Stochastic Analysis and Control of Fluid Flows

Lecture 1

School of Mathematics –IISER-TVM

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Introduction to feedback stabilization – Stabilizability of F.D.S.

Jean-Pierre Raymond – Institut Mathématiques de Toulouse

Q1. Controllability. How to drive a nonlinear system from a given initial state to another prescribed state by the action of some control during the time interval $(0, T)$?

Q2. Stabilizability. How to maintain a system at an unstable position, in the presence of perturbation by using some measurements to estimate the state at each time t, and by using the estimated state in a control law ?

Plan of Lecture 1

1. Introduction

1.1. Problems and models. The inverted pendulum – A finite dimensional model

Two infinite dimensional models. Fluid flows in a neighbourhood of unstable solutions. A Burgers type equation

1.2. Feedback stabilization with total and partial information

2. Finite dimensional linear systems

2.1. The Duhamel formula

2.2. Controllability and Reachability

2.3. Stabilizability and its characterization

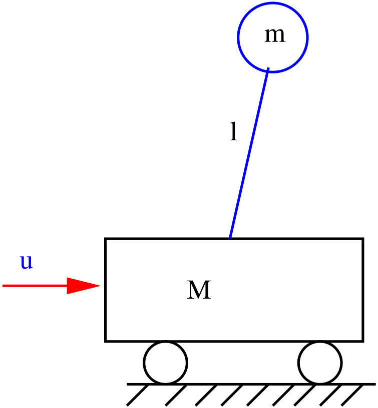
2.4. Construction of a Feedback

3. Algorithms for solving Riccati equations

1. Introduction

1.1. Models

1.1.i. The inverted pendulum



The nonlinear system

$$(M+m)x'' + m\,\ell\,\theta''\cos\theta - m\,\ell|\theta'|^2\sin\theta = u,$$

$$m\,\ell\,\theta'' + m\,x''\,\cos\theta = m\,g\,\sin\theta.$$

M is the mass of the car,

m is the mass of the pendulum,

$x(t)$ is the position of the car at time t,

$\theta(t)$ is the angle between the unstable position and the pendulum measured clockwise,

u is a force applied to the car. The Lagrance equations read as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x'}\right) - \frac{\partial L}{\partial x} = u,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \theta'}\right) - \frac{\partial L}{\partial \theta} = 0,$$

where the Lagrangian $L = T - V$, T is the kinetic energy and V is the potential energy.

$$L = \frac{1}{2}(M+m)|x'|^2 + m\,\ell\,x'\theta'\cos\theta + \frac{1}{2}m\,\ell^2\,|\theta'|^2 - mg\ell\cos\theta.$$

$$x'' = -\frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} + \frac{m\ell}{M + m\sin^2\theta}|\theta'|^2\sin\theta + \frac{1}{M + m\sin^2\theta}u,$$

$$\theta'' = -\frac{m}{M + m\sin^2\theta}|\theta'|^2\sin\theta\cos\theta + g\frac{M + m}{M\ell + m\ell\sin^2\theta}\sin\theta$$

$$-\frac{\cos\theta}{M\ell + m\ell\sin^2\theta}\,u.$$

As a first order system, we write

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ x' \\ \theta \\ \theta' \end{pmatrix} = F(x, x', \theta, \theta') + \begin{pmatrix} 0 \\ \frac{1}{M + m\sin^2\theta} \\ 0 \\ \frac{-\cos\theta}{M\ell + m\ell\sin^2\theta} \end{pmatrix} u = F(x, x', \theta, \theta') + G(\theta)\,u,$$

with

$$F(x, x', \theta, \theta') =$$

$$\begin{pmatrix} x' \\ -\dfrac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} + \dfrac{m\ell}{M+m}|\theta'|^2\sin\theta \\ \theta' \\ -\dfrac{m}{M + m\sin^2\theta}|\theta'|^2\sin\theta\cos\theta + g\dfrac{M+m}{M\ell + m\ell\sin^2\theta}\sin\theta \end{pmatrix}.$$

We linearize F about $(0,0,0,0)$, we write

$$F(x,x',\theta,\theta') = F(0,0,0,0) + DF(0,0,0,0)\begin{pmatrix} x \\ x' \\ \theta \\ \theta' \end{pmatrix} + \widetilde{N}(\theta,\theta'),$$

with $F(0,0,0,0) = (0,0,0,0)^T$ and

$$DF(0,0,0,0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{g(M+m)}{M\ell} & 0 \end{pmatrix}.$$

Similarly

$$G(\theta)u = G(0)u + \widetilde{R}(\theta)u,$$

with

$$G(0) = \begin{pmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{M\ell} \end{pmatrix}.$$

Setting $z = (x, x', \theta, \theta')^T$, we have

$$z' = Az + Bu + N(z) + R(z)u, \quad z(0) = z_0,$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{g(M+m)}{M\ell} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{M\ell} \end{pmatrix},$$

$$z_0 = (x_0, x_1, \theta_0, \theta_1)^T$$

and the two nonlinear terms N and R obey $N(0) = 0$ and $R(0) = 0$.

1.1.ii. Fluid flows in a neighbourhood of unstable solutions

• We consider a fluid flow governed by the N.S.E. (w and q are the volocity and the pressure)

• Given an unstable stationary solution w_s.

• Find a boundary control u able to stabilize w exponentially when $w(0) = w_s + z_0$.

