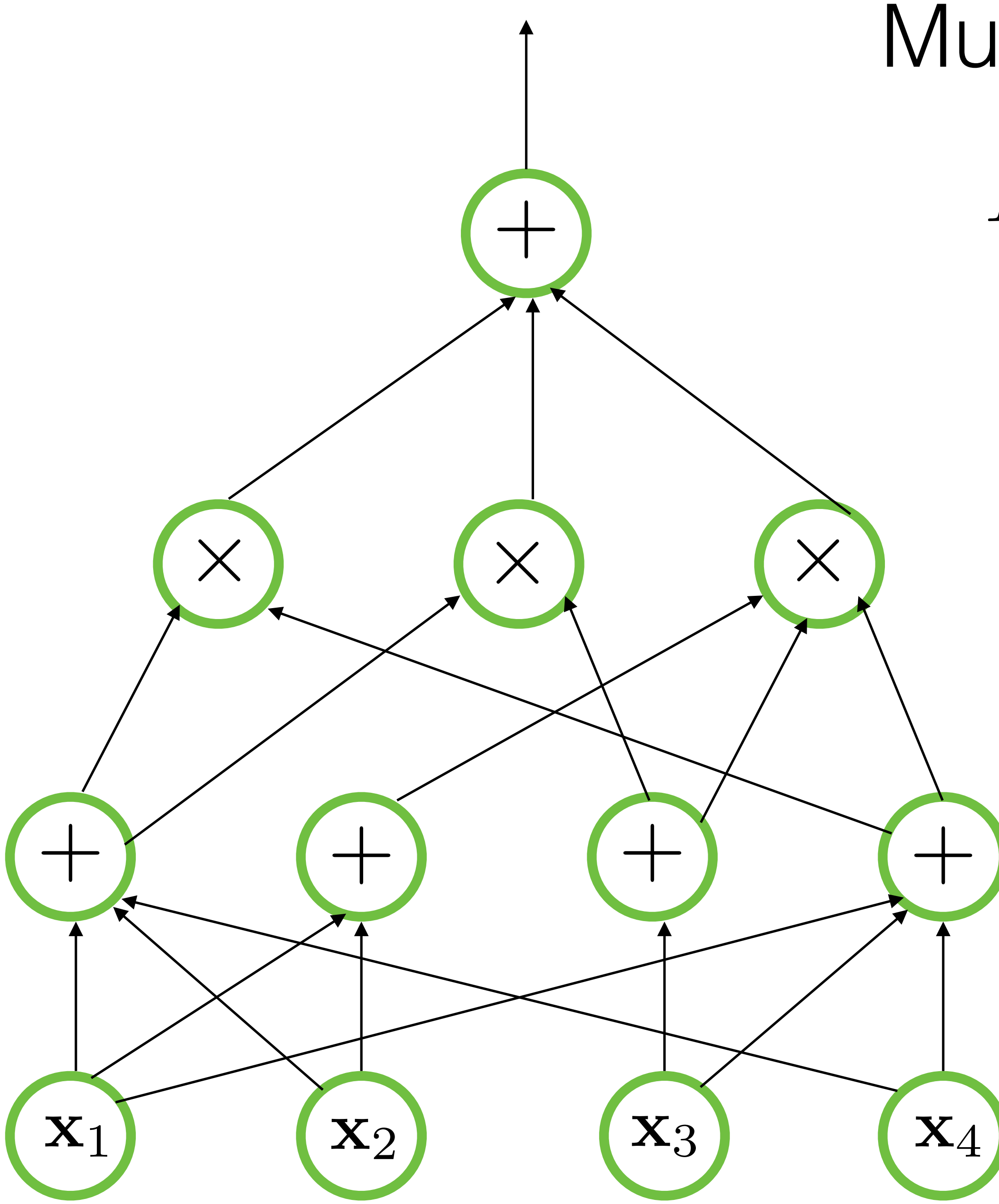
Some Closure Results for Polynomial Factorization and Applications

