On Parameterized Lower Bounds for Multilinear Algebraic Models

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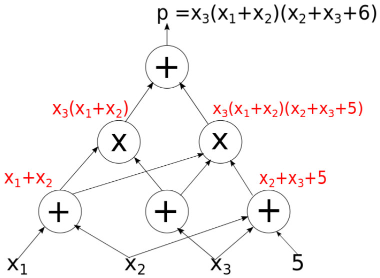
Arithmetic Circuits



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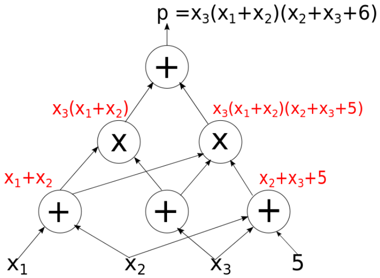

- Model of computation for polynomials.
- *Lower Bound Problem:* Minimum size of circuit $C \in \mathcal{C}$ required to compute hard polynomial p.

Known Lower Bounds

- Superlinear ($\Omega(n \log d)$) lower bound for general arithmetic circuits computing $\sum_{i=1}^{n} x_i^{d-1}$ [Baur and Strassen, 1983].

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- $\Sigma\Pi^{O(\sqrt{n})}\Sigma\Pi^{O(\sqrt{n})}$ circuits computing Determinant, Permanent have $2^{\Omega(\sqrt{n})}$ size [Gupta et al., 2014].

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Need for Parameterization

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- *Notion of tractability*: $f(k)\text{poly}(n)$, where f is a computable function in the parameter k, known as *fixed parameter tractable* or FPT.
- Study of parameterized view of algebraic complexity is interesting [Engels, 2016]

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- Polynomials parameterized by degree can be computed in FPT-size.
- Improving the circuits to formulas would improve the space complexity of the parameterized algorithms in [Fomin et al., 2012], [Williams, 2009],[Koutis and Williams, 2009].

Motivation

For multilinear models

- Results using polynomials parameterized by degree test for multilinear monomials present in the polynomial [Williams, 2009],[Koutis and Williams, 2009].
- Lower bounds on multilinear models in the classical setting do not translate to $n^{\Omega(k)}$ lower bounds [Raz, 2009].

Degree as Parameter

- There is a polynomial p of degree k, with FPT-size $\Sigma\Pi\Sigma\Pi$ circuit [G Prakash Rao 2017].
- $\Sigma\bigwedge\Sigma\bigwedge\Sigma$ circuits with top power gate fan-in restricted to $o(k)$ require $n^{\Omega(k)}$ size to compute p.

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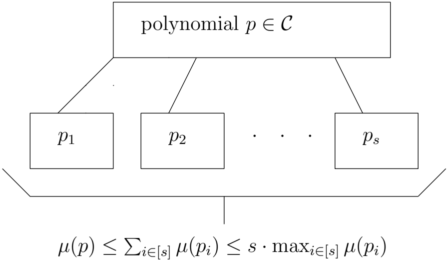
- $\Sigma\bigwedge\Sigma\bigwedge\Sigma$ circuits with top power gate fan-in restricted to $o(k)$ require $n^{\Omega(k)}$ size to compute p.
- Depth-reduction in the classical setting [Agrawal and Vinay, 2008] doesn't translate directly to parameterized setting.

Degree as Parameter

- Same lower bound also obtained against $\Sigma \bigwedge^{o(k)} \Sigma \bigwedge \Sigma$ circuits computing SYM_n^k.
- Parameterized separation of depth-3 and depth-4 circuits.

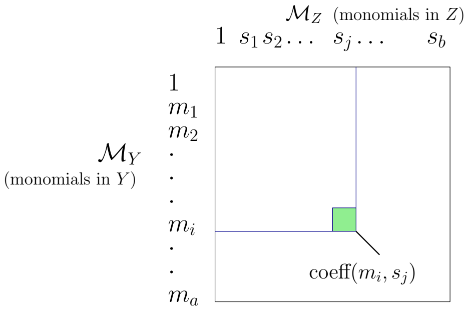
Approach for Lower Bounds

- Consider class $\mathcal{C}$ of arithmetic circuits of degree k.
- Fix complexity measure $\mu : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{R}^+$, sub-additive and sub-multiplicative.