We consider two different problems:
- Control of the wake behind a circular cylinder,
- The control of a flow in an open cavity.

Control of the wake behind an obstacle $Re = u_s D/\nu$

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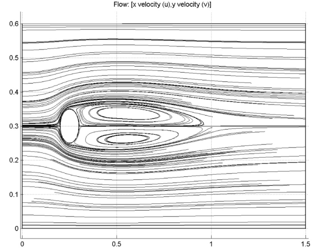
<td>
</td>
<td>$5 < Re < 50$</td>
<td>A fixed pair of vortices</td>
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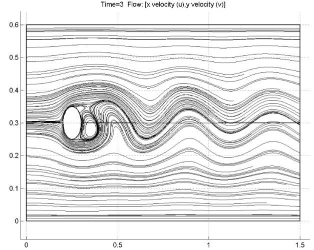
<td>
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<td>$50 < Re < 150$</td>
<td>Vortex street</td>
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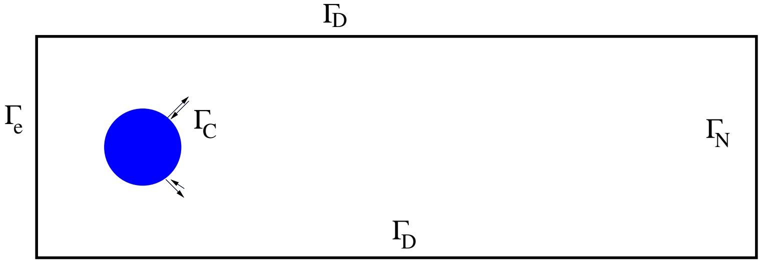
Flow around a cylinder with an outflow boundary condition

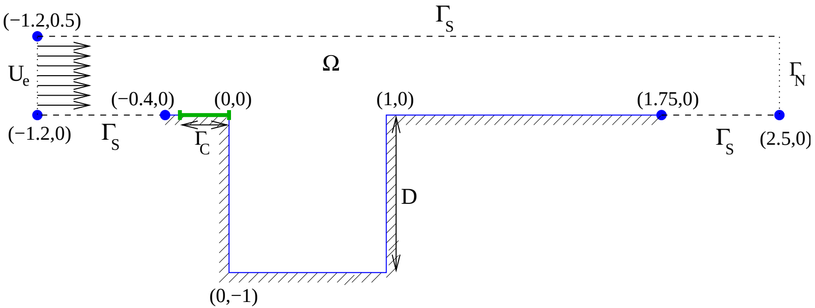


$$\nu \frac{\partial z}{\partial n} - pn = 0 \quad \text{on} \quad \Gamma_N$$

or

$$\sigma(z,p)n = \nu \left(\nabla z + (\nabla z)^T \right) n - pn = 0 \quad \text{on} \quad \Gamma_N.$$

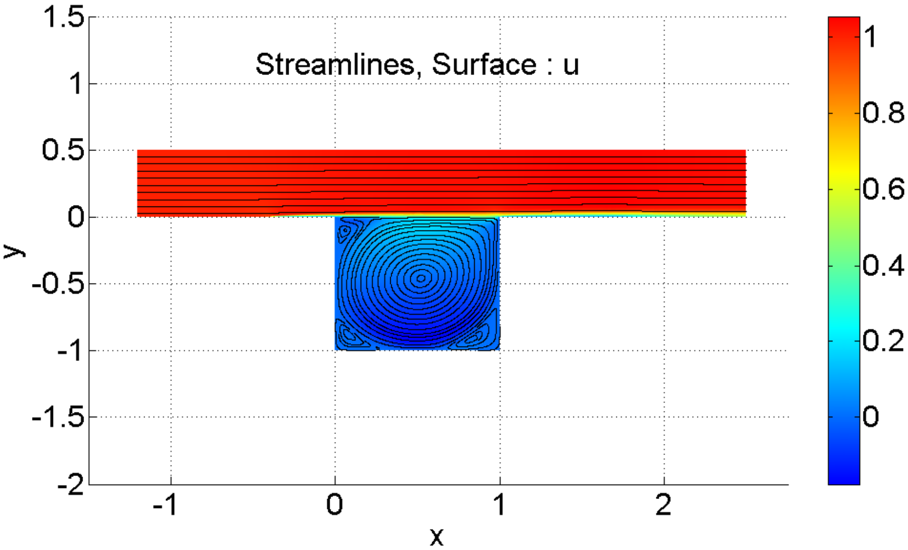
Flow in an open cavity



The boundary conditions on Γ_S are

$$w \cdot n = 0 \quad \text{and} \quad \sigma(w, q)\,\tau = 0.$$

Stationary solution of the open cavity



The unstable stationary solution w_s of the N.S.E.

$$-\nu\Delta w_s + (w_s\cdot\nabla)w_s + \nabla q_s = 0, \quad \text{in } \Omega,$$

$$\text{div } w_s = 0 \quad \text{in } \Omega, \quad w_s = u_s \text{ on } \Gamma_e, \quad + \text{ Other B.C. on } \Gamma\setminus\Gamma_e.$$