Chi-Ning Chou

Harvard University

Joint work with **Mrinal Kumar** and **Noam Solomon**

Arithmetic Circuits



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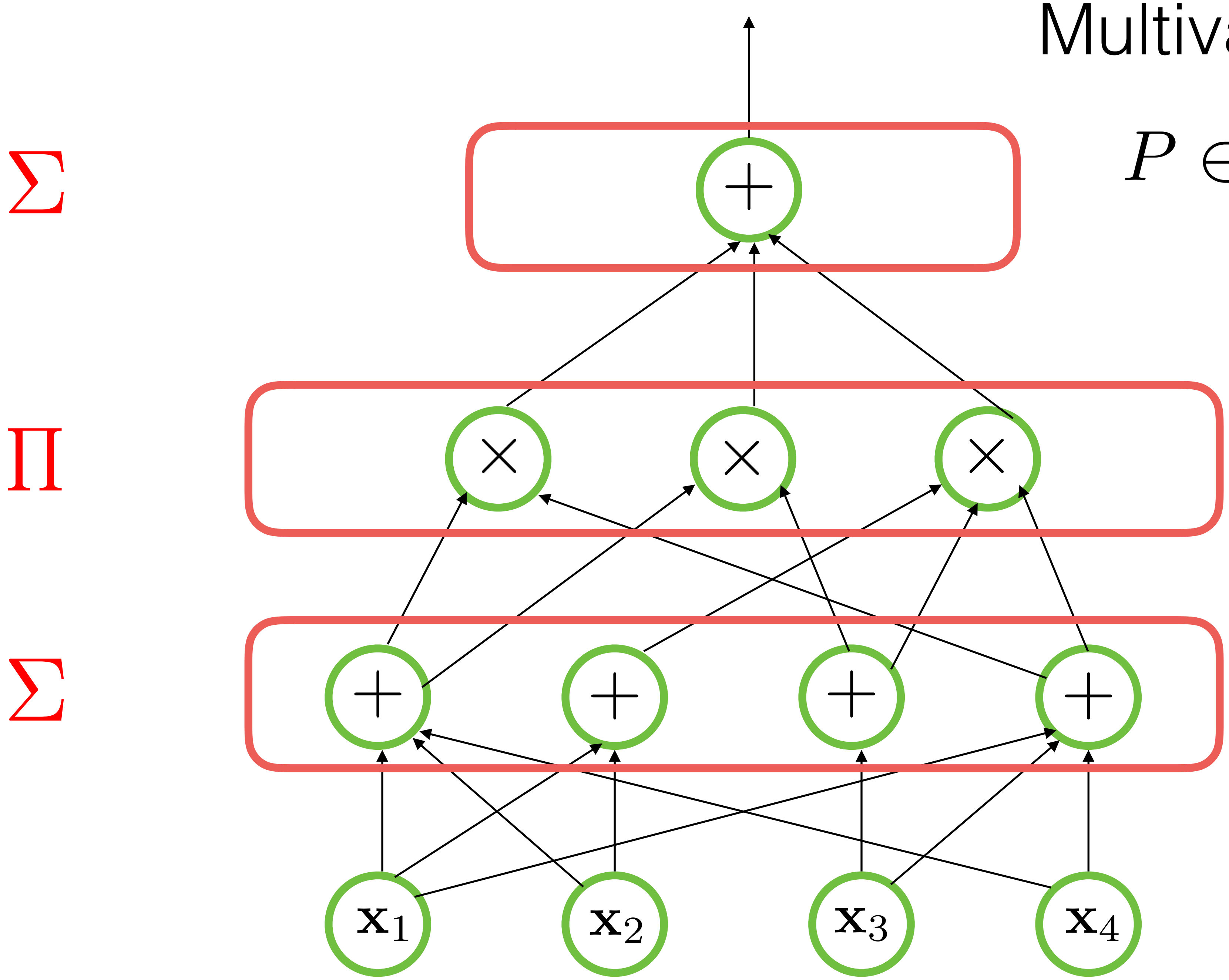

