Proving super-polynomial lower bounds for syntactic multilinear branching programs
Approaches and Challenges

Ramya C
CMI, Chennai, INDIA.
(Joint work with B.V.Raghavendra Rao, IIT Madras)

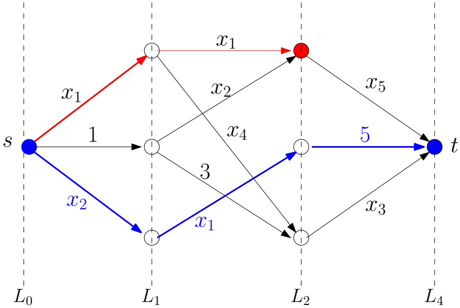
March 21, 2019

Syntactic Multilinear ABP (smABP)

- Multilinear polynomial - every variable has degree at most one in any monomial. E.g.: $\det_n$ is multilinear.

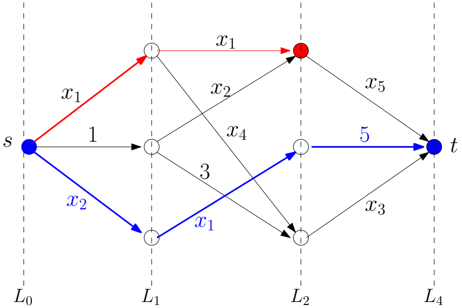
Syntactic Multilinear ABP (smABP)

- Multilinear polynomial - every variable has degree at most one in any monomial. E.g.: $\det_n$ is multilinear.
- Syntactic Multilinear ABP (smABP): along any path a variable as an edge label appears atmost once.



Syntactic Multilinear ABP (smABP)

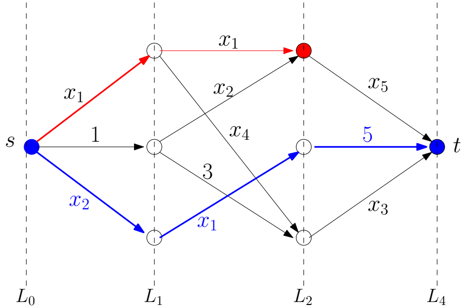
- Multilinear polynomial - every variable has degree at most one in any monomial. E.g.: $\det_n$ is multilinear.
- Syntactic Multilinear ABP (smABP): along any path a variable as an edge label appears atmost once.



- Alon et al. (2017) showed $\Omega\left(\frac{n^2}{\log^2 n}\right)$ size lower bound for smABPs computing an explicit n-variate multilinear polynomial.

Syntactic Multilinear ABP (smABP)

- Multilinear polynomial - every variable has degree at most one in any monomial. E.g.: $\det_n$ is multilinear.
- Syntactic Multilinear ABP (smABP): along any path a variable as an edge label appears atmost once.



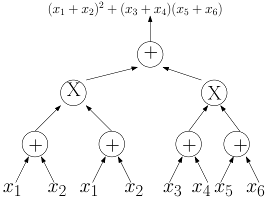
- Alon et al. (2017) showed $\Omega\left(\frac{n^2}{\log^2 n}\right)$ size lower bound for smABPs computing an explicit n-variate multilinear polynomial.
- Goal: Prove super-polynomial lower bounds for smABPs.

Multilinear Formulas

- Arithmetic Formulas - circuits whose underlying graph is tree.

Multilinear Formulas

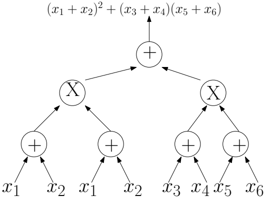
- Arithmetic Formulas - circuits whose underlying graph is tree.



Formula F

Multilinear Formulas

- Arithmetic Formulas - circuits whose underlying graph is tree.

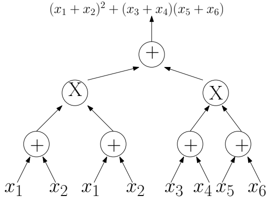


Formula F

- **Multilinear formula** : every gate computes a multilinear polynomial.

Multilinear Formulas

- Arithmetic Formulas - circuits whose underlying graph is tree.

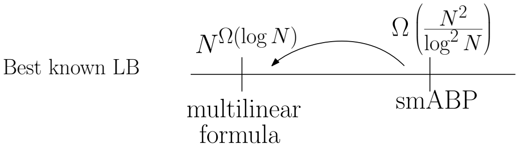


Formula F

- **Multilinear formula** : every gate computes a multilinear polynomial.
- Raz (2009) showed that multilinear formulas computing det_n or perm_n must have size $n^{\Omega(\log n)}$.

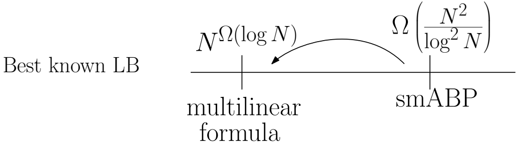
Possible Approaches to prove smABP Lower Bounds

Goal: Prove super-polynomial lower bounds for smABPs.



Possible Approaches to prove smABP Lower Bounds

Goal: Prove super-polynomial lower bounds for smABPs.

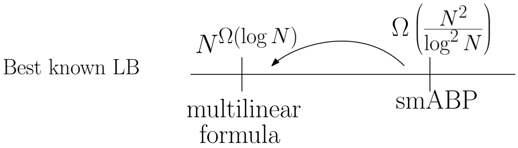


smABPs to multilinear formula

An smABP of size $N^{O(1)}$ computing an N-variate polynomial f can be converted into multilinear formula of size $N^{O(\log N)}$ computing f.