The stabilization problem

$$\text{Find } u \text{ s.t. } \|w(t) - w_s\|_Z \longrightarrow 0 \quad \text{as } t\longrightarrow\infty,$$

$$\frac{\partial w}{\partial t} - \nu\Delta w + (w\cdot\nabla)w + \nabla q = 0, \quad \text{div } w = 0 \quad \text{in } Q,$$

$$w = u_s \text{ on } \Sigma_e = \Gamma_e\times(0,\infty), \quad w = Mu \text{ on } \Sigma_c = \Gamma_c\times(0,\infty),$$

$$+ \text{ Other B.C. on } \Sigma\setminus(\Sigma_e\cup\Sigma_c), \quad w(0) = w_0 \text{ in } \Omega.$$

Set $z = w - w_s$, $p = q - q_s$. The linearized (resp. nonlinear) equation is

$$\frac{\partial z}{\partial t} - \nu \Delta z + (w_s \cdot \nabla) z + (z \cdot \nabla) w_s + (z \cdot \nabla) z + \nabla p = 0,$$

$$\text{div } z = 0 \quad \text{in } Q, \quad z = Mu \quad \text{on } \Sigma_D,$$

$$+ \text{ Other B.C. on } \Sigma \setminus (\Sigma_D), \qquad z(0) = z_0 \quad \text{in } \Omega\,.$$

with

$$\text{supp}\, M \subset \Gamma_c.$$

1.1.iii. A simplified model – A Burgers type equation

$$\frac{\partial w}{\partial t} - \Delta w + \partial_i (w^2) = 0 \quad \text{in } Q,$$

$$w = Mu + g \quad \text{on } \Sigma, \quad z(0) = z_0 \quad \text{in } \Omega\,.$$

Let w_s be the stationnary solution to

$$-\Delta w_s + \partial_i (w_s^2) = 0 \quad \text{in } \Omega, \qquad w_s = g \quad \text{on } \Gamma\,.$$

The L. E. and the nonlinear equation satisfied by $z = w - w_s$ are

$$\frac{\partial z}{\partial t} - \Delta z + 2\,\partial_i w_s\, z + 2\,\partial_i z\, w_s \; + \; \partial_i (z^2) = 0,$$

$$z = Mu \quad \text{on } \Sigma, \quad z(0) = z_0 = w_0 - w_s \quad \text{in } \Omega\,.$$

1.2. Feedback stabilization

1.2.i Representation of systems

- The model of the inverted pendulum can be written in the form

$$z' = Az + Bu + N(z) + R(z)u, \quad z(0) = z_0,$$

where $z = (x, x', \theta, \theta')$, the two nonlinear terms N and R obey $N(0) = 0$, $N'(0) = 0$, and $R(0) = 0$.

- The Burgers eq. can be written in the form

$$z' = Az + Bu + F(z), \quad z(0) = z_0,$$

where $z = w - w_s$, w is the velocity of the instationary flow, while w_s is the stationary solution.

- The Navier-Stokes equations can be written in the form

$$\Pi z' = A\Pi z + Bu + F(z), \quad \Pi z(0) = \Pi z_0,$$

$$(I - \Pi)z = (I - \Pi)Du.$$

The Leray projector Π is used to eliminate the pressure p in the equations.

Thus we have to deal with linear systems either of the form

$$z' = Az + Bu, \quad z(0) = z_0,$$

or of the form

$$\Pi z' = A\Pi z + Bu, \quad \Pi z(0) = \Pi z_0,$$

$$(I - \Pi)z = (I - \Pi)Du.$$

This last system is called a descriptor system.

The approximate system obtained by a F.E.M. of the L.N.S.E. will be of the same type.

1.2.ii. The open loop stabilization problem

For these models, the problem consists in finding a control $u \in L^2(0,\infty;U)$ such that

$$\|z_{z_0,u}(t)\|_Z \le Ce^{-\omega t}\|z_0\|_Z, \quad C > 0, \quad \omega > 0,$$

provided that $\|z_0\|_Z$ is small enough.

Existence of a stabilizing control. Except for finite dimensional systems, there is no general result for the stabilizability of nonlinear systems.

Stabilization of the linearized model. Similar problems can be studied for the linearized model

$$z' = Az + Bu, \quad z(0) = z_0.$$

If we find a control $u \in L^2(0,\infty;U)$ able to stabilize the linearized system (e.g. by solving an optimal control problem), there is no reason that u also stabilizes the nonlinear system.

1.2.iii. Stabilization by feedback with full information

One way for finding a control able to stabilize the nonlinear system is to look for a control stabilizing the linear system in feedback form, that is such that

$$u(t) = K\, z_{z_0,u}(t), \qquad K \in \mathcal{L}(Z,U).$$

For that we consider the optimal control problem $(\mathcal{P})$

$$\text{Minimize } \; J(z,u) = \frac{1}{2}\int_0^\infty \Big\|Cz(t)\Big\|_Y^2\, dt + \frac{1}{2}\int_0^\infty \Big\|u(t)\Big\|_U^2\, dt$$

$$z' = Az + Bu, \qquad z(0) = z_0,$$

where $C \in \mathcal{L}(Z,Y)$.

The issues

• What are the conditions on (A, B) so that the above control problem has a solution ?

• How to determine the operator K ?

• How to choose the control problem so that the closed loop nonlinear system

$$z' = Az + BKz + F(z), \quad z(0) = z_0,$$

admits a solution exponentially decreasing ?

