On geometric complexity theory: Multiplicity obstructions are stronger than occurrence obstructions

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2. Multiplicity obstructions exist
3. No occurrence obstructions exist
4. Towards studying multiplicities: Connecting orbits with their closures

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[Dörfler, I, Panova 2019] brings good news: There are very natural group varieties that

- cannot be separated with occurrence obstructions, but
- can be separated with multiplicity obstructions.

The paper shows how symmetric functions can be used to determine multiplicities.

Indeed, it proves new cases of Foulkes' conjecture on plethysm coefficients.

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Example: Every element in $\mathrm{Poly}^2 \mathbb{C}^2$ is of the form $ax^2 + bxy + cy^2$.
$\mathrm{Poly}^2 \mathrm{Poly}^2 \mathbb{C}^2$ contains the **discriminant** $b^2 - 4ac$.
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- Assume for the sake of contradiction that $X \subseteq Y$.
- Then there is a surjection $\mathbb{C}[Y]_d \twoheadrightarrow \mathbb{C}[X]_d$
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Therefore, if there exists λ such that $\mathrm{mult}_\lambda \mathbb{C}[Y]_d < \mathrm{mult}_\lambda \mathbb{C}[X]_d$, then $X \not\subseteq Y$. This is called a **multiplicity obstruction**.
If $\mathrm{mult}_\lambda \mathbb{C}[Y]_d = 0 < \mathrm{mult}_\lambda \mathbb{C}[X]_d$, then this is called an **occurrence obstruction**.

Impossibility of "Factoring power sums"

Two GL_m-varieties:

- Product of homogeneous linear forms:

$$\mathsf{Ch}_m^n := \{\ell_1 \cdots \ell_n \mid \ell_i \in \mathrm{Poly}^1\mathbb{C}^m\} \subseteq \mathrm{Poly}^n\mathbb{C}^m.$$

- Border Waring rank $\leq k$ polynomials:

$$\mathsf{Pow}_{m,k}^n := \overline{\{\ell_1^n + \cdots + \ell_k^n \mid \ell_i \in \mathrm{Poly}^1\mathbb{C}^m\}} \subseteq \mathrm{Poly}^n\mathbb{C}^m.$$

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- More generally: $\mathrm{Pow}_{m,2}^n \subseteq \mathrm{Ch}_m^n$:

$$(\ell_1 + \zeta\ell_2)(\ell_1 + \zeta^2\ell_2)\cdots(\ell_1 + \zeta^n\ell_2) = \ell_1^n + \zeta^{\frac{n(n+1)}{2}}\ell_2^n$$

  for $\zeta^n = 1$, $\zeta \neq 1$.

Separation:

- But for $n \geq 2$, $m \geq 3$, $k \geq 3$: $\boxed{\mathrm{Pow}_{m,k}^n \not\subseteq \mathrm{Ch}_m^n}$

This can be interpreted as an upper bound on the border Waring subrank.

Theorem [Dörfler, I, Panova 2019]

For any $m \geq 3$, $n \geq 2$, let $k = d = n+1$, $\lambda = (n^2-2, n, 2)$. Then

$$\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ch}_m^n]_d) < \mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Pow}_{m,k}^n]_d),$$

i.e., λ is a multiplicity obstruction that shows $\mathrm{Pow}_{m,k}^n \not\subseteq \mathrm{Ch}_m^n$.

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In two finite cases we can rule out the existence of occurrence obstructions:

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For the cases $(n, m) = (6, 3)$ and $(n, m) = (7, 4)$ we have:
For all μ: If $\text{mult}_\mu(\mathbb{C}[\text{Pow}^n_{m,k}]_d) > 0$, then $\text{mult}_\mu(\mathbb{C}[\text{Ch}^n_m]_d) > 0$.

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- Many multiplicity obstructions in low degrees d are actually occurrence obstructions!

The parameters are carefully chosen so that they are

1. large enough so that no occurrence obstructions occur, and
2. small enough so that the multiplicities can be computed.

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Proposition [Bürgisser, I, Panova 2016]

If $k \geq d$, then $\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Pow}_{m,k}^n]_d) = a_\lambda(d, n)$.

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Proof goes via multilinear algebra and inclusion/exclusion.

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The proof is similar to the proof of the classical result:

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Two-step process: $\mathscr{V}_\lambda^H = (\mathscr{V}_\lambda^{Q_n})^{\mathfrak{S}_n}$.

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1. $\dim \mathscr{V}_\lambda^{Q_n} = \#$ semistandard tableaux of shape λ in which each number $1,\ldots,n$ appears equally often.
2. $\dim(\mathscr{V}_\lambda^{Q_n})^{\mathfrak{S}_n} = a_\lambda(n,d)$ for $|\lambda| = nd$

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This finishes the proof:

$$\begin{aligned} \text{mult}_\lambda(\mathbb{C}[\text{Ch}^n_m]_d) &= \text{mult}_\lambda(\mathbb{C}[\overline{\text{GL}_n(x_1 \cdots x_n)}]_d) \le \text{mult}_\lambda(\mathbb{C}[\text{GL}_n(x_1 \cdots x_n)]_d) \\ &= a_\lambda(n,d) < a_\lambda(d,n) = \text{mult}_\lambda(\mathbb{C}[\text{Pow}^n_{m,k}]_d) \end{aligned}$$

Therefore $\text{Pow}^n_{m,k} \not\subseteq \text{Ch}^n_m$.

1. Introduction
2. Multiplicity obstructions exist
3. No occurrence obstructions exist
4. Towards studying multiplicities: Connecting orbits with their closures

No occurrence obstructions for $n = 6$, $m = 3$

- Goal: If $\text{mult}_\mu(\mathbb{C}[\mathrm{Poly}^6\mathbb{C}^3]) > 0$, then $\text{mult}_\mu(\mathbb{C}[\mathsf{Ch}_3^6]) > 0$.