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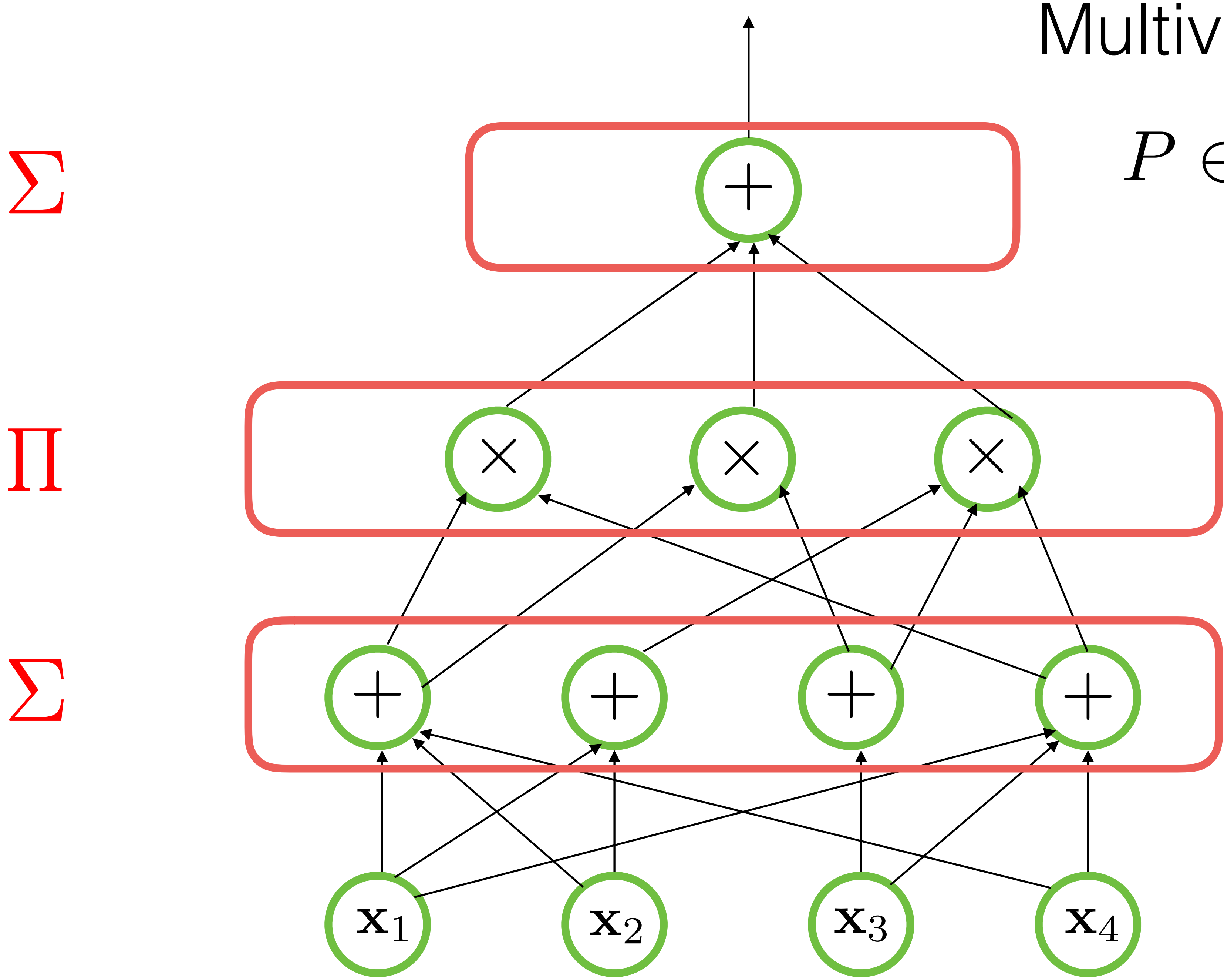