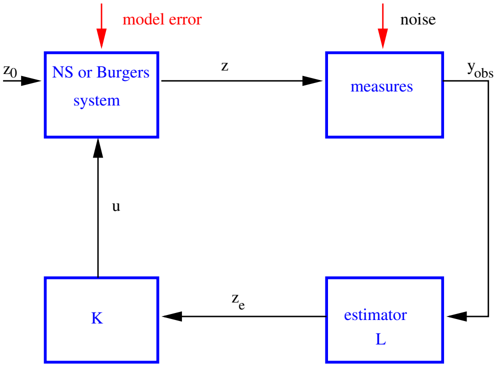
• Is this programme useful in pratical applications ?

1.2.iv. Stabilization by feedback with partial information

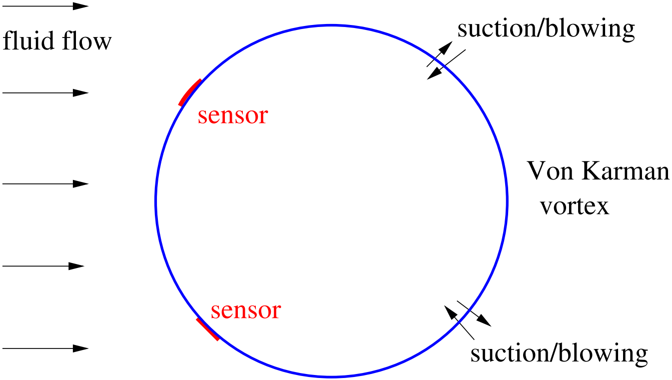
If the feedback K also stabilizes the nonlinear system, by solving

$$z' = Az + BKz + F(z) + \mu, \quad z(0) = z_0 + \mu_0,$$

we know a control $u(t) = Kz_{c\ell n,z_0}(t)$ stabilizing the nonlinear system. But in pratice μ_0 is not necessarily known, we can also have other types of uncertainties μ. This is why we have to look for an estimation z_e of the state z in function of measurements, and use the control law $u(t) = K\, z_e(t)$.



The complete problem: estimation + feedback control



2. Finite Dimensional Linear Systems

2.1. The Duhamel formula

When $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, the solution to system

$$z' = Az + Bu, \quad z(0) = z_0,$$

is defined by

$$(E) \qquad z_{z_0,u}(t) = z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A}Bu(s)\,ds.$$

The same formula holds true in $\mathbb{C}^n$. If $(\lambda_i)_{1\le i\le r}$ are the complex eigenvalues of A, we can define

$$E(\lambda_j) = \mathrm{Ker}(A - \lambda_j I), \quad \dim E(\lambda_j) = \ell_j = \text{geometric multiplicity of } \lambda_j,$$

$$G(\lambda_j) = \mathrm{Ker}((\lambda_j I - A)^{m(\lambda_j)}), \quad \text{the generalized eigenspace ass. to } \lambda_j,$$

$$\dim G(\lambda_j) = N(\lambda_j) = \text{algebraic multiplicity of } \lambda_j.$$

We have

$$\mathbb{C}^n = \oplus_{j=1}^{r} G(\lambda_j), \quad A = Q\widehat{A}Q^{-1}, \quad AG(\lambda_j) \subset G(\lambda_j),$$

$$\widehat{A} = \begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & 0 \\ & 0 & \ddots & \\ & & & \Lambda_r \end{pmatrix}, \quad \Lambda_j = \begin{pmatrix} J_j^1 & & & \\ & J_j^2 & & 0 \\ & 0 & \ddots & \\ & & & J_j^{\ell_j} \end{pmatrix},$$

and

$$e^{tA} = Qe^{t\widehat{A}}Q^{-1} = Q \begin{pmatrix} e^{t\Lambda_1} & & & \\ & e^{t\Lambda_2} & & 0 \\ & 0 & \ddots & \\ & & & e^{t\Lambda_r} \end{pmatrix} Q^{-1}.$$

In particular

$$e^{tA}G(\lambda_j) \subset G(\lambda_j).$$

We can rewrite the equation

$$z' = Az + Bu, \quad z(0) = z_0,$$

in the form

$$Q^{-1}z' = Q^{-1}AQQ^{-1}z + Q^{-1}Bu, \quad Q^{-1}z(0) = Q^{-1}z_0,$$

that is

$$\widehat{z}' = \widehat{A}\,\widehat{z} + \widehat{B}u, \quad \widehat{z}(0) = \widehat{z}_0,$$

where $\widehat{z} = Q^{-1}z$, $\widehat{B} = Q^{-1}B$, $\widehat{z}_0 = Q^{-1}z_0$.

The vector $\widehat{z}$ may be written as

$$\widehat{z} = \oplus_{j=1}^{r}\widehat{z}^i = \begin{pmatrix} \widehat{z}_1 \\ \vdots \\ \widehat{z}_r \end{pmatrix} \quad \text{with} \quad \widehat{z}^i = \begin{pmatrix} \vdots \\ \widehat{z}_i \\ \vdots \end{pmatrix}.$$

Setting

$$z = \oplus_{j=1}^{r} Q\hat{z}^i = \oplus_{j=1}^{r} z^i,$$

we can notice that z^i is the projection of z onto $G(\lambda_i)$ along $\oplus_{j=1,\, j\neq i}^{r} G(\lambda_j)$.