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- Partitions: $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{N}^3, \quad \mu_1 \geq \mu_2 \geq \mu_3$

Proposition (Semigroup properties)

Let μ and ν be partitions with $\text{mult}_\mu(\mathbb{C}[\text{Poly}^6\mathbb{C}^3]) > 0$ and $\text{mult}_\nu(\mathbb{C}[\text{Poly}^6\mathbb{C}^3]) > 0$. Then $\text{mult}_{\mu+\nu}(\mathbb{C}[\text{Poly}^6\mathbb{C}^3]) > 0$.

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Conclusion: $\{\mu \mid \text{mult}_\mu(\mathbb{C}[\text{Poly}^6\mathbb{C}^3]) > 0\}$ and $\{\mu \mid \text{mult}_\mu(\mathbb{C}[\text{Ch}_3^6]) > 0\}$ are semigroups.

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$\{\mu \mid \text{mult}_\mu(\mathbb{C}[\text{Poly}^6\mathbb{C}^3]) > 0\}$ has 89 generators:

(6), (6, 6), (8, 4), (10, 2), (6, 6, 6), (8, 6, 4), (10, 4, 4), (9, 6, 3), (8, 8, 2), (10, 6, 2), (11, 5, 2), (10, 7, 1), (12, 4, 2), (11, 6, 1), (10, 8), (14, 2, 2), (13, 4, 1), (13, 5), (15, 3), (8, 8, 8), (10, 8, 6), (11, 7, 6), (10, 9, 5), (11, 8, 5), (10, 10, 4), (12, 7, 5), (11, 9, 4), (13, 6, 5), (12, 8, 4), (11, 10, 3), (13, 7, 4), (12, 9, 3), (13, 8, 3), (12, 10, 2), (15, 5, 4), (14, 7, 3), (13, 9, 2), (13, 10, 1), (16, 5, 3), (15, 7, 2), (14, 9, 1), (17, 4, 3), (15, 8, 1), (15, 9), (19, 3, 2), (18, 5, 1), (17, 7), (10, 10, 10), (11, 10, 9), (12, 10, 8), (13, 9, 8), (12, 11, 7), (13, 10, 7), (14, 9, 7), (13, 11, 6), (15, 8, 7), (13, 12, 5), (16, 7, 7), (15, 9, 6), (14, 11, 5), (13, 13, 4), (15, 10, 5), (15, 11, 4), (14, 13, 3), (16, 11, 3), (15, 13, 2), (15, 14, 1), (17, 13), (13, 12, 11), (14, 11, 11), (13, 13, 10), (15, 11, 10), (14, 13, 9), (16, 11, 9), (15, 13, 8), (15, 14, 7), (18, 9, 9), (15, 15, 6), (17, 17, 2), (18, 17, 1), (26, 5, 5), (15, 14, 13), (16, 13, 13), (15, 15, 12), (17, 17, 8), (18, 15, 15), (17, 17, 14), (25, 23), (45, 45).

For each generator μ we construct an occurrence of $\mathscr{V}_\mu$ in $\mathbb{C}[\text{Ch}_3^6]$ by computer.

Summary

- $\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ch}_m^n]_d) < \mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Pow}_{m,k}^n]_d)$, therefore $\mathrm{Pow}_{m,k}^n \not\subseteq \mathrm{Ch}_m^n$.
- Proof based on relationship "orbit vs orbit closure":
  $\mathrm{mult}_\lambda(\mathbb{C}[\overline{\mathrm{GL}_n(x_1 \cdots x_n)}]) \leq \mathrm{mult}_\lambda(\mathbb{C}[\mathrm{GL}_n(x_1 \cdots x_n)])$.
- In finite cases we verified by computer:
  there are no occurrence obstructions showing $\mathrm{Pow}_{m,k}^n \not\subseteq \mathrm{Ch}_m^n$, but multiplicity obstructions work

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4 Towards studying multiplicities: Connecting orbits with their closures

[Bürgisser, I 2017] connects orbit and orbit closure more closely:

- Let $0 \neq \Phi \in \mathbb{C}[\overline{\mathsf{GL}_m(x_1^n + \cdots + x_m^n)}]$ be invariant under SL_m

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Theorem [Bürgisser, I 2017]

For all d there is e:

$$\mathbb{C}[\mathrm{GL}_m(x_1^n+\cdots+x_m^n)]_d \overset{\gamma}{\hookrightarrow} \mathbb{C}[\overline{\mathrm{GL}_m(x_1^n + \cdots + x_m^n)}]_{d+em} \subseteq \mathbb{C}[\mathrm{GL}_m(x_1^n+\cdots+x_m^n)]_{d+em},$$

where $\gamma(f) := \Phi^e f$.

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- Proof idea of I-Kandasamy: Given a tableau S, construct a slightly larger tableau T such that f_T and f_S coincide on $\mathrm{SL}_m(x_1^n + \cdots + x_m^n)$.

Summary

- The representation theory of $\mathbb{C}[\mathrm{GL}_m p]$ can usually be much better understood than the representation theory of $\mathbb{C}[\overline{\mathrm{GL}_m p}]$
- In many cases of interest: the representation theory of $\mathbb{C}[\mathrm{GL}_m p]$ and $\mathbb{C}[\overline{\mathrm{GL}_m p}]$ is connected by a fundamental invariant Φ
- In the case of power sums, this connection is very close
- The hope is that $\mathbb{C}[\mathrm{GL}_m p]$ and $\mathbb{C}[\overline{\mathrm{GL}_m p}]$ are closely related in more involved cases

Thank you for your attention!