Algorithmic Applications of An Approximate Version of Carathéodory's Theorem

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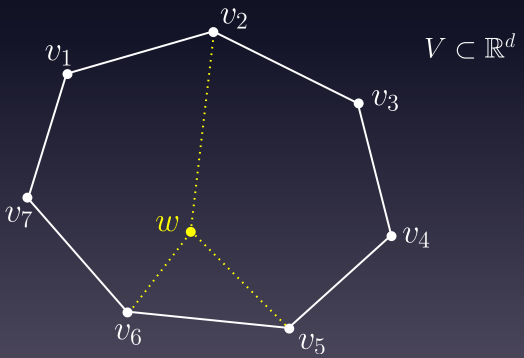
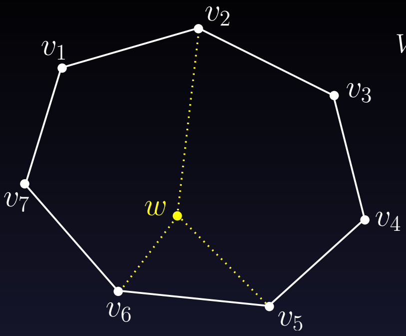
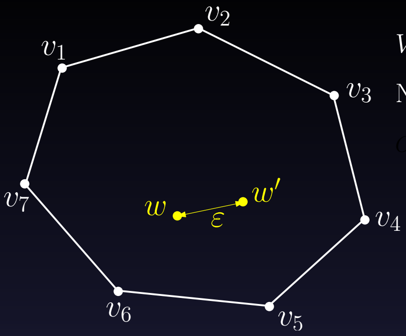
Carathéodory's Theorem

Any vector in the convex hull of a set V in $\mathbb{R}^d$ can be expressed as a convex combination of at most $d+1$ vectors of V.





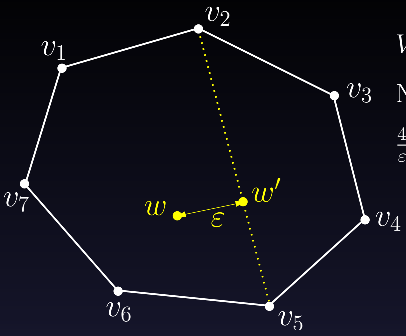
$V \subset \mathbb{R}^d$



$V \subset \mathbb{R}^d$

Norm $p \geq 2$

$O\left(\frac{p}{\varepsilon^2}\right)$ vectors



$V \subset \mathbb{R}^d$

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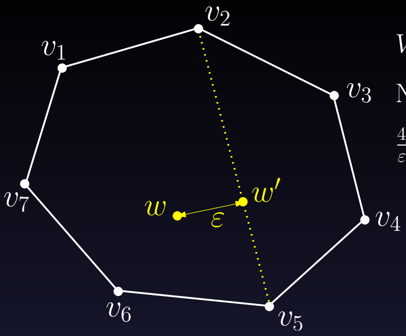

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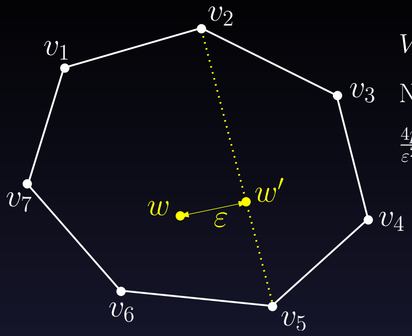
Approx. Carathéodory's Theorem

Given set V in the p-unit ball with norm $p \geq 2$, for every vector in the convex hull of V there exists an ε-close (under p-norm distance) vector that is a convex combination of at most $\frac{4p}{\varepsilon^2}$ vectors of V.



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Proof: Instantiating Maurey's Lemma.

Alternatively, via Khintchine inequality.

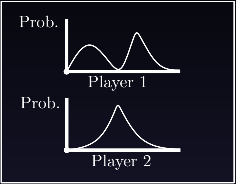
Application I: Approximating Nash Equilibria

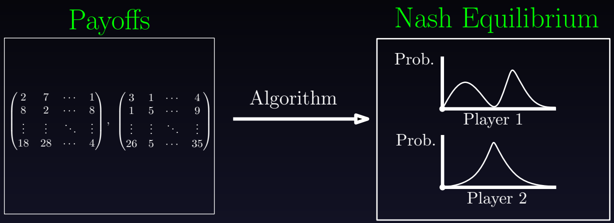
Payoffs

$$\begin{pmatrix} 2 & 7 & \cdots & 1 \\ 8 & 2 & \cdots & 8 \\ \vdots & \vdots & \ddots & \vdots \\ 18 & 28 & \cdots & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 & \cdots & 4 \\ 1 & 5 & \cdots & 9 \\ \vdots & \vdots & \ddots & \vdots \\ 26 & 5 & \cdots & 35 \end{pmatrix}$$

Algorithm $\longrightarrow$

Nash Equilibrium





Nash equilibrium in two-player games is PPAD-hard [GP06, DGP06, CD06, CDT09].