Rank of the Coefficient Matrix

Coefficient matrix M_f of a multilinear polynomial $f = \sum_{i,j} \text{coeff}(m_i, s_j) m_i s_j$:



- Y, Z are partitions of X.
- Rank of M_f can indicate hardness of f.

Example of Rank Calculation

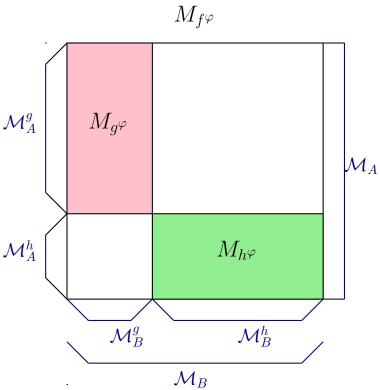
$$
\begin{aligned}
f &= (x_1^2 + 2x_1)(x_2 + 3) + (x_3^2 + 4x_3)(x_4 + 1) \\
&= x_1^2x_2 + 3x_1^2 + 2x_1x_2 + 6x_1 + x_3^2x_4 + x_3^2 + 4x_3x_4 + 4x_3
\end{aligned}
$$

	1	x_2	x_3	x_2^2	x_3^2
1	0	0	4	0	1
x_1	6	2	0	0	0
x_4	0	0	4	0	1
x_1^2	3	1	0	0	0
x_4^2	0	0	0	0	0

Here, $X = \{x_1, x_2, x_3, x_4\},\ Y = \{x_1, x_4\},\ Z = \{x_2, x_3\}$

Properties of Rank

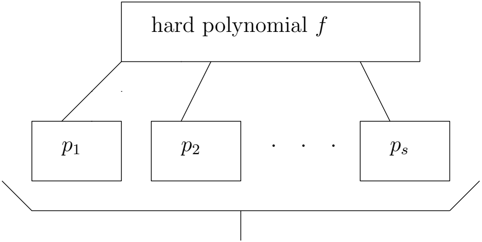
- Let g, h be polynomials, $\mathrm{var}(g) \cap \mathrm{var}(h) = \phi$.
- Let $f = g + h$, then $\mathrm{rank}(M_{f^\varphi}) \leq \mathrm{rank}(M_{g^\varphi}) + \mathrm{rank}(M_{h^\varphi})$.



Properties of Rank

- $f = g \times h,\ \text{rank}(M_{f^\varphi}) \leq \text{rank}(M_{g^\varphi}) \times \text{rank}(M_{h^\varphi})$.
- *Observation*: For any multilinear polynomial f of degree k, $\text{rank}_\varphi(f) \leq k\binom{n/2}{k/2}$, for any partition φ.

Approach to Lower bounds using Rank



$$\forall \varphi',\ \mathsf{rank}_{\varphi'}(f) \leq s \cdot \max_{\varphi} \max_{i \in [s]} \mathsf{rank}_{\varphi}(p_i)$$

Our Result

Theorem

For the parameterized polynomial family $f = (f_{n,2k})_{n,k \geq 0}$,

$$\mathrm{rank}_{\varphi}(f_{n,2k}) = \Omega\left(\frac{n^k}{(2k)^{2k}}\right)$$

for every equi-partition $\varphi : X \to Y \cup Z$.

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Corollary

Any ROABP computing the polynomial family $f = (f_{n,2k})$ *has size* $\Omega(n^k/(2k)^{2k})$*.*

Constructing High Rank Polynomial of degree $2k$

Main Idea

- Use the polynomial $p = \prod_{j \in [k/2]} (x_{1,j} w_{1,j} + \cdots + x_{\frac{n}{k},j} w_{\frac{n}{k},j})$.
- p has rank $n^{k/2}/k^{k/2}$ under one partition, but low rank with many others.

Constructing High Rank Polynomial of degree $2k$

- Consider $K_{2k} = (V, E)$, define polynomial $p_{i,j}$ for edge (i, j) in K_{2k}.

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- Hard polynomial $f = \sum_{M \in \mathcal{M}} a_M \prod_{(i,j) \in M} (1 + p_{i,j})$, $\mathcal{M}$ is family of all perfect matchings in K_{2k}.