Possible Approaches to prove smABP Lower Bounds

Goal: Prove super-polynomial lower bounds for smABPs.



smABPs to multilinear formula

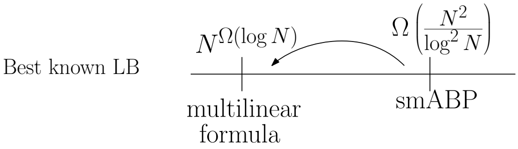
An smABP of size $N^{O(1)}$ computing an N-variate polynomial f can be converted into multilinear formula of size $N^{O(\log N)}$ computing f.

Possible Approaches to prove smABP Lower Bounds

1. Convert an smABP into multilinear formula of size $N^{o(\log N)}$ - Raz's $N^{\Omega(\log N)}$ lower bound is sufficient.

Possible Approaches to prove smABP Lower Bounds

Goal: Prove super-polynomial lower bounds for smABPs.



smABPs to multilinear formula

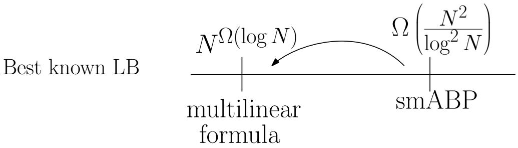
An smABP of size $N^{O(1)}$ computing an N-variate polynomial f can be converted into multilinear formula of size $N^{O(\log N)}$ computing f.

Possible Approaches to prove smABP Lower Bounds

1. Convert an smABP into multilinear formula of size $N^{o(\log N)}$ - Raz's $N^{\Omega(\log N)}$ lower bound is sufficient.
2. Prove $N^{\omega(\log N)}$ lower bound for multilinear formulas - The above conversion is sufficient.

Possible Approaches to prove smABP Lower Bounds

<u>Goal:</u> Prove super-polynomial lower bounds for smABPs.



smABPs to multilinear formula

An smABP of size $N^{O(1)}$ computing an N-variate polynomial f can be converted into multilinear formula of size $N^{O(\log N)}$ computing f.

Possible Approaches to prove smABP Lower Bounds

1. Convert an smABP into multilinear formula of size $N^{o(\log N)}$ - Raz's $N^{\Omega(\log N)}$ lower bound is sufficient.
2. Prove $N^{\omega(\log N)}$ lower bound for multilinear formulas - The above conversion is sufficient.

Both these approaches seem difficult with the current techniques!

Preliminaries - Partial Derivative Matrix

Let $f \in \mathbb{F}[y_1, \ldots, y_m, z_1, \ldots, z_m]$ be a multilinear polynomial.
$M_f[p, q] = A$ iff A is the coefficient of monomial pq in f.

- $\text{rank}(M_{f+g}) \leq \text{rank}(M_f) + \text{rank}(M_g)$; $\text{rank}(M_{fg}) \leq \text{rank}(M_f) \cdot \text{rank}(M_g)$.

Preliminaries - Partial Derivative Matrix

Let $f \in \mathbb{F}[y_1, \ldots, y_m, z_1, \ldots, z_m]$ be a multilinear polynomial.
$M_f[p, q] = A$ iff A is the coefficient of monomial pq in f.

- $\text{rank}(M_{f+g}) \leq \text{rank}(M_f) + \text{rank}(M_g)$; $\text{rank}(M_{fg}) \leq \text{rank}(M_f) \cdot \text{rank}(M_g)$.
- E.g. : $f = (x_1 + x_2)(x_3 + x_4) \cdots (x_{N-1} + x_N)$;
  $X = \{x_1, \ldots, x_N\}$; $Y = \{y_1, \ldots, y_{N/2}\}$; $Z = \{z_1, \ldots, z_{N/2}\}$.
  Consider $\varphi, \varphi' : X \to Y \cup Z$.
  $f^{\varphi} = (y_1 + z_1)(y_2 + z_2) \cdots (y_{N/2} + z_{N/2})$, $rank(M_{f^{\varphi}}) = 2^{N/2}$
  $f^{\varphi'} = (y_1 + y_2)(z_1 + z_2) \cdots (y_{N/2} + z_{N/2})$, $rank(M_{f^{\varphi'}}) = 1$

Preliminaries - Partial Derivative Matrix

Let $f \in \mathbb{F}[y_1, \ldots, y_m, z_1, \ldots, z_m]$ be a multilinear polynomial.
$M_f[p, q] = A$ iff A is the coefficient of monomial pq in f.

- $\text{rank}(M_{f+g}) \leq \text{rank}(M_f) + \text{rank}(M_g)$; $\text{rank}(M_{fg}) \leq \text{rank}(M_f) \cdot \text{rank}(M_g)$.
- E.g. : $f = (x_1 + x_2)(x_3 + x_4) \cdots (x_{N-1} + x_N)$;
  $X = \{x_1, \ldots, x_N\}$; $Y = \{y_1, \ldots, y_{N/2}\}$; $Z = \{z_1, \ldots, z_{N/2}\}$.
  Consider $\varphi, \varphi' : X \to Y \cup Z$.
  $f^{\varphi} = (y_1 + z_1)(y_2 + z_2) \cdots (y_{N/2} + z_{N/2})$, $rank(M_{f^{\varphi}}) = 2^{N/2}$
  $f^{\varphi'} = (y_1 + y_2)(z_1 + z_2) \cdots (y_{N/2} + z_{N/2})$, $rank(M_{f^{\varphi'}}) = 1$
- When φ has $|Y| = |Z| = |X|/2$, $rank(M_{f^{\varphi}}) \leq 2^{N/2}$.

Formulas obtained from smABPs

- The multilinear formula obtained from an smABP has more structure than an arbitrary multilinear formula of super-polynomial size.
- This talk: A finer analysis of techniques developed by Raz(2009) for these multilinear formulas.