We can also decompose $\mathbb{R}^n$ into *real generalized eigenspaces*

$$\mathbb{R}^n = \oplus_{j=1}^{r} G_{\mathbb{R}}(\lambda_j), \quad G_{\mathbb{R}}(\lambda_j) = G_{\mathbb{R}}(\bar{\lambda}_j) = \text{vec}\{\text{Re}\, G(\lambda_j), \text{Im}\, G(\lambda_j)\},$$
$$AG_{\mathbb{R}}(\lambda_j) \subset G_{\mathbb{R}}(\lambda_j).$$

The projection onto $G_{\mathbb{R}}(\lambda_i)$ along $\oplus_{j=1,\, j\neq i}^{r} G_{\mathbb{R}}(\lambda_j)$ can be defined accordingly.

The same analysis can be done for parabolic partial differential equations.

2.2. Controllability and Reachability of Finite Dimensional Systems

In this part, $Z = \mathbb{R}^n$ or $Z = \mathbb{C}^n$ and $U = \mathbb{R}^m$ or $U = \mathbb{C}^m$. We make the identifications $Z = Z^*$ and $U = U^*$.

The operator $L_T \ : \ L^2(0, T; U) \longmapsto Z$

$$L_T u = \int_0^T e^{(T-s)A} B u(s)\, ds.$$

The reachable set from z_0 at time T

$$R_T(z_0) = e^{TA} z_0 + \operatorname{Im} L_T.$$

Exact controllability. The pair (A, B) is exactly controllable at time T if $R_T(z_0) = Z$ for all $z_0 \in Z$.

Reachability. A state z_T is reachable from $z_0 = 0$ at time $T < \infty$ if there exists $u \in L^2(0, T; U)$ such that

$$L_T u = z_{0,u}(T) = \int_0^T e^{(T-s)A} B u(s)\, ds = z_T.$$

The system (A, B) is reachable at time $T < \infty$ iff

$$\operatorname{Im} L_T = Z.$$

A finite dimensional system is reachable at time T iff it is controllable at time T.

The system is reachable (or exactly controllable) at time $T < \infty$ iff the matrix (called 'controllability Gramian')

$$W_{A,B}^T = \int_0^T e^{tA} B B^* e^{tA^*}\, dt$$

is invertible.

Idea of the proof. L_T is surjective iff its adjoint operator $L_T^* \in \mathcal{L}(Y, L^2(0, T; U))$,

$$(L_T^* \phi)(\cdot) = B^* e^{(T-\cdot)A^*} \phi,$$

is injective. This last condition is equivalent to the existence of $\alpha > 0$ such that

$$\int_0^T \|B^* e^{sA^*} \phi\|_U^2\, ds = \|L_T^* \phi\|_{L^2(0,T;U)}^2 \geq \alpha \|\phi\|_Z^2, \quad \forall \phi \in Z.$$

Assume that the system (E) is reachable at time T. For a given $z_T \in Z$, the control

$$\bar{u}(t) = -B^* e^{(T-t)A^*} (W_{A,B}^T)^{-1} e^{TA} z_T$$

is such that

$$z_{0,\bar{u}}(T) = \int_0^T e^{(T-s)A} B\bar{u}(s)\, ds = z_T.$$

Moreover

$$\|\bar{u}\|_{L^2(0,T;U)}^2 = ((W_{A,B}^T)^{-1} z_T, z_T)_Z$$

and

$$\|\bar{u}\|_{L^2(0,T;U)} \leq \|u\|_{L^2(0,T;U)},$$

for all u such that

$$z_{0,u}(T) = \int_0^T e^{(T-s)A} Bu(s)\, ds = z_T.$$

Finite Dimensional System. The system (A, B) is controllable at time T if and only if one of the following conditions is satisfied.

$$W_{A,B}^T = \int_0^T e^{tA} B B^* e^{tA^*}\, dt > 0, \quad \forall T > 0,$$

$$\text{rank}\left[B \,|\, AB \,|\, \dots \,|\, A^{n-1} B\right] = n,$$

$$\forall \lambda \in \mathbb{C}, \quad \text{rank}\,[A - \lambda I \,|\, B] = n,$$

$$\forall \lambda \in \mathbb{C}, \quad A^* \phi = \lambda \phi \quad \text{and} \quad B^* \phi = 0 \Rightarrow \phi = 0,$$

$$\exists K \in \mathcal{L}(Z, U), \quad \sigma(A + BK) \text{ can be freely assign}$$

(with the cond. that complex eigenvalues are in conjugate pairs).

Applications. Let us study the controllability of the linearized inverted pendulum.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{g(M+m)}{M\ell} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{M\ell} \end{pmatrix},$$

The controllability matrix is

$$[B \mid AB \mid A^2B \mid A^3B] = \begin{pmatrix} 0 & \dfrac{1}{M} & 0 & \dfrac{mg}{M^2\ell} \\ \dfrac{1}{M} & 0 & \dfrac{mg}{M^2\ell} & 0 \\ 0 & -\dfrac{1}{M} & 0 & -\frac{g(M+m)}{M^2\ell^2} \\ -\frac{1}{M} & 0 & -\dfrac{g(M+m)}{M^2\ell^2} & 0 \end{pmatrix}.$$

Its determinent is

$$
\begin{aligned}
&\det[B \mid AB \mid A^2 B \mid A^3 B] \\
&= -\frac{g(M+m)}{M^3 \ell^2}\left(-\frac{g(M+m)}{M^3 \ell^2} + \frac{mg}{M^3 \ell^2}\right) \\
&\quad + \frac{mg}{M^3 \ell^2}\left(-\frac{g(M+m)}{M^3 \ell^2} + \frac{mg}{M^3 \ell^2}\right) = \frac{g^2}{M^4 \ell^4}.
\end{aligned}
$$

The linearized inverted pendulum is controllable at any time $T > 0$.