Payoffs

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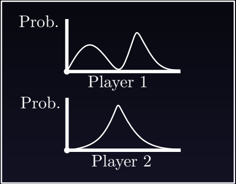
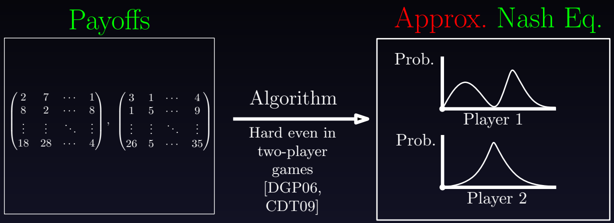
Algorithm →

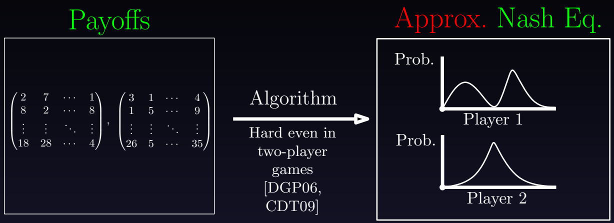
Hard even in two-player games [DGP06, CDT09]

Nash Equilibrium





Focus: Two-Player Games



Focus: Two-Player Games

Two-Player Games model settings in which two self-interested entities *simultaneously* select actions to maximize their own payoffs.

Payoff matrices A and B of size $n \times n$

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix} \end{array}$$

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Approximate Nash equilibrium (x, y): No player can benefit more than ε by unilateral deviation

$$
\begin{aligned}
e_i^T A y &\leq x^T A y + \varepsilon && \forall i \in [n] \text{ and} \\
x^T B e_j &\leq x^T B y + \varepsilon && \forall j \in [n]
\end{aligned}
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Computation of Eq. in Two-Player Games

Nash Equilibria

General Games: Exp. time
[Lemke & Howson 1964]

Zero-Sum Games: Poly. time
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General Games: $n^{O\left(\log n/\varepsilon^2\right)}$
[Lipton et al. 2003]

Low-Rank Games: $(1/\varepsilon)^{\mathrm{rank}}$
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This Talk: Sparsity

Definition (Sparsity of a Game)

The sparsity of a game (A, B) is defined to be the maximum number of non-zero entries in any column of $A + B$.

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- Sparsity $= 0$ in zero-sum games
- In general, sparsity is at most n

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Theorem

In a two-player s-sparse game an ε-Nash equilibrium can be computed in time $n^{O\left(\log s/\varepsilon^2\right)}$.

Payoff matrices normalized $A, B \in [-1, 1]^{n\times n}$.

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Implications:

- When s is a fixed constant we get a polynomial-time algorithm
- For general games $(s \leq n)$ the running time matches the best-known upper bound: $n^{O(\log n/\varepsilon^2)}$ [LMM'03].

Nash eq: $e_i^T Ay \leq x^T Ay \quad \forall i$ and

$$x^T Be_j \leq x^T By \quad \forall j$$

Bilinear Program for Nash Eq. [MS'64]

$$\begin{aligned} \text{maximize} \quad & x^T(A+B)y - \pi_1 - \pi_2 \\ \text{subject to} \quad & x^T B \leq \pi_2 \quad \text{and} \quad Ay \leq \pi_1 \\ & x, y \in \Delta^n \quad \text{and} \quad \pi_1, \pi_2 \in [-1, 1] \end{aligned}$$

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Say (x^*, y^*) is a Nash eq. Given $u^* = C y^*$ we get an LP.

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A vector *close* to Cy^* is sufficient to find an approx. Nash eq.

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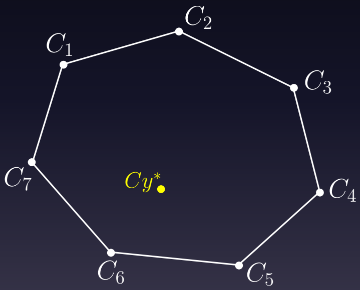

Matrix C

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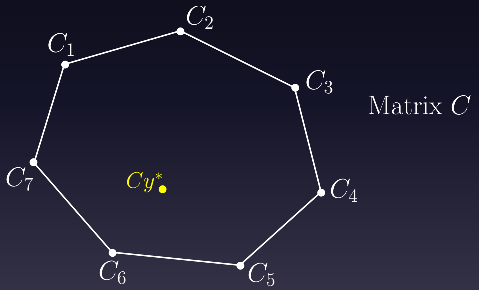

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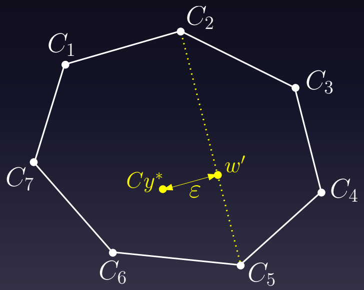