Formulas obtained from smABPs

- The multilinear formula obtained from an smABP has more structure than an arbitrary multilinear formula of super-polynomial size.
- <u>This talk:</u> A finer analysis of techniques developed by Raz(2009) for these multilinear formulas.
- Upper bounding rank of the partial derivative matrix of a polynomial computed by an smABP seems to be difficult.

Formulas obtained from smABPs

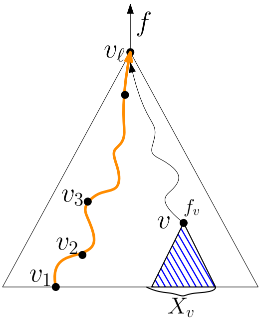
- The multilinear formula obtained from an smABP has more structure than an arbitrary multilinear formula of super-polynomial size.
- This talk: A finer analysis of techniques developed by Raz(2009) for these multilinear formulas.
- Upper bounding rank of the partial derivative matrix of a polynomial computed by an smABP seems to be difficult.
- Identify limitations of the formula from smABPs to prove upper bound on the rank of the partial derivative matrix of f under a random partition of the variables.

Formulas obtained from smABPs

- The multilinear formula obtained from an smABP has more structure than an arbitrary multilinear formula of super-polynomial size.
- This talk: A finer analysis of techniques developed by Raz(2009) for these multilinear formulas.
- Upper bounding rank of the partial derivative matrix of a polynomial computed by an smABP seems to be difficult.
- Identify limitations of the formula from smABPs to prove upper bound on the rank of the partial derivative matrix of f under a random partition of the variables.
- We know hard polynomials that have full rank under any partition of the variables.

Central Signatures in a multilinear formula Φ

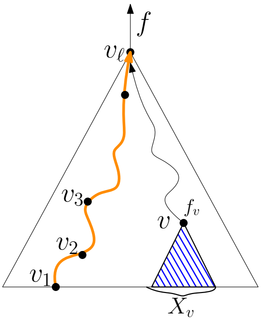
- f_v:polynomial computed at v, X_v:variables in formula rooted at v. Here, $X_{v_1} \subseteq X_{v_2} \subseteq \ldots \subseteq X_{v_\ell}$.



- For any path $\rho = (v_1, \ldots, v_\ell)$, signature$(\rho) = (X_{v_1}, \ldots, X_{v_\ell})$.

Central Signatures in a multilinear formula Φ

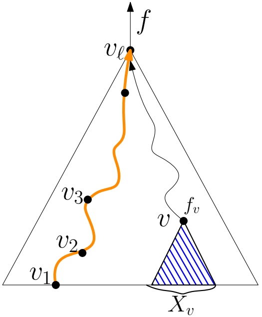
- f_v:polynomial computed at v, X_v:variables in formula rooted at v. Here, $X_{v_1} \subseteq X_{v_2} \subseteq \ldots \subseteq X_{v_\ell}$.



- For any path $\rho = (v_1, \ldots, v_\ell)$, signature$(\rho) = (X_{v_1}, \ldots, X_{v_\ell})$.
- A signature is *central* if $|X_{v_{i+1}}| \leq 2|X_{v_i}|$ for all $i \in [\ell - 1]$.

Central Signatures in a multilinear formula Φ

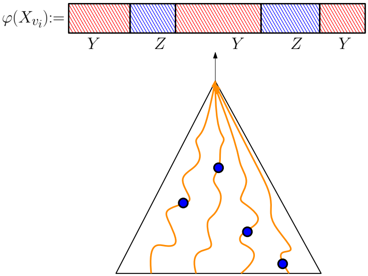
- f_v:polynomial computed at v, X_v:variables in formula rooted at v. Here, $X_{v_1} \subseteq X_{v_2} \subseteq \ldots \subseteq X_{v_\ell}$.



- For any path $\rho = (v_1, \ldots, v_\ell)$, signature$(\rho) = (X_{v_1}, \ldots, X_{v_\ell})$.
- A signature is *central* if $|X_{v_{i+1}}| \leq 2|X_{v_i}|$ for all $i \in [\ell - 1]$.
- There is at least one central signature in Φ.

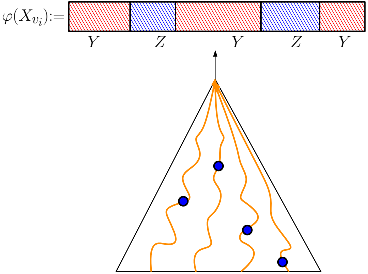
High Imbalance implies Low Rank

- For any $\varphi : X \to Y \cup Z$, $\varphi(X_{v_i}) \in Y \cup Z$.
- A central signature is *k-unbalanced* w.r.t φ if for some $i \in [\ell]$, $||\varphi(X_{v_i}) \cap Y| - |\varphi(X_{v_i}) \cap Z|| > k$.



High Imbalance implies Low Rank

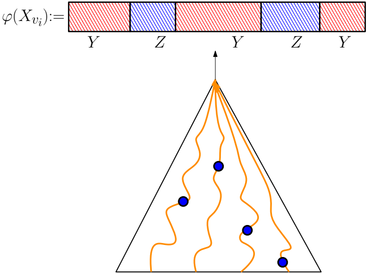
- For any $\varphi : X \to Y \cup Z$, $\varphi(X_{v_i}) \in Y \cup Z$.
- A central signature is *k-unbalanced* w.r.t φ if for some $i \in [\ell]$, $||\varphi(X_{v_i}) \cap Y| - |\varphi(X_{v_i}) \cap Z|| > k$.



- Let $f = (x_1 + x_2)(x_3 + x_4)$ and φ, φ' be partitions.
  $f^{\varphi} = (y_1 + y_2)(y_3 + y_4)$, imbalance $= 4$, rank$(M_{f^{\varphi}}) = 1$.
  $f^{\varphi} = (y_1 + z_1)(y_2 + z_2)$, imbalance $= 0$, rank$(M_{f^{\varphi}}) = 4$.