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- Hard polynomial $f = \sum_{M \in \mathcal{M}} a_M \prod_{(i,j) \in M} (1 + p_{i,j})$, $\mathcal{M}$ is family of all perfect matchings in K_{2k}.
- Under any φ, there is $M \in \mathcal{M}$, $\prod_{(i,j) \in M} (1 + p_{i,j})$ is of rank $\Omega(n^k / g(k))$.

Constructing High Rank Polynomial of degree $2k$

- For each vertex $i \in V$, $V_i = \{x_i, x_{i+1}, \ldots, x_{i+\frac{n}{2k}-1}\}$, where $x_a \preceq x_b$ iff $a \leq b$.

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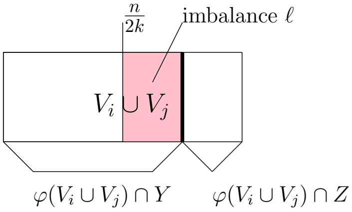
- $p_{i,j} = p(x_i, \ldots, x_{j+\frac{n}{2k}-1})$ is defined on n/k ordered variables, like an interval.
- We want edge polynomial $p_{i,j}$ to be of full rank.

$$p_{i,j} = p(x_i, \ldots, x_{j+\frac{n}{2k}-1}) = \sum_{a<b} \omega_{a,b} x_a x_b$$

Rank lower bound for $p_{i,j}$

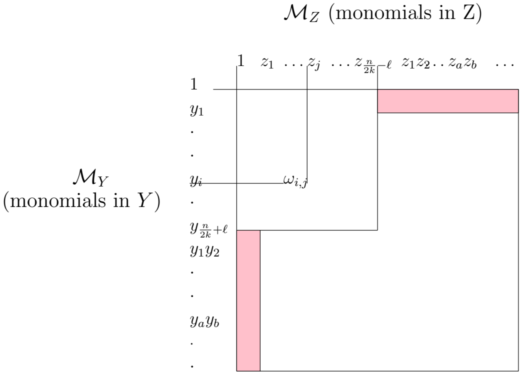


Theorem

If $V_i \cup V_j$ is ℓ-unbalanced with respect to a partition $\varphi : X \to Y \cup Z$, then $\text{rank}_\varphi(p_{ij}) = \Omega(n/2k - |\ell|)$.

Proof: Rank lower bound for $p_{i,j}$

Coefficient matrix for $M_\varphi(p_{i,j})$, where $p_{i,j} = \sum_{a<b} \omega_{a,b} x_a x_b$:



Analyzing rank of the polynomial f

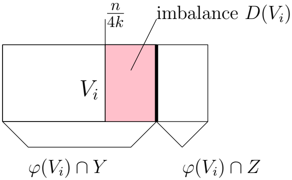
- Let $p_M = \prod_{(i,j)\in M}(1 + p_{i,j})$.
- *Main idea*: Fix a partition φ. From any matching M, we obtain a matching N, p_N is full rank under φ.
- Assign weights $\text{wt}(e) = |\ell_e|$ for $e \in M$. *Bad edges* have weight above threshold $t = n/2k - n/2k(k-1)$.

Analyzing rank of the polynomial f

- If M has no bad edges, then each edge polynomial has rank $\Omega(n/2k - wt(e)) = \Omega(n/2k(k-1))$, $\text{rank}_\varphi(f) = \Omega(n^k/g(k))$.
- Otherwise, repeatedly swap end-points of bad edge (i, j) with good edge (i', j').
- Matching with new edges $(i, j'), (i', j)$ or $(i, i'), (j, j')$ has one bad edge lesser than M.

Analyzing rank of the polynomial f

Proof Outline



$$\mathrm{wt}(e) = |\ell| = |D(V_i) + D(V_j)|$$

$$D(V_i) = |\varphi(V_i) \cap Y| - \frac{n}{4k}$$

$$D(V_i) \in [-\frac{n}{4k}, \frac{n}{4k}]$$

- e is a bad edge, $\mathrm{wt}(e) > t = n/2k - n/2k(k-1)$.
- $\sum_{e' \in M} sgn(e')\mathrm{wt}(e') = \sum_{i \in [2k]} D(V_i) = 0$.
- By averaging, there is an edge e_1, $sgn(e_1)\mathrm{wt}(e_1) < -t/(k-1) = -n/2k(k-1) + n/2k(k-1)^2$.