2.3. Stabilizability of F.D.S.

2.3.i. Open loop stabilizability. System (A, B) is *open loop stabilizable in* Z when for any initial condition $z_0 \in Z$, there exists a control $u \in L^2(0, \infty; U)$ s.t.

$$\int_0^\infty \|z_{z_0,u}(t)\|_Z^2 \, dt < \infty.$$

Stabilizability by feedback. System (A, B) is *stabilizable by feedback* when there exists an operator $K \in \mathcal{L}(Z, U)$ s. t. $A + BK$ is exponentially stable in Z.

Open loop stabilizability is equivalent to *stabilizability by feedback* for finite dimensional systems or for parabolic systems with Dirichlet boundary conditions like the heat equation, the linearized Burgers equation, the Stokes equation, the linearized Navier-Stokes equations.

The precise assumptions are

- the semigroup generated by $(A, D(A))$ on Z is analytic,
- $B \in \mathcal{L}(U, (D(A^*))')$

2.3.ii. Characterization of the stabilizability of F.D.S.

The finite dimensional case. System (E) is stabilizable if and only if one of the following conditions is satisfied.

$$(i)\ \forall\lambda,\ \mathrm{Re}\lambda \geq 0, \quad \mathrm{rank}\,[A - \lambda I \,|\, B] = n,$$

$$(ii)\ \forall\lambda,\ \mathrm{Re}\lambda \geq 0, \quad A^*\phi = \lambda\phi \quad \text{and} \quad B^*\phi = 0 \Rightarrow \phi = 0,$$

$$(iii)\ \forall\lambda,\ \mathrm{Re}\lambda \geq 0, \quad \mathrm{Ker}(\lambda I - A^*) \cap \mathrm{Ker}(B^*) = \{0\},$$

$$(iv)\ \exists K \in \mathcal{L}(Z, U), \quad \sigma(A + BK) \text{ is stable.}$$

Conditions (ii), (iii), (iv) are equivalent to the stabilizability of (E) for **I**nfinite **D**imensional **S**ystems under the previous conditions on (A, B) (analyticity, compactness, degree of unboundness of B).

FDS Example of a stabilizing feedback. The system (A, B) is controllable iff

$$W^T_{-A,B} = \int_0^T e^{-tA}BB^*e^{-tA^*}\,dt,$$

is invertible for all $T > 0$. Indeed

$$e^{TA}W^T_{-A,B}e^{TA^*} = \int_0^T e^{(T-t)A}BB^*e^{(T-t)A^*}\,dt = \int_0^T e^{\tau A}BB^*e^{\tau A^*}\,d\tau = W^T_{A,B}.$$

Assume that (A, B) is controllable, then

$$K = -B^*(W^T_{-A,B})^{-1}$$

is a stabilizing feedback.

Idea of the proof. The mapping

$$z \longmapsto ((W^T_{-A,B})^{-1}z, z)_Z$$

is a Lyapunov function of the closed loop linear system.

Proof.

FDS – Characterization of the stabilizability in terms of Gramians.

We assume that

$$Z = Z_s \oplus Z_u, \quad Z_u = \oplus_{j=1}^{N_u} G_{\mathbb{R}}(\lambda_j), \quad Z_s = \oplus_{j=N_u+1}^{r} G_{\mathbb{R}}(\lambda_j),$$

$$\text{Re}\lambda_j > 0 \quad \text{if } 1 \le j \le N_u,$$

$$\text{Re}\lambda_j < 0 \quad \text{if } N_u + 1 \le j \le r.$$

Recall that

$$e^{tA} Z_u \subset Z_u \quad \text{and} \quad e^{tA} Z_s \subset Z_s.$$

We also have

$$Z = Z^* = Z_s^* \oplus Z_u^*, \quad Z_u^* = \oplus_{j=1}^{N_u} G_{\mathbb{R}}^*(\lambda_j), \quad Z_s^* = \oplus_{j=N_u+1}^{r} G_{\mathbb{R}}^*(\lambda_j),$$

$$e^{tA^*} Z_u^* \subset Z_u^* \quad \text{and} \quad e^{tA^*} Z_s^* \subset Z_s^*.$$

Let π_u the projection onto Z_u along Z_s and set $\pi_s = I - \pi_u$. The system

$$\pi_u z' = z_u' = A_u z_u + \pi_u B u, \quad \pi_s z' = z_s' = A_s z_s + \pi_s B u,$$

can be written as a system in $Z \times Z$ of the form (using matrix notation)

$$\begin{pmatrix} z_u' \\ z_s' \end{pmatrix} = \begin{pmatrix} A_u & 0 \\ 0 & A_s \end{pmatrix} \begin{pmatrix} z_u \\ z_s \end{pmatrix} + \begin{pmatrix} \pi_u B u \\ \pi_s B u \end{pmatrix}, \quad \begin{pmatrix} z_u(0) \\ z_s(0) \end{pmatrix} = \begin{pmatrix} \pi_u z_0 \\ \pi_s z_0 \end{pmatrix}.$$

If the system (A, B) is stabilizable then the system $(A_u, B_u) = (\pi_u A, \pi_u B)$ is also stabilizable. The converse is true. Assume that $(A_u, B_u) = (\pi_u A, \pi_u B)$ is stabilizable and let $K \in \mathcal{L}(Z_u, U)$ be a stabilizing feedback. Then the system

$$\begin{pmatrix} z_u' \\ z_s' \end{pmatrix} = \begin{pmatrix} A_u + B_u K & 0 \\ B_s K & A_s \end{pmatrix} \begin{pmatrix} z_u \\ z_s \end{pmatrix}$$

is also stable.