High Imbalance implies Low Rank

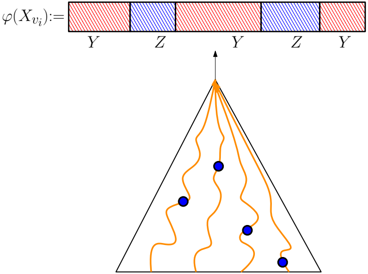
- For any $\varphi : X \to Y \cup Z$, $\varphi(X_{v_i}) \in Y \cup Z$.
- A central signature is *k-unbalanced* w.r.t φ if for some $i \in [\ell]$, $||\varphi(X_{v_i}) \cap Y| - |\varphi(X_{v_i}) \cap Z|| > k$.



- Let $f = (x_1 + x_2)(x_3 + x_4)$ and φ, φ' be partitions.
  $f^{\varphi} = (y_1 + y_2)(y_3 + y_4)$, imbalance $= 4$, $\text{rank}(M_{f^{\varphi}}) = 1$.
  $f^{\varphi} = (y_1 + z_1)(y_2 + z_2)$, imbalance $= 0$, $\text{rank}(M_{f^{\varphi}}) = 4$.
- Φ is k-**weak** w.r.t. φ if every central signature in Φ is k-unbalanced w.r.t. φ.

High Imbalance implies Low Rank

- For any $\varphi : X \to Y \cup Z$, $\varphi(X_{v_i}) \in Y \cup Z$.
- A central signature is *k-unbalanced* w.r.t φ if for some $i \in [\ell]$, $||\varphi(X_{v_i}) \cap Y| - |\varphi(X_{v_i}) \cap Z|| > k$.



- Let $f = (x_1 + x_2)(x_3 + x_4)$ and φ, φ' be partitions.
  $f^{\varphi} = (y_1 + y_2)(y_3 + y_4)$, imbalance $= 4$, $\text{rank}(M_{f^{\varphi}}) = 1$.
  $f^{\varphi} = (y_1 + z_1)(y_2 + z_2)$, imbalance $= 0$, $\text{rank}(M_{f^{\varphi}}) = 4$.
- Φ is k-**weak** w.r.t. φ if every central signature in Φ is k-unbalanced w.r.t. φ.
- If Φ is k-**weak** w.r.t. φ, then $\text{rank}(M_{f^{\varphi}})$ is low.

Theorem

If Φ is k-weak w.r.t φ, then $\text{rank}(M_{f^{\varphi}}) \leq |\Phi| \cdot 2^{N/2 - k/2}$.

Super-polynomial Lower bounds for smABPs
Approach 1 : Via Central Signatures

Step 1 Convert size $\mathcal{S}$ smABP P into $\mathcal{S}^{O(\log N)}$ size multilinear formula Φ.

Super-polynomial Lower bounds for smABPs
Approach 1 : Via Central Signatures

Step 1 Convert size $\mathcal{S}$ smABP P into $\mathcal{S}^{O(\log N)}$ size multilinear formula Φ.

Step 2 $\mathcal{D}$ be distribution of partitions with $|Y| = |Z| = |X|/2$ and $\varphi \sim \mathcal{D}$.

Super-polynomial Lower bounds for smABPs
Approach 1 : Via Central Signatures

Step 1 Convert size $\mathcal{S}$ smABP P into $\mathcal{S}^{O(\log N)}$ size multilinear formula Φ.

Step 2 $\mathcal{D}$ be distribution of partitions with $|Y| = |Z| = |X|/2$ and $\varphi \sim \mathcal{D}$.

$$\Phi \text{ is } k\text{-weak w.r.t } \varphi \implies \text{rank}(M_{f^\varphi}) \leq |\Phi| \cdot 2^{N/2-k/2}.$$

Super-polynomial Lower bounds for smABPs
Approach 1 : Via Central Signatures

Step 1 Convert size $\mathcal{S}$ smABP P into $\mathcal{S}^{O(\log N)}$ size multilinear formula Φ.

Step 2 $\mathcal{D}$ be distribution of partitions with $|Y| = |Z| = |X|/2$ and $\varphi \sim \mathcal{D}$.

$$\Phi \text{ is } k\text{-weak w.r.t } \varphi \implies \text{rank}(M_{f^\varphi}) \leq |\Phi| \cdot 2^{N/2 - k/2}.$$

Step 3

$$\Pr_{\varphi \sim D}[\Phi \text{ is not } k\text{-weak}] \leq \Pr[\exists \text{ central signature not } k\text{-unbalanced}]$$

$$\leq (\sharp \text{ of central signatures}) \cdot \Pr[\text{central sign. not } k\text{-unbalanced}]$$

$\sharp$ of central signatures: $N^{o(\log N)}$ when $S = N^{o(1)}$

Pr[central sign. not k-unbalanced]: $N^{-\Omega(\log N)}$ for $k = N^{1/5}$ (via. hypergeometric distribution)

< 1

Super-polynomial Lower bounds for smABPs
Approach 1 : Via Central Signatures

Step 1 Convert size $\mathcal{S}$ smABP P into $\mathcal{S}^{O(\log N)}$ size multilinear formula Φ.

Step 2 $\mathcal{D}$ be distribution of partitions with $|Y| = |Z| = |X|/2$ and $\varphi \sim \mathcal{D}$.

$$\Phi \text{ is } k\text{-weak w.r.t } \varphi \implies \text{rank}(M_{f^\varphi}) \leq |\Phi| \cdot 2^{N/2-k/2}.$$

Step 3

$$\Pr_{\varphi\sim D}[\Phi \text{ is not } k\text{-weak}] \leq \Pr[\exists \text{ central signature not } k\text{-unbalanced}]$$

$$\leq \underbrace{(\sharp \text{ of central signatures})}_{N^{o(\log N)} \text{ when } S = N^{o(1)}} \cdot \underbrace{\Pr[\text{central sign. not } k\text{-unbalanced}]}_{N^{-\Omega(\log N)} \text{ for } k = N^{1/5} \text{ (via. hypergeometric distribution)}}$$

$$< 1$$

Step 4 With prob. $1 - o(1)$, Φ is k-weak w.r.t. φ.

