\Delta w(t) = 0, \quad \text{in } \Omega, \qquad w(t) = u(t) \text{ on } \Gamma.$$

We set

$$z = y + w.$$

Equation satisfied by y

$$\frac{\partial y}{\partial t} - \Delta y = -w',$$
$$y = 0 \quad \text{on } \Sigma, \qquad y(0) = z_0 - w(0).$$

Evolution equation satisfied by y

$$y'(t) = Ay - w'(t), \qquad y(0) = z_0 - w(0).$$

With the Heat semigroup we obtain

$$y(t) = e^{tA}(z_0 - w(0)) - \int_0^t e^{(t-\tau)A} w'(\tau)\, d\tau.$$

Integrating by parts

$$y(t) = e^{tA} z_0 + \int_0^t (-A) e^{(t-\tau)A} w(\tau)\, d\tau - w(t).$$

Therefore

$$z(t) = y(t) + w(t) = e^{tA} z_0 + \int_0^t (-A) e^{(t-\tau)A} Du(\tau)\, d\tau.$$

This means that

$$z' = Az + (-A)Du = Az + Bu, \qquad z(0) = z_0.$$

(We have to extend the semigroup to $(\mathcal{D}(A^*))'$). The equation is equivalent to

$$(z', \phi)_{L^2(\Omega)} = (z, A^*\phi)_{L^2(\Omega)} + (u, B^*\phi)_{L^2(\Gamma)}, \quad \forall \phi \in \mathcal{D}(A^*).$$

We say that z is a weak solution to this evolution equation when

$$\frac{d}{dt} \int_\Omega z(t)\, \phi = \int_\Omega z(t)\, A\phi - \int_\Gamma u(t)\, \frac{\partial \phi}{\partial n}, \qquad \forall \phi \in \mathcal{D}(A^*).$$

Here we have used that $A = A^*$. Thus

$$B^*\phi = -\frac{\partial \phi}{\partial n}.$$

2.2. The Burgers equation with non homogeneous B.C.

The linearized control system is

$$\frac{\partial z}{\partial t} - \Delta z + 2\,\partial_i w_s\, z + 2\,\partial_i z\, w_s = 0,$$

$$z = u \quad \text{on } \Sigma, \quad z(0) = z_0 \quad \text{in } \Omega\,.$$

It is of the form

$$z' = Az + Bu, \qquad z(0) = z_0.$$

As before $B \notin \mathcal{L}(U, Z)$, but $B \in \mathcal{L}(U, (\mathcal{D}(A^*))')$.

$\mathcal{D}(A) = \mathcal{D}(A^*) = H^2(\Omega) \cap H_0^1(\Omega)$, $A^*\phi = \Delta\phi + 2\partial_i\phi\, w_s$.

The Dirichlet operator for the linearized Burgers equation

$Du(t) = w(t)$ is the solution to

$$\lambda_0 w(t) - \Delta w(t) + 2\,\partial_i w_s\, w(t) + 2\,\partial_i w(t)\, w_s = 0, \quad \text{in } \Omega,$$
$$w(t) = u(t) \ \text{ on } \Gamma.$$

We set

$$z = y + w.$$

Equation satisfied by y:

$$\frac{\partial y}{\partial t} - \Delta y + 2\,\partial_i w_s\, y + 2\,\partial_i y\, w_s = \lambda_0 w - w',$$
$$y = 0 \quad \text{on } \Sigma, \qquad y(0) = z_0 - w(0).$$

Evolution equation satisfied by y:

$$y'(t) = Ay + \lambda_0 w - w'(t), \qquad y(0) = z_0 - w(0).$$

With the Stokes semigroup we obtain

$$y(t) = e^{tA}(y_0 - w(0)) - \int_0^t e^{(t-\tau)A}\,(-\lambda_0 w(\tau) + w'(\tau))\,d\tau.$$

Integrating by parts

$$y(t) = e^{tA} z_0 + \int_0^t (\lambda_0 I - A) e^{(t-\tau)A}\, w(\tau)\,d\tau - w(t).$$

Therefore

$$z(t) = y(t) + w(t) = e^{tA}y_0 + \int_0^t (\lambda_0 I - A)e^{(t-\tau)A}\, Du(\tau)\, d\tau.$$

This means that

$$z' = Az + (\lambda_0 I - A)Du = Az + Bu, \qquad z(0) = z_0.$$

(we have to extend the semigroup to $(\mathcal{D}(A^*))'$)

We say that z is a weak solution to this evolution equation when

$$\frac{d}{dt}\int_\Omega z(t)\,\phi = \int_\Omega z(t)\,A^*\phi - \int_\Gamma u(t)\Big(\frac{\partial\phi}{\partial n} + 2w_s\,\phi\,n_i\Big), \qquad \forall\phi \in \mathcal{D}(A^*).$$

Here $B^*\phi = -\dfrac{\partial\phi}{\partial n} - 2w_s\,\phi\,n_i$.