- Let $D(V_i) = a$, $D(V_j) = b$, $D(V_{i_1}) = c$, $D(V_{j_1}) = d$. Assume $a, b > 0$.
- Since $c + d < 0$, either both $c, d < 0$ or one of them is negative.

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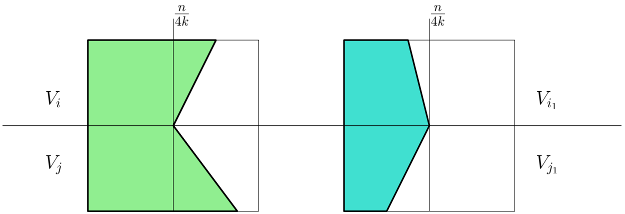
- Since $c + d < 0$, either both $c, d < 0$ or one of them is negative.
- *Case 1:* If $c, d < 0$ then $|a + b| + |c + d| > |a + c| + |b + d|$ and matching with edges $(i, i_1), (j, j_1)$ has lower weight.

Case 1

Edges e and e_1 are represented as:



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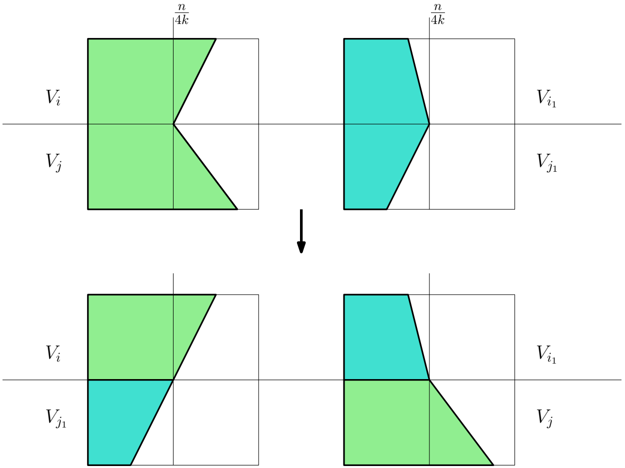
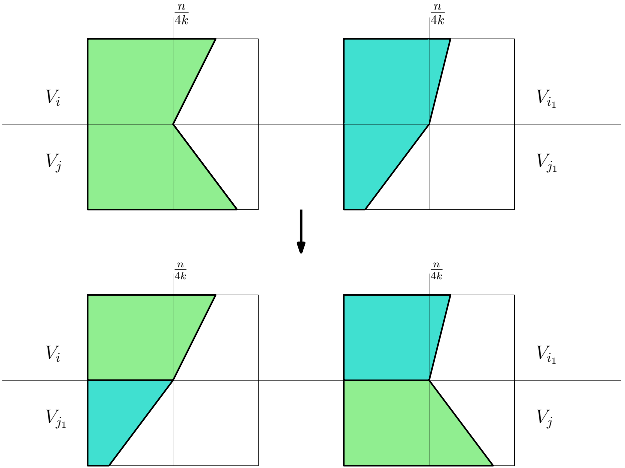


New edges e' and e'_1 are more balanced.

Case 2: Proof Idea when $c \geq 0, \ d < 0$



- Let $c \geq 0, d < 0$. Now, $c + d < -n/2k(k-1) + n/2k(k-1)^2$ and $d > -n/4k$. So, $c \leq n/4k - n/2k(k-1) + n/2k(k-1)^2$.
- Then if $c > a, b$,

$$a + b < 2c \leq n/2k - n/k(k-1) + n/k(k-1)^2.$$

But $a + b \geq n/2k - n/2k(k-1)$.

Analyzing rank of hard polynomial f

Proof Outline:Case 2

- *Subcase 1:* If $a > c$, $a + b > b + c$, matching with edge pair $(i, j_1), (j, i_1)$ is preferred.
- *Subcase 2:* Similarly for $b > c$, $a + b > a + c$ matching with edges $(i, i_1), (j_1, j)$ has higher rank.

Application: Size Lower Bound on ROABP

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- Width is bounded by size s. By previous theorem, we obtain the required bound.

More Results

- The family of hard polynomials $f_{n,2k}$ is of depth-4 and FPT size.
- We also construct hard polynomial h that is sum of 3 read-once polynomials.
- h achieves similar rank with a loss of constant factor in the exponent of n.

Thank you!
Questions?