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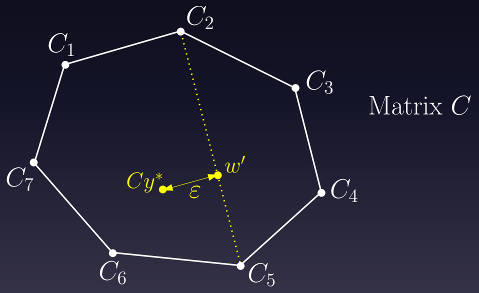

Idea: Exhaustively search for w'

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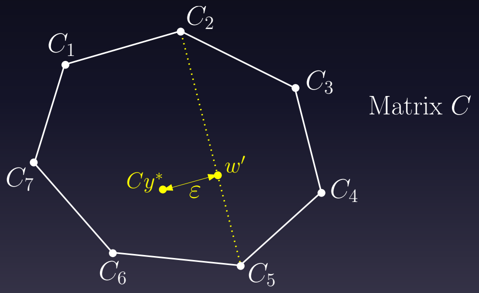

Idea: Exhaustively search for w',
by enumerating subsets of columns of C.

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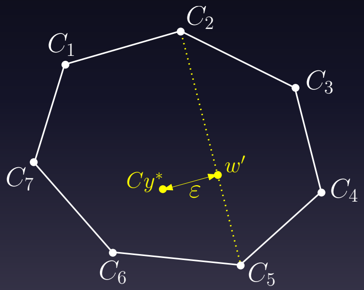

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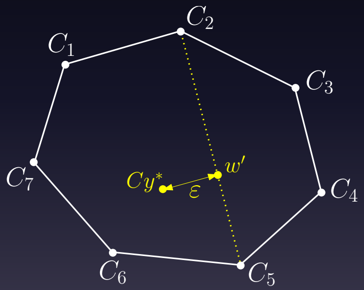


Matrix C

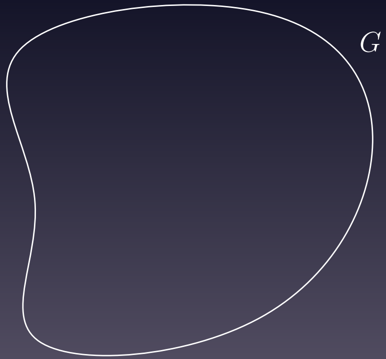
General Result

We can efficiently approximate any sparse bilinear or quadratic form over the simplex.

Application II: Approximation Algorithm for Densest Subgraph

Normalized Densest Subgraph Problem

Given: Graph G and size parameter k



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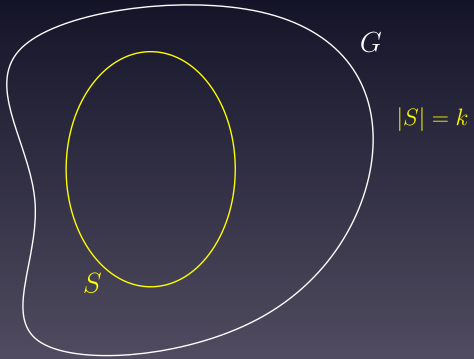
Objective: Find vertex subset S of size k such that density(S) is maximized.



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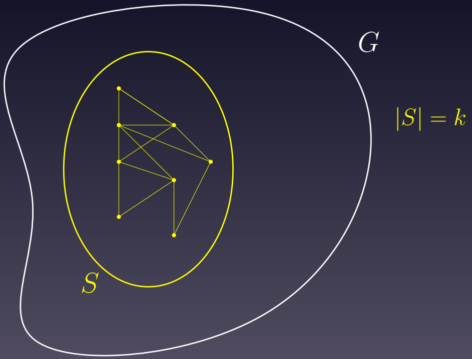
$$\text{density}(S) := \frac{\#\text{ edges in } S}{k^2}$$



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Theorem

In a degree d graph, an ε additive approximation for the densest bipartite subgraph problem can be computed in time

$$n^{O\left(\varepsilon^{-2}\log(d/k)\right)}.$$

✓ Application I: Approximating Nash Equilibria

✓ Application II: Approximating Dense Subgraphs

General Result

We can efficiently approximate any sparse bilinear or quadratic form over the simplex.

Extensions

- Convex hull of matrices with entrywise norm and Schatten p-norm
- Shapley-Folkman Lemma
- Colorful Carathéodory Theorem

Thank You!

Khintchine Inequality

Let $r_1, r_2, \ldots, r_m$ be a sequence of i.i.d. random variables with $\Pr(r_i = \pm 1) = \frac{1}{2}$
In addition, let $u_1, u_2, \ldots, u_m \in \mathbb{R}^d$ be a deterministic sequence of vectors. Then, for $2 \leq p < \infty$

$$\mathbb{E} \left\| \sum_{i=1}^{m} r_i u_i \right\|_p \leq \sqrt{p} \, \left(\sum_{i=1}^{m} \|u_i\|_p^2 \right)^{\frac{1}{2}}$$