2.4. Characterization of the stabilizability of F.D.S.

The following conditions are equivalent

(a) (A, B) is stabilizable,

(b) $(A_u, B_u) = (\pi_u A, \pi_u B)$ is stabilizable,

(c) The Gramian

$$W^{\infty}_{-A_u,B_u} = \int_0^{\infty} e^{-tA_u} B_u B_u^* e^{-tA_u^*} \, dt$$

is invertible.

(d) There exists $\alpha > 0$ such that for all $\phi \in Z_u^*$,

$$(O.I.) \qquad (W^{\infty}_{-A_u,B_u}\phi, \phi)_Z = \int_0^{\infty} \left\| B_u^* e^{-tA_u^*}\phi \right\|_U^2 dt \ \geq \ \alpha \|\phi\|_Z^2.$$

The operator

$$P_u = (W^{\infty}_{-A_u,B_u})^{-1} \in \mathcal{L}(Z_u, Z_u^*), \quad P_u = P_u^* \geq 0,$$

provides a stabilizing feedback

$$A_u - B_u B_u^* P_u \quad \text{is exponentially stable.}$$

Recall that

$$B_u = \pi_u B \quad \text{and} \quad B_u^* = B^* \pi_u^*,$$

where π_u^* is the projection onto Z_u^* along Z_s^*.

The operator P_u satisfies the following Algebraic Bernoulli equation (a degenerate Algebraic Riccati equation)

$$P_u A_u + A_u^* P_u - P_u B_u B_u^* P_u = 0.$$

If we set

$$P = \pi_u^* P_u \pi_u.$$

Then $P \in \mathcal{L}(Z)$ is such that $P = P^* \geq 0$ and solves the following ($A.B.E.$)

$$A.B.E. \qquad \begin{aligned} & P \in \mathcal{L}(Z), \quad P = P^* \geq 0, \\ & A^* P + PA - PBB^* P = 0, \\ & A - BB^* P \text{ generates} \\ & \text{an exponentially stable semigroup.} \end{aligned}$$

Moreover, the feedback $-B^* P$ provides the control of minimal norm in $L^2(0,\infty;U)$

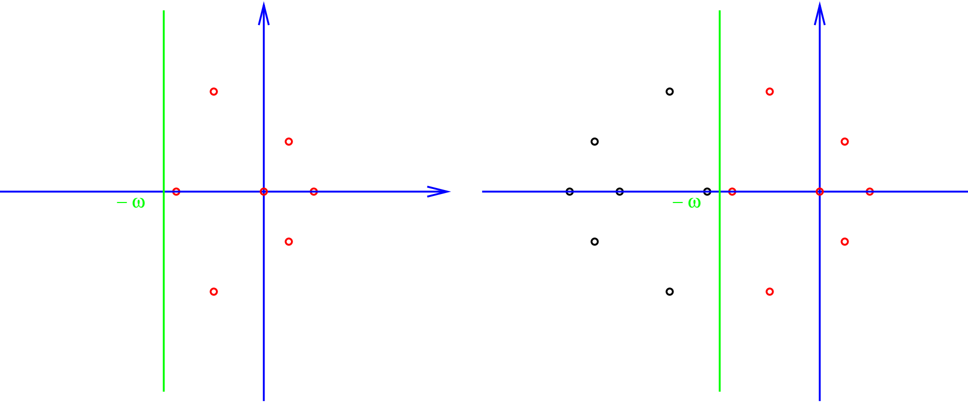
$$u(t) = -B^* P e^{t(A-BB^* P)} z_0,$$

and the spectrum of $A - BB^* P$ is

$$\sigma(A - BB^* P) = \{-\mathrm{Re}\lambda + i\,\mathrm{Im}\lambda \mid \lambda \in \sigma(A),\ \mathrm{Re}\lambda > 0\}$$
$$\cup\{\lambda \mid \lambda \in \sigma(A),\ \mathrm{Re}\lambda < 0\}.$$

To obtain a better exponential decay we can replace A by $A + \omega I$ and determine the corresponding feedback $-B^* P_\omega$. In that case the spectrum of $A - BB^* P_\omega$ is as follows

Spectrum of A and of $A - BB^* P_\omega$



4. An algorithm for solving Riccati equations

We consider the A.R.E.

$$A.R.E. \qquad \begin{array}{l} P \in \mathcal{L}(Z), \quad P = P^* \geq 0, \\ A^*P + PA - PBR^{-1}B^*P + C^*C = 0, \\ A - BR^{-1}B^*P \text{ generates} \\ \text{an exponentially stable semigroup,} \end{array}$$

where the unknown P belongs to $\mathcal{L}(\mathbb{R}^n)$, $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$, where $\mathbb{R}^m$ is the discrete control space, $C \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^\ell)$, where $\mathbb{R}^\ell$ is the observation space.

We make the following assumption.

(H_1) The pair (A, B) is stabilizable

(H_2) The pair (A, C) is detectable or A has no eigenvalue on the imaginary axis.