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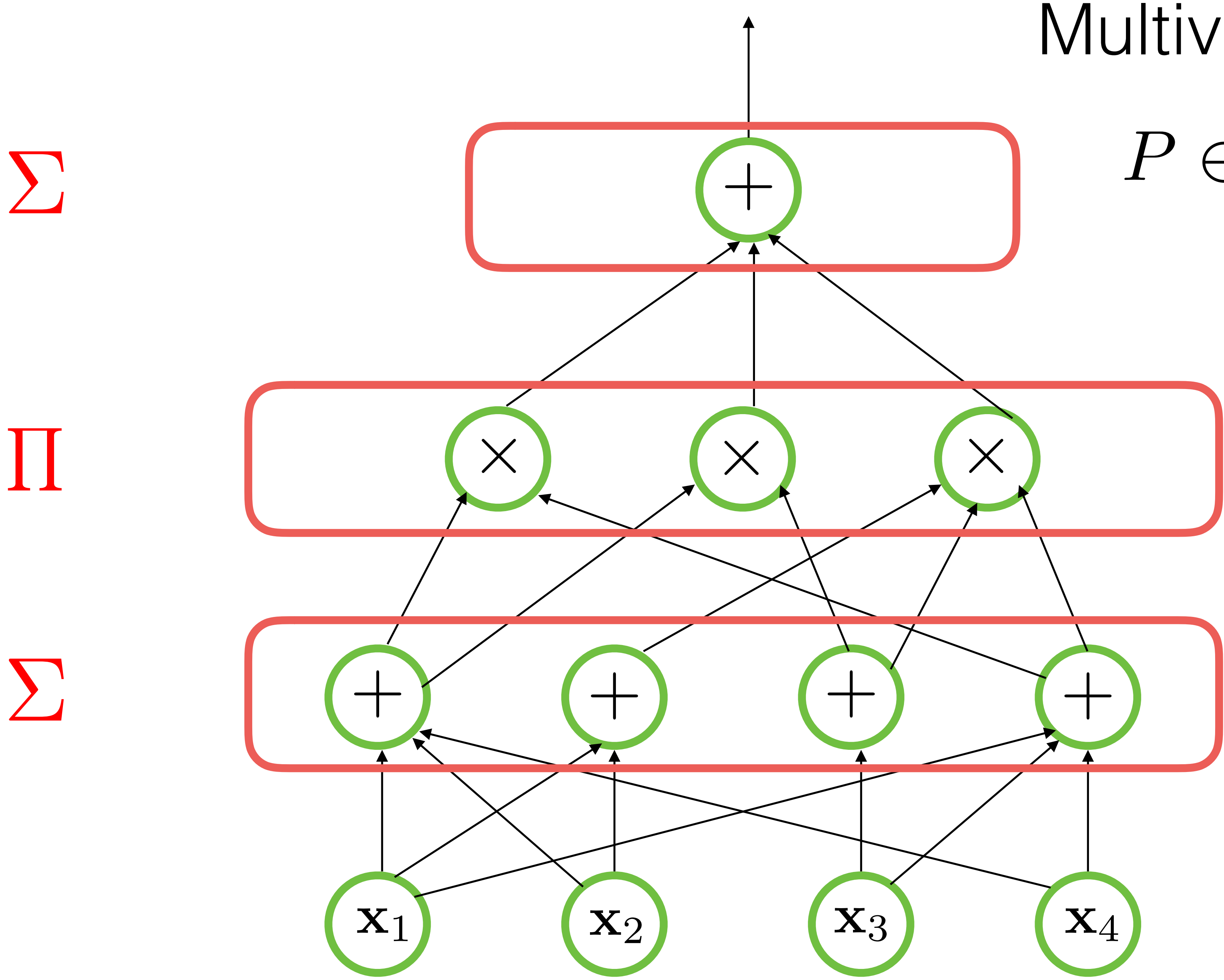

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