With prob. $1 - o(1)$, $\text{rank}(M_{f^\varphi}) < 2^{N/2}$.

($\Rightarrow\Leftarrow$ as P computed a full rank polynomial) $\therefore S = N^{\Omega(1)}$.

Need: "small" formulas have "small" number of central signatures.

Super-polynomial Lower bounds for smABPs
Approach 1 : Via Central Signatures

Step 1 Convert size $\mathcal{S}$ smABP P into $\mathcal{S}^{O(\log N)}$ size multilinear formula Φ.

Step 2 $\mathcal{D}$ be distribution of partitions with $|Y| = |Z| = |X|/2$ and $\varphi \sim \mathcal{D}$.

$$\Phi \text{ is } k\text{-weak w.r.t } \varphi \implies \text{rank}(M_{f^\varphi}) \le |\Phi| \cdot 2^{N/2-k/2}.$$

Step 3

$$\Pr_{\varphi \sim D}[\Phi \text{ is not } k\text{-weak}] \le \Pr[\exists \text{ central signature not } k\text{-unbalanced}]$$

$$\le \underbrace{(\sharp \text{ of central signatures})}_{N^{o(\log N)} \text{ when } S = N^{o(1)}} \cdot \underbrace{\Pr[\text{central sign. not } k\text{-unbalanced}]}_{N^{-\Omega(\log N)} \text{ for } k = N^{1/5}}$$

(via. hypergeometric distribution)

< 1

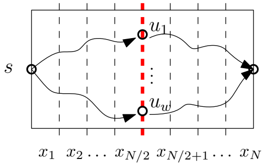
Step 4 With prob. $1 - o(1)$, Φ is k-weak w.r.t. φ.

With prob. $1 - o(1)$, $\text{rank}(M_{f^\varphi}) < 2^{N/2}$.

($\Rightarrow\Leftarrow$ as P computed a full rank polynomial) $\therefore S = N^{\Omega(1)}$.

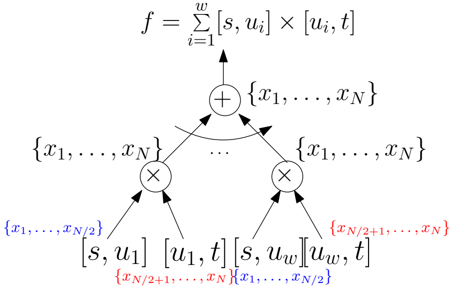
<u>Need:</u> "small" formulas have "small" number of central signatures.
<u>Roadblock:</u> For ROABPs, *#central signatures* could be $N^{\Omega(\log N)}$.

The Case of ROABPs



ROABP size $\mathcal{S}$

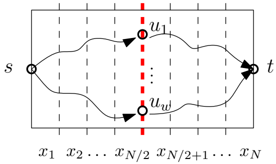
$[u,v]$= polynomial computed by subprogram with source u and sink v



Formula Φ size $\mathcal{S}^{O(\log N)}$

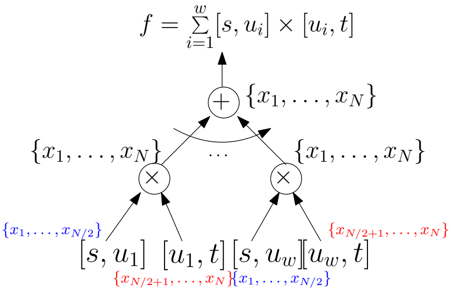
- For gate v in Φ,let S_v be interval assoc. with gate v. Note, $X_v \subseteq S_v$.

The Case of ROABPs



ROABP size $\mathcal{S}$

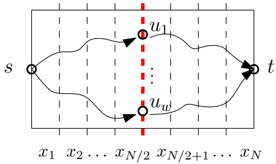
$[u, v]$= polynomial computed by subprogram with source u and sink v



Formula Φ size $\mathcal{S}^{O(\log N)}$

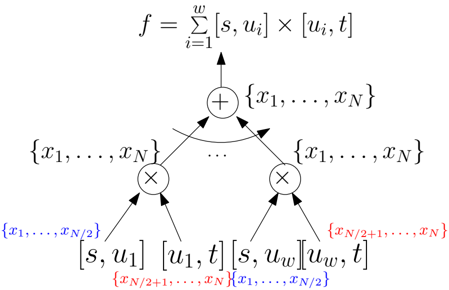
- For gate v in Φ,let S_v be interval assoc. with gate v. Note, $X_v \subseteq S_v$.
- When $v = v_1 + \cdots + v_w$, $S_v = S_{v_1} = \cdots = S_{v_w}$.
- When $v = v_1 \times v_2$, $|S_v| = 2|S_{v_1}|$ and $|S_v| = 2|S_{v_2}|$.

The Case of ROABPs



ROABP size $\mathcal{S}$

$[u, v]$= polynomial computed by subprogram with source u and sink v



Formula Φ size $\mathcal{S}^{O(\log N)}$

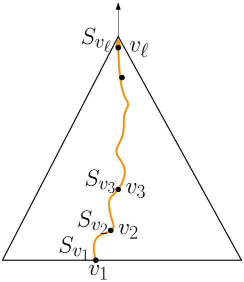
- For gate v in Φ,let S_v be interval assoc. with gate v. Note, $X_v \subseteq S_v$.
- When $v = v_1 + \cdots + v_w$, $S_v = S_{v_1} = \cdots = S_{v_w}$.
- When $v = v_1 \times v_2$, $|S_v| = 2|S_{v_1}|$ and $|S_v| = 2|S_{v_2}|$.
- For a path $\rho = (v_1, \dots, v_\ell)$, ext-sign$(\rho) = (S_{v_1}, S_{v_2}, \dots, S_{v_\ell})$.