2.3. The Stokes equation with non homogeneous boundary conditions

$$\frac{\partial z}{\partial t} - \Delta z + \nabla p = 0, \qquad \mathrm{div} z = 0 \quad \text{in } \Omega \times (0, T),$$

$$z = u \qquad \text{on } \Gamma_D \times (0, T),$$

$$\sigma(z, p)n = 0 \qquad \text{on } \Gamma_N \times (0, T),$$

$$z(0) = z_0 \text{ in } \Omega.$$

Before writing this system in the form

$$\Pi z' = A\Pi z + Bu, \quad \Pi z(0) = \Pi z_0,$$

$$(I - \Pi)z = (I - \Pi)Du,$$

we notice that p is the solution to

$$\Delta p(t) = 0 \quad \text{in } \Omega, \quad \frac{\partial p(t)}{\partial n} = \frac{\partial z}{\partial t} \cdot n - \Delta z \cdot n \quad \text{on } \Gamma_D,$$

$$p(t) = \nu(\nabla z + (\nabla z)^T)n \cdot n \quad \text{on } \Gamma_N.$$

The Dirichlet operator for the Stokes equation

$Du(t) = w(t)$ is the solution to

$$-\Delta w(t) + \nabla \rho(t) = 0,\ \operatorname{div} w(t) = 0 \quad \text{in } \Omega, \quad w(t) = u(t) \ \text{on } \Gamma_D$$
$$\sigma(w, \rho)n = 0 \text{ on } \Gamma_N.$$

We set

$$z = y + w \quad \text{and} \quad p = q + \rho.$$

Equation satisfied by y:

$$\frac{\partial y}{\partial t} = \Delta y - \nabla q - w', \quad \operatorname{div} y = 0,$$
$$y = 0 \quad \text{on } \Sigma_D, \quad \sigma(y, q)n = 0 \quad \text{on } \Sigma_N, \quad y(0) = z_0 - w(0).$$

Evolution equation satisfied by y:

$$y'(t) = Ay - \Pi w'(t), \qquad y(0) = \Pi(z_0 - w(0)).$$

With the Stokes semigroup we obtain

$$y(t) = S(t)(z_0 - w(0)) - \int_0^t S(t-\tau)\,\Pi w'(\tau)\,d\tau.$$

Integrating by parts

$$y(t) = S(t)z_0 + \int_0^t (-A)S(t-\tau)\,\Pi w(\tau)\,d\tau - \Pi w(t).$$

Therefore

$$\Pi z(t) = y(t) + \Pi w(t) = S(t)z_0 + \int_0^t (-A)S(t-\tau)\,\Pi D u(\tau)\,d\tau.$$

This means that

$$\Pi z' = A\Pi z + (-A)\Pi D u, \qquad \Pi z(0) = z_0.$$

(we have to extend the semigroup to $(\mathcal{D}(A))'$)

What is the equation satisfied by $(I-\Pi)z$?

$$(I-\Pi)z(t) = (I-\Pi)w(t) = (I-\Pi)Du(t).$$

The system satisfied by z is finally :

$$\Pi z' = A\Pi z + (-A)\Pi D u, \qquad \Pi z(0) = z_0,$$

$$(I-\Pi)z = (I-\Pi)Du = (I-\Pi)D(u\cdot n\,n).$$

The P.D.E. satisfied by Πz

$$\frac{\partial \Pi z}{\partial t} - \Delta \Pi z + \nabla \tilde{p} = 0, \qquad \mathrm{div} \Pi z = 0,$$

$$\Pi z = \gamma_\tau u - \nabla_\tau q \qquad \text{on } \Gamma_D \times (0, T),$$

$$\sigma(\Pi z, \tilde{p}) n = 0 \qquad \text{on } \Gamma_N \times (0, T),$$

$$\Pi z(0) = \Pi z_0 \text{ in } \Omega,$$

where

$$\Delta q(t) = 0 \text{ in } \Omega, \quad \frac{\partial q(t)}{\partial n} = u(t) \cdot n \text{ in } \Gamma_D, \quad q(t) = 0 \text{ on } \Gamma_N.$$