Equation $(A.R.E)$ admits a unique solution.

The matrix

$$\mathcal{H} = \begin{bmatrix} A & -BR^{-1}B^* \\ CC^* & -A^* \end{bmatrix}$$

is called the Hamiltonian matrix associated with equation $(A.R.E.)$.

It is a symplectic matrix

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}^{-1} \mathcal{H} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = -\mathcal{H}^*.$$

Therefore the matrix $\mathcal{H}$ and $-\mathcal{H}^*$ are similar and they have the same eigenvalues. On the other hand $\mathcal{H}$ and $\mathcal{H}^*$ have also the same set of eigenvalues. Thus if λ is an eigenvalue of H, then $-\lambda$ is also an eigenvalue of $\mathcal{H}$ with the same multiplicity.

Let us denote by $-\lambda_1, -\lambda_2, \ldots, -\lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_n$, the eigenvalues of $\mathcal{H}$, where $\mathrm{Re}(\lambda_i) \geq 0$ for $i = 1, \ldots, n$. Under assumptions (H_1) and (H_2), it can be shown that the matrix $\mathcal{H}$ has no pure imaginary eigenvalues, that is $\mathrm{Re}(\lambda_i) > 0$ for $i = 1, \ldots, n$.

There exists a matrix V whose columns are eigenvectors, or generalized eigenvectors of $\mathcal{H}$, such that

$$V^{-1}\mathcal{H}V = \begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix},$$

where $-J$ is composed of Jordan blocks corresponding to eigenvalues with negative real part, and

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix},$$

is such that $V = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix}$, is a matrix whose columns are eigenvectors corresponding to eigenvalues with negative real parts. It may be proved that V_{11} is nonsingular and the unique positive semidefinite solution of equation ($A.R.E.$) is given by

$$P = V_{21} V_{11}^{-1}.$$

A very simple example

Set

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with $\lambda > 0$, $R^{-1} = I$, and $C = 0$. Then

$$\mathcal{H} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & -1 & -\lambda \end{pmatrix}.$$

In that case ($C = 0$), we replace the condition '(A, C) is detectable' by the condition '$A - BB^* P$ is stable'.

We have $\mathcal{H} V_1 = -\lambda V_1$ with

$$V_1 = (-1, 2\lambda, 0, 4\lambda^2)^T.$$

The solution to

$$(\mathcal{H} + \lambda I) V_2 = V_1,$$

is

$$V_2 = (-1/\lambda, 1, -4\lambda^2, 0)^T.$$

We obtain the solution to the $(A.B.E.)$ (the degenerate $(A.R.E.)$)

$$P = \begin{pmatrix} 0 & -4\lambda^2 \\ 4\lambda^2 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1/\lambda \\ 2\lambda & 1 \end{pmatrix}^{-1},$$

$$P = \begin{pmatrix} 0 & -4\lambda^2 \\ 4\lambda^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1/\lambda \\ -2\lambda & -1 \end{pmatrix} = \begin{pmatrix} 8\lambda^3 & 4\lambda^2 \\ 4\lambda^2 & 4\lambda \end{pmatrix}.$$

Moreover

$$A - BB^* P = \begin{pmatrix} \lambda & 1 \\ -4\lambda^2 & -3\lambda \end{pmatrix}.$$

Another simple example – A reduced inverted pendulum

Instead of studying the equations of the inverted pendulum, we can consider the simple model

$$\theta'' - \sin\theta = u,$$

where θ is the angular displacement from the unstable vertical equilibrium, and u is taken as a control. The linearized system about 0 is

$$z' = Az + Bu, \quad z(0) = z_0,$$

where

$$z = (\theta, \zeta)^T, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \zeta = \theta'.$$

We choose $R^{-1} = I$ and $C = 0$. Then

$$\mathcal{H} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We can verify the $\lambda = 1$ and $\lambda = -1$ are the two eigenvalues of $\mathcal{H}$ of multiplicity 2.

Rather than computing the matrix Riccati equation, we can equivalently determine directly the control of minimal norm stabilizing the system.

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$. They are both of multiplicity 1. We have

$$E(\lambda_1) = \{(\theta,\zeta) \in \mathbb{R}^2 \mid \theta = -\zeta\} \quad \text{and} \quad E(\lambda_2) = \{(\theta,\zeta) \in \mathbb{R}^2 \mid \theta = \zeta\}.$$

We can rewrite the system as follows

$$\begin{aligned}
\left(\frac{\theta+\zeta}{2}\right)' &= \left(\frac{\theta+\zeta}{2}\right) + \frac{1}{2}u,\\
\left(\frac{\zeta-\theta}{2}\right)' &= -\left(\frac{\zeta-\theta}{2}\right) + \frac{1}{2}u.
\end{aligned}$$

The first equation corresponds to the projected system onto the unstable subspace. The feedback of minimal norm stabilizing the unstable system is obtained by solving the one dimensional Riccati equation

$$p > 0, \quad 2p - p^2/4 = 0.$$

Thus $p = 8$ and the feedback law is

$$u(t) = -\frac{1}{2}\,8\,\left(\frac{\theta + \zeta}{2}\right) = -2(\theta + \zeta).$$

Thus the closed loop linear system is

$$\begin{pmatrix} \theta \\ \zeta \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \theta \\ \zeta \end{pmatrix}.$$

We notice that the two eigenvalues of the generator of this system are $\lambda = -1$.