Extended central signatures in Φ

- For any $i \in [\ell - 1]$, $|S_{v_{i+1}}| = |S_{v_i}|$ or $|S_{v_{i+1}}| = 2|S_{v_i}|$. Every ext-signature is central.

Extended central signatures in Φ

- For any $i \in [\ell - 1]$, $|S_{v_{i+1}}| = |S_{v_i}|$ or $|S_{v_{i+1}}| = 2|S_{v_i}|$. Every ext-signature is central.
- For any Φ obtained from ROABP, #ext-sign $= 2^{O(\log N)} = O(N)$.



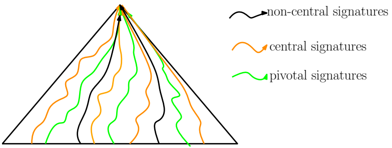
Theorem

Let $f_1, \ldots f_m$ be polynomials computed by ROABPs of size $S_1, \ldots, S_m$. There is an explicit polynomial g such that if $g = f_1 + \cdots + f_m$ then either $m = N^{\Omega(1)}$ or there is an $i \in [m]$ such that $S_i = 2^{\Omega(N^{1/100})}$.

- Exponential lower bounds for sum of ROABPs is known.

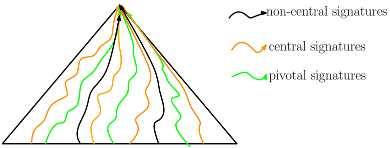
Covering central signatures

- Let P be smABP and Φ be multilinear formula obtained from P. Let $\mathcal{R}$ be the set of central signatures in Φ.



Covering central signatures

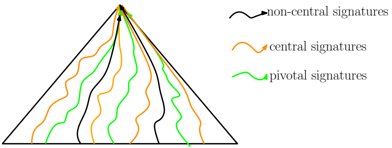
- Let P be smABP and Φ be multilinear formula obtained from P. Let $\mathcal{R}$ be the set of central signatures in Φ.



  - Find a set $\mathcal{R}' \subseteq \mathcal{R}$ s.t.
    $\forall \gamma \in \mathcal{R}'$, γ is k-unbalanced $\Rightarrow \forall \gamma' \in \mathcal{R}$, γ is $(k - \delta)-$unbalanced.

Covering central signatures

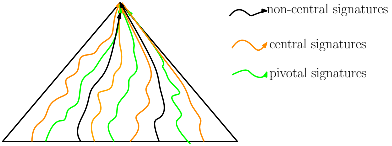
- Let P be smABP and Φ be multilinear formula obtained from P. Let $\mathcal{R}$ be the set of central signatures in Φ.



  - Find a set $\mathcal{R}' \subseteq \mathcal{R}$ s.t. $\forall \gamma \in \mathcal{R}'$, γ is k-unbalanced $\Rightarrow \forall \gamma' \in \mathcal{R}$, γ is $(k-\delta)-$unbalanced.
  - Need a "small" set of central sign. that "cover" all central signatures.

Covering central signatures

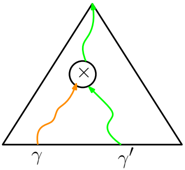
- Let P be smABP and Φ be multilinear formula obtained from P. Let $\mathcal{R}$ be the set of central signatures in Φ.



- Find a set $\mathcal{R}' \subseteq \mathcal{R}$ s.t.
  $\forall \gamma \in \mathcal{R}'$, γ is k-unbalanced $\Rightarrow \forall \gamma' \in \mathcal{R}$, γ is $(k-\delta)-$unbalanced.
- Need a "small" set of central sign. that "cover" all central signatures.

Signature Cover $\mathcal{C}$ of Φ

For every central sign. γ in Φ, either $\gamma \in \mathcal{C}$ or there is a $\gamma' \in \mathcal{C}$ such that the paths meet at $\times$ gate.



Theorem

If there exists a signature cover $\mathcal{C}$ of Φ s.t. every central sign. in $\mathcal{C}$ is k-unbalanced w.r.t. φ then $\text{rank}(M_{f^{\varphi}}) \leq |\Phi| \cdot 2^{N/2-k/2}$.

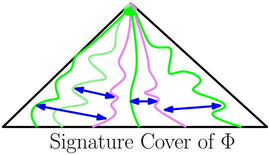
Clustering central signatures

Let $\mathcal{C}$ be a signature cover of Φ. If every central signature in $\mathcal{C}$ is k-unbalanced then rank is low.

Clustering central signatures

Let $\mathcal{C}$ be a signature cover of Φ. If every central signature in $\mathcal{C}$ is k-unbalanced then rank is low.

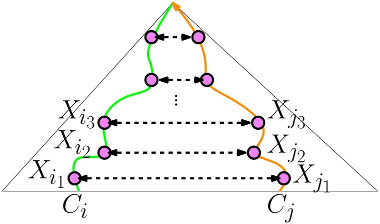
- Define a measure of "closeness" among central signatures in $\mathcal{C}$.



Clustering central signatures

Let $\mathcal{C}$ be a signature cover of Φ. If every central signature in $\mathcal{C}$ is k-unbalanced then rank is low.