3. Stabilizability of Infinite Dimensional Systems

3.1. Characterization of the stabilizability

Theorem. Assume that the semigroup generated by $(A, \mathcal{D}(A))$ is analytic on Z, the resolvent of A, i.e. $R(\lambda) = (\lambda I - A)^{-1}$ for some λ, is compact, and the spectrum of A obeys

$$\ldots < \mathrm{Re}\lambda_{N_u+1} < -\omega < \mathrm{Re}\lambda_{N_u} \leq \mathrm{Re}\lambda_{N_u-1} \leq \ldots \leq \mathrm{Re}\lambda_1 .$$

For $1 \leq j \leq N_u$, let $(\Phi_j^k)_{1\leq k\leq \ell_j}$ a basis of $\mathrm{Ker}(A^* - \lambda_j I)$.

The pair $(A + \omega I, B)$ is stabilizable iff, for all $1 \leq j \leq N_u$,

$$\text{the family} \quad (B^*\Phi_j^k)_{1\leq k\leq \ell_j} \quad \text{is linearly independent.}$$

The stabilizability condition reduces to showing that, for all $1 \leq j \leq N_u$,

$$A^*\Phi = \lambda_j \Phi \quad \text{and} \quad B^*\Phi = 0 \quad \text{implies} \quad \Phi = 0.$$

3.2. Boundary stabilizability of the Heat equation

We want to stabilize the heat equation

$$\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q = \Omega \times (0, \infty),$$
$$z = m\,u \quad \text{in } \Sigma = \Gamma \times (0, \infty),$$
$$z(0) = z_0,$$

with a prescribed exponential decay rate $-\omega$, by a control u of finite dimension

$$u(x,t) = \sum_{i=1}^{K} v_i(t) w_i(x).$$

We assume that m is regular, non negative and that $m(x) = 1$ for $x \in \Gamma_c \subset \Gamma$.

We have seen that the heat equation can be written as

$$z' = Az + Bu, \quad z(0) = z_0,$$

with

$$A = \Delta, \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad B = (-A)\,D\,m.$$

In that case $\mathcal{D}(A^*) = \mathcal{D}(A)$ and $A = A^*$.

Expression of B^*.

We say that z is a weak solution of the P.D.E. iff

$$\frac{d}{dt}\int_\Omega z(x,t)\,\phi(x)\,dx = \int_\Omega z(x,t)\,\Delta\phi(x)\,dx + \int_\Gamma m(x)u(x,t)\frac{\partial\phi(x)}{\partial n}d\gamma(x),$$

for all $\phi \in \mathcal{D}(A^*) = H^2(\Omega)\cap H^1_0(\Omega)$.

Similarly, we say that z is a weak solution of the evolution equation iff

$$\frac{d}{dt}\int_\Omega z(x,t)\,\phi(x)\,dx = \int_\Omega z(x,t)\,A\phi(x)\,dx + \int_\Gamma u(x,t)B^*\phi(x)\,d\gamma(x),$$

for all $\phi \in \mathcal{D}(A^*) = H^2(\Omega)\cap H^1_0(\Omega)$.

This means that

$$B^*\phi = -m\,\frac{\partial\phi}{\partial n}.$$

In particular

$$B^*\phi = 0, \quad \text{implies} \quad \frac{\partial\phi}{\partial n} = 0 \quad \text{on } \Gamma_c \quad (m > 0 \text{ on } \Gamma_c).$$

The stabilizability condition consists in showing that, for all $1 \leq j \leq N_u$, if

$$-\Delta\phi = \lambda_j \phi \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \Gamma,$$

and

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma_c,$$

then

$$\phi = 0.$$

In the case when $\Omega = (0, a) \times (0, b)$, the eigenfunctions are known

$$\phi_{j,\ell}(x_1, x_2) = \frac{2}{\sqrt{ab}} \sin\left(\frac{j\pi x_1}{a}\right) \sin\left(\frac{\ell \pi x_2}{b}\right)$$

and

$$\lambda_{j,\ell} = -\frac{j^2\pi^2}{a^2} - \frac{\ell^2\pi^2}{b^2}.$$

Assume that $m = 1$ on $\{a\} \times (b/4, 3b/4)$. On Γ_c, we have

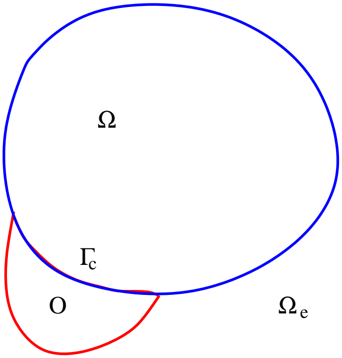
$$B^*\phi_{j,\ell}(a, x_2) = \frac{\partial \phi_{j,\ell}}{\partial n}(a, x_2) = \frac{2a}{\sqrt{ab}} \sin\left(\frac{\ell\pi x_2}{b}\right).$$

The condition of linear independence of the family

$$\left(B^*\phi_{j,\ell}(a, \cdot)|_{\Gamma_c}\right)_{(j,\ell) \ s.t. \ \lambda_{j,\ell} \geq -\omega},$$

is obvious.

For an arbitrary bounded and regular domain Ω, we use an extension procedure



$-\Delta\phi = \lambda_j\,\phi$ in Ω, $\phi = 0$ on Γ,

and $\dfrac{\partial\phi}{\partial n} = 0$ on Γ_c.

We extend ϕ by zero to $\mathcal{O}$.

$-\Delta\phi = \lambda_j\,\phi$ in $\Omega_e = \Omega \cup \mathcal{O} \cup \Gamma_c$.

If

$$-\Delta\phi = \lambda\phi \quad \text{in } \Omega, \quad -\Delta\phi = \lambda\phi \quad \text{in } \mathcal{O},$$

$$\phi|_{\partial\Omega} = \phi|_{\partial\mathcal{O}}, \quad \frac{\partial\phi}{\partial n}|_{\partial\Omega} = \frac{\partial\phi}{\partial n}|_{\partial\mathcal{O}},$$

then

$$-\Delta\phi = \lambda\phi \quad \text{in } \Omega_e.$$

In particular, if $\phi = 0$ in $\mathcal{O}$, we have $\phi = 0$ in Ω (consequence of the Holmgren Theorem).

For the other problems (linearized Burgers equation, Stokes equations, Linearized Navier-Stokes equations), the unique continuation property relies on Carleman type estimates.

Fabre-Lebeau (Stokes, Linearized N.S.E.), Fursikov-Imanuvilov (Convection-diffusion, Linearized N.S.E.), Lebeau-Robbiano (elliptic equation) (see lecture 3).

4. Feedback stabilizabilition of Infinite Dimensional Linear Systems

As for finite dimensional systems, we can decompose Z and $Z^* \equiv Z$ as follows

$$Z = Z_{\omega,s} \oplus Z_{\omega,u}, \quad Z_{\omega,u} = \oplus_{j=1}^{N_u} G_{\mathbb{R}}(\lambda_j), \quad Z_{\omega,s} \text{ is invariant under } A,$$

$$Z^* = Z^*_{\omega,s} \oplus Z^*_{\omega,u}, \quad Z^*_{\omega,u} = \oplus_{j=1}^{N_u} G^*_{\mathbb{R}}(\lambda_j), \quad Z^*_{\omega,s} \text{ is invariant under } A^*.$$

Here, $Z_{\omega,u}$ is the unstable space for $A_\omega = A + \omega I$ and $Z^*_{\omega,u}$ is the unstable space for $A^*_\omega = A^* + \omega I$.
Let $\pi_{\omega,u}$ the projection onto $Z_{\omega,u}$ along $Z_{\omega,s}$ and set $\pi_{\omega,s} = I - \pi_{\omega,u}$.

Similarly let $\pi^*_{\omega,u}$ the projection onto $Z^*_{\omega,u}$ along $Z^*_{\omega,s}$ and set $\pi^*_{\omega,s} = (I - \pi_{\omega,u})^*$.

The system $(A + \omega I, B)$ is stabilizable iff the system $(A_{\omega,u}, B_{\omega,u}) = (\pi_{\omega,u} A_\omega, \pi_{\omega,u} B)$ is stabilizable.