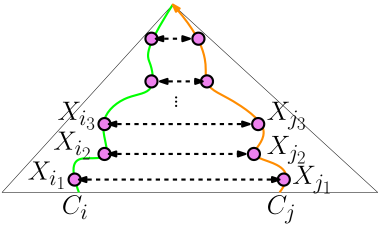
- $\Delta(C_i, C_j) = \max_{k \in [\ell]} \Delta(X_{i_k}, X_{j_k})$



Clustering central signatures

Let $\mathcal{C}$ be a signature cover of Φ. If every central signature in $\mathcal{C}$ is k-unbalanced then rank is low.

- $\Delta(C_i, C_j) = \max_{k \in [\ell]} \Delta(X_{i_k}, X_{j_k})$

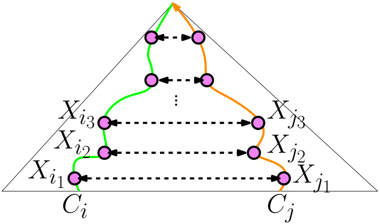


For $\delta > 0$, a δ-cluster of $\mathcal{C}$ is a set of signatures in $\mathcal{C}$ s.t. for every $C \in \mathcal{C}$, there is a $C_j \in \mathcal{C}$ with $\Delta(C, C_j) \leq \delta$.

Clustering central signatures

Let $\mathcal{C}$ be a signature cover of Φ. If every central signature in $\mathcal{C}$ is k-unbalanced then rank is low.

- $\Delta(C_i, C_j) = \max_{k \in [\ell]} \Delta(X_{i_k}, X_{j_k})$



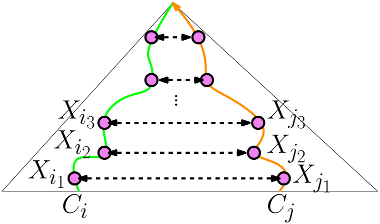
For $\delta > 0$, a δ-cluster of $\mathcal{C}$ is a set of signatures in $\mathcal{C}$ s.t. for every $C \in \mathcal{C}$, there is a $C_j \in \mathcal{C}$ with $\Delta(C, C_j) \leq \delta$.

Observation: If all central sign. in δ-cluster of $\mathcal{C}$ are k-unbalanced, then every signature in $\mathcal{C}$ is at least $k - 2\delta$-unbalanced.

Clustering central signatures

Let $\mathcal{C}$ be a signature cover of Φ. If every central signature in $\mathcal{C}$ is k-unbalanced then rank is low.

- $\Delta(C_i, C_j) = \max_{k \in [\ell]} \Delta(X_{i_k}, X_{j_k})$



For $\delta > 0$, a δ-cluster of $\mathcal{C}$ is a set of signatures in $\mathcal{C}$ s.t. for every $C \in \mathcal{C}$, there is a $C_j \in \mathcal{C}$ with $\Delta(C, C_j) \leq \delta$.

Observation: If all central sign. in δ-cluster of $\mathcal{C}$ are k-unbalanced, then every signature in $\mathcal{C}$ is at least $k - 2\delta$-unbalanced.

- Enough to find a signature cover with a "small" δ-cluster in Φ.

Lower bounds for smABPs

Approach via Central Signatures

Show that every smABP of size S can be converted to a multilinear formula of size $S^{O(\log N)}$ such that there is a δ-cluster of a signature cover with at most $S^{o(\log N)}$ signatures for any $\delta << N^{1/5}$.

Lower bounds for smABPs

Approach via Central Signatures

Show that every smABP of size S can be converted to a multilinear formula of size $S^{O(\log N)}$ such that there is a δ-cluster of a signature cover with at most $S^{o(\log N)}$ signatures for any $\delta << N^{1/5}$.

Challenge

Are there multilinear formulas of size S where any δ-cluster of a signature cover has $S^{\Omega(\log N)}$ signatures ?

Super-polynomial Lower bounds for smABPs
Approach 2 : Via Ordered Algebraic Branching Programs

- Along any path in an smABP a variable appears at most once.

Super-polynomial Lower bounds for smABPs
Approach 2 : Via Ordered Algebraic Branching Programs

- Along any path in an smABP a variable appears at most once.
- The variables on $s \rightsquigarrow t$ paths are permutations(orders) in S_n.

Super-polynomial Lower bounds for smABPs
Approach 2 : Via Ordered Algebraic Branching Programs

- Along any path in an smABP a variable appears at most once.
- The variables on $s \rightsquigarrow t$ paths are permutations(orders) in S_n.
- $\mathcal{L}$-**ordered ABPs** : Given $\mathcal{L}$ permutations in S_n, every path reads variables in one of the $\mathcal{L}$ orders.

Super-polynomial Lower bounds for smABPs
Approach 2 : Via Ordered Algebraic Branching Programs

- Along any path in an smABP a variable appears at most once.
- The variables on $s \rightsquigarrow t$ paths are permutations(orders) in S_n.
- $\mathcal{L}$-**ordered ABPs** : Given $\mathcal{L}$ permutations in S_n, every path reads variables in one of the $\mathcal{L}$ orders.
- 1-ordered ABPs $\equiv_{\mathrm{P}}$ ROABP [Jansen et. al 2008]

Super-polynomial Lower bounds for smABPs
Approach 2 : Via Ordered Algebraic Branching Programs

- Along any path in an smABP a variable appears at most once.
- The variables on $s \rightsquigarrow t$ paths are permutations(orders) in S_n.
- $\mathcal{L}$**-ordered ABPs** : Given $\mathcal{L}$ permutations in S_n, every path reads variables in one of the $\mathcal{L}$ orders.
- 1-ordered ABPs $\equiv_{\mathrm{P}}$ ROABP [Jansen et. al 2008]

Lower bound for $\mathcal{L}$-ordered ABPs [R.,Raghavendra Rao 2018]

Let $\mathcal{L} \leq 2^{n^{1/2-\epsilon}}, \epsilon > 0$ and $f_1, \ldots, f_m$ be $\mathcal{L}$-ordered ABPs of size $S_1, \ldots, S_m$. There exists an explicit polynomial g such that if $g = f_1 + \cdots + f_m$, $m = 2^{\Omega(N^{1/40})}$ or $\exists\ i \in [m]$ with $S_i = 2^{\Omega(N^{1/40})}$.