4.1. Boundary stabilization of the 1D heat equation

We consider the equation

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{in } (0,1) \times (0,\infty),$$
$$z(0,t) = 0 \quad \text{and} \quad z(1,t) = u(t) \quad \text{for } t \in (0,\infty),$$
$$z(x,0) = z_0(x) \quad \text{in } (0,1).$$

Here $U = \mathbb{R}$. The control is necessarily of dimension 1. We look for u stabilizing the equation with the exponential decay rate e^{-10t}. $\omega = 10$.

The eigenvalues of $A = d^2/dx^2$, with Dirichlet boundary conditions, and the corresponding eigenfunctions are

$$\lambda_j = -\pi^2 j^2, \quad e_j(x) = \sqrt{2}\sin(j\,\pi\,x).$$

The only unstable eigenvalue for $A + \omega I$, with $\omega = 10$ is $\lambda_1 + 10 = 10 - \pi^2$. Thus

$$Z_{\omega,u} = \mathbb{R}\, e_1 \quad \text{and} \quad Z_{\omega,s} = \bigoplus_{i=2}^{\infty} \mathbb{R}\, e_i.$$

Let us define the projectors associated with this decomposition

$$\pi_{\omega,u} f = (f, e_1)_{L^2(\Omega)} e_1,$$
$$\pi_{\omega,s} f = \sum_{i=2}^{\infty} (f, e_i)_{L^2(\Omega)} e_i.$$

Thus

$$\pi_{\omega,u} Bu = \langle Bu, e_1 \rangle e_1 = (u, B^* e_1)_{L^2(\Gamma)} e_1 = -u\, e_1'(1)\, e_1,$$
$$\pi_{\omega,s} Bu = \sum_{i=2}^{\infty} \langle Bu, e_i \rangle e_i = \sum_{i=2}^{\infty} (u, B^* e_i)_{L^2(\Gamma)} e_i = -u \sum_{i=2}^{\infty} e_i'(1)\, e_i.$$

and

$$e_i'(1) = \sqrt{2}\, i\, \pi\, (-1)^i.$$

The series

$$\sum_{i=2}^{\infty} e_i'(1)\, e_i$$

converges in $(D(A^*))'$, but not in $L^2(\Omega)$.

To determine the control of minimal norm, we project the equation $z' = (A + \omega I)z + Bu$ onto $Z_{\omega,u}$. If $\pi_{\omega,u} z = z_1 e_1$, the equation for z_1 is

$$z_1' = (10 - \pi^2)z_1 - u(t)e_1'(1) = (10 - \pi^2)z_1 + u(t)\,\pi\sqrt{2},$$
$$z_1(0) = (z_0, e_1)_{L^2(0,1)}.$$

The Bernoulli equation for this system is

$$p > 0, \quad 2(10 - \pi^2)p - (\pi\sqrt{2})^2 p^2 = 0.$$

Thus

$$p = \frac{2(10 - \pi^2)}{2\pi^2},$$

and the control of minimal norm obeys the feedback law

$$u(t) = -\pi\sqrt{2}\frac{2(10 - \pi^2)}{2\pi^2}\, z_1(t).$$

The closed loop linear system for satisfied by z_1 is

$$z_1' = -(10 - \pi^2)z_1, \quad z_1(0) = (z_0, e_1)_{L^2(0,1)}.$$

The closed loop system for $e^{10t} z = \widehat{z}$ is

$$\frac{\partial \widehat{z}}{\partial t} - \frac{\partial^2 \widehat{z}}{\partial x^2} - 10\widehat{z} = 0 \quad \text{in } (0,1) \times (0,\infty),$$

$$\widehat{z}(0,t) = 0 \quad \text{and} \quad \widehat{z}(1,t) = -\pi\sqrt{2}\frac{2(10-\pi^2)}{2\pi^2}\,(\widehat{z}(t), e_1)_{L^2(0,1)}, \quad \text{for } t \in (0,\infty$$

$$\widehat{z}(x,0) = z_0(x) \quad \text{in } (0,1).$$

Since this system is stable, it means that the solution z to the equation

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{in } (0,1) \times (0,\infty),$$

$$z(0,t) = 0 \quad \text{and} \quad z(1,t) = -e^{-10t}\pi\sqrt{2}\frac{2(10-\pi^2)}{2\pi^2}\,(z(t), e_1)_{L^2(0,1)}, \quad \text{for } t \in$$

$$z(x,0) = z_0(x) \quad \text{in } (0,1),$$

obeys

$$\|z(t)\|_{L^2(0,1)} \le Ce^{-10t}\|z_0\|_{L^2(0,1)}.$$

4.2. Boundary stabilization of the 2D heat equation in a rectangle

We now consider the control system

$$
\begin{aligned}
&\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q = \Omega \times (0,\infty),\\
&z = u \quad \text{in } \Sigma_c = \Gamma_c \times (0,\infty),\\
&z = 0 \quad \text{in } \Sigma \setminus \Sigma_c, \quad \text{with } \Sigma = \Gamma \times (0,\infty),\\
&z(0) = z_0 \quad \text{in } \Omega,
\end{aligned}
$$

where $\Omega = (0,\pi) \times (0,1)$, $\Gamma_c = \{\pi\} \times (0,1)$.

The eigenvalues of $A = \Delta$, with Dirichlet boundary conditions, and the corresponding eigenfunctions are

$$
\lambda_{j,\ell} = -j^2 - \pi^2\,\ell^2, \quad e_{j,\ell}(x) = \sqrt{2/\pi}\,\sin(j\,x_1)\sqrt{2}\,\sin(\ell\,\pi\,x_2).
$$

If we want to stabilize with the exponential decay rate e^{-11t}, we have only an unstable eigenvalue

$$
\lambda_{1,1} = 11 - 1 - \pi^2 = 10 - \pi^2.
$$

We write the system for $e^{11t}z = \widehat{z}$ and $e^{11t}u = \widehat{u}$ in the form

$$\widehat{z}' = A\widehat{z} + B\widehat{u}, \quad \widehat{z}(0) = z_0.$$

Thus

$$B^* e_{1,1} = -\frac{\partial e_{1,1}}{\partial n}.$$

We choose the control $\widehat{u}(x_2, t)$ of the form

$$\widehat{u}(x_2, t) = v(t) B^* e_{1,1} = -v(t) \frac{2}{\sqrt{\pi}} \sin(\pi x_2).$$

We set $\pi_{u,\omega}\widehat{z} = z_{1,1}e_{1,1}$. We obtain the equation for $z_{1,1}$ by projecting the state equation the equation satisfied by $\widehat{u}$ onto $Z_{\omega,u} = \mathbb{R}\, e_{1,1}$. We have

$$
\begin{aligned}
z_{1,1}' &= (10-\pi^2)z_{1,1} + (\widehat{u}, B^* e_{1,1})_{L^2(\Gamma_c)} \\
&= (10-\pi^2)z_{1,1} + v(t)\beta, \quad \text{with } \beta = \int_{\Gamma_c} \frac{4}{\pi}\sin^2(\pi\, x_2)\, dx_2, \\
z_{1,1}(0) &= (z_0, e_{1,1})_{L^2(\Omega)}.
\end{aligned}
$$

The Bernoulli equation for this system is

$$p > 0, \quad 2(10-\pi^2)p - \beta^2 p^2 = 0.$$

Thus

$$p = \frac{2(10-\pi^2)}{\beta^2}.$$

The closed loop linear system for $z_{1,1}$ is

$$z_{1,1}' = -(10-\pi^2)z_{1,1}, \quad z_{1,1}(0) = (z_0, e_{1,1})_{L^2(\Omega)},$$

$$v(t) = -\frac{2}{\beta}(10-\pi^2)z_{1,1}(t),$$

and

$$\widehat{u}(x_2,t) = \frac{4}{\beta\sqrt{\pi}}\sin(\pi x_2)(10-\pi^2)z_{1,1}.$$

The closed loop linear system for z is

$$\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q = \Omega\times(0,\infty),$$

$$z = e^{-11t}\frac{4}{\beta\sqrt{\pi}}\sin(\pi x_2)(10-\pi^2)\int_\Omega z\, e_{1,1}\, dx \quad \text{in } \Sigma_c = \Gamma_c\times(0,\infty),$$

$$z = 0 \quad \text{in } \Sigma\setminus\Sigma_c, \quad \text{with } \Sigma = \Gamma\times(0,\infty),$$

$$z(0) = z_0 \quad \text{in } \Omega.$$

Conclusion

• Now, we have to study the boundary feedback stabilization of the linearized Navier-Stokes equations.

• We have to choose the feedback control law for the L.N.S.E. in such a way that it also stabilizes, locally, the Navier-Stokes equations.