- How many orders can a poly(N) size smABP admit ?

Low-depth formulas

smABP P
size $n^{O(1)}$
compuitng f
n variables

→ [Agrawal-Vinay 2008]
[Tavenas 2013]

$\Sigma\Pi^{[O(\sqrt{n})]}\Sigma\Pi^{[O(\sqrt{n})]}$
multilinear formula
size $2^{O(\sqrt{n}\log n)}$
computing f

Low-depth formulas

smABP P
size $n^{O(1)}$
compuitng f
n variables

$\longrightarrow$ [Agrawal-Vinay 2008] [Tavenas 2013]

$\Sigma\Pi^{[O(\sqrt{n})]}\Sigma\Pi^{[O(\sqrt{n})]}$
multilinear formula
size $2^{O(\sqrt{n}\log n)}$
computing f

- In $\Sigma\Pi^{[O(\sqrt{n})]}\Sigma\Pi^{[O(\sqrt{n})]}$ the polynomials computed by bottom sum gates are of degree $O(\sqrt{n})$ in n variables.

Low-depth formulas

smABP P
size $n^{O(1)}$
compuitng f
n variables

→ [Agrawal-Vinay 2008] [Tavenas 2013] →

$\Sigma\Pi^{[O(\sqrt{n})]}\Sigma\Pi^{[O(\sqrt{n})]}$
multilinear formula
size $2^{O(\sqrt{n}\log n)}$
computing f

- In $\Sigma\Pi^{[O(\sqrt{n})]}\Sigma\Pi^{[O(\sqrt{n})]}$ the polynomials computed by bottom sum gates are of degree $O(\sqrt{n})$ in n variables.
- Can we ensure that the bottom sum gates compute polynomials in smaller number of variables say $O(\sqrt{n})$?

Super-polynomial Lower bounds for smABPs
Approach 3 : Via Depth-Reduction

Theorem [R.,Raghavendra Rao 2018]

smABP P
size $n^{O(1)}$
computing f
n-variate

$\Longrightarrow$

$\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{O(\sqrt{n})}$
multilinear formula
size $2^{O(\sqrt{n}\log^2 n)}$
computing f

$\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{O(\sqrt{n})}$ - sum of products of $O(\sqrt{n})$ polynomials each $O(\sqrt{n})$-variate

Super-polynomial Lower bounds for smABPs

Approach 3 : Via Depth-Reduction

Theorem [R.,Raghavendra Rao 2018]

smABP P size $n^{O(1)}$ computing f n-variate	$\Longrightarrow$	$\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{O(\sqrt{n})}$ multilinear formula size $2^{O(\sqrt{n}\log^2 n)}$ computing f

$\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{O(\sqrt{n})}$ - sum of products of $O(\sqrt{n})$ polynomials each $O(\sqrt{n})$-variate

Corollary

$2^{\omega(\sqrt{n}\log^2 n)}$ lower bound for $\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{O(\sqrt{n})}$ multilinear formulas	$\Longrightarrow$	$n^{\omega(1)}$ lower bound for smABPs

Super-polynomial Lower bounds for smABPs
Approach 3 : Via Depth-Reduction

Theorem [R.,Raghavendra Rao 2018]

smABP P
size $n^{O(1)}$
computing f
n-variate

⟹

$\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{O(\sqrt{n})}$
multilinear formula
size $2^{O(\sqrt{n}\log^2 n)}$
computing f

$\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{O(\sqrt{n})}$ - sum of products of $O(\sqrt{n})$ polynomials each $O(\sqrt{n})$-variate

Corollary

$2^{\omega(\sqrt{n}\log^2 n)}$ lower bound
for $\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{O(\sqrt{n})}$
multilinear formulas

⟹

$n^{\omega(1)}$ lower bound
for smABPs

- Kumar, Saraf (2015) and Kayal,Saha (2015) showed a $2^{\Omega(\sqrt{n}\log n)}$ lower bound for $\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{[O(\sqrt{n})]}$ circuits.

Super-polynomial Lower bounds for smABPs
Approach 3 : Via Depth-Reduction

Theorem [R.,Raghavendra Rao 2018]

smABP P
size $n^{O(1)}$
computing f
n-variate

$\Longrightarrow$

$\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{O(\sqrt{n})}$
multilinear formula
size $2^{O(\sqrt{n}\log^2 n)}$
computing f

$\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{O(\sqrt{n})}$ - sum of products of $O(\sqrt{n})$ polynomials each $O(\sqrt{n})$-variate

Corollary

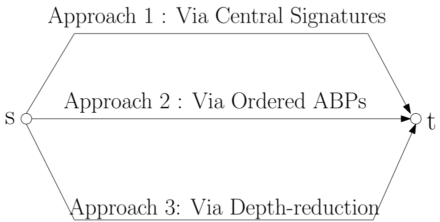
$2^{\omega(\sqrt{n}\log^2 n)}$ lower bound
for $\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{O(\sqrt{n})}$
multilinear formulas

$\Longrightarrow$

$n^{\omega(1)}$ lower bound
for smABPs

- Kumar, Saraf (2015) and Kayal,Saha (2015) showed a $2^{\Omega(\sqrt{n}\log n)}$ lower bound for $\Sigma\Pi^{[O(\sqrt{n})]}(\Sigma\Pi)^{[O(\sqrt{n})]}$ circuits.
- A similar depth reduction for multilinear circuits is known [Kumar et. al 2019]

Summary



Proving super-polynomial lower bound for smABPs

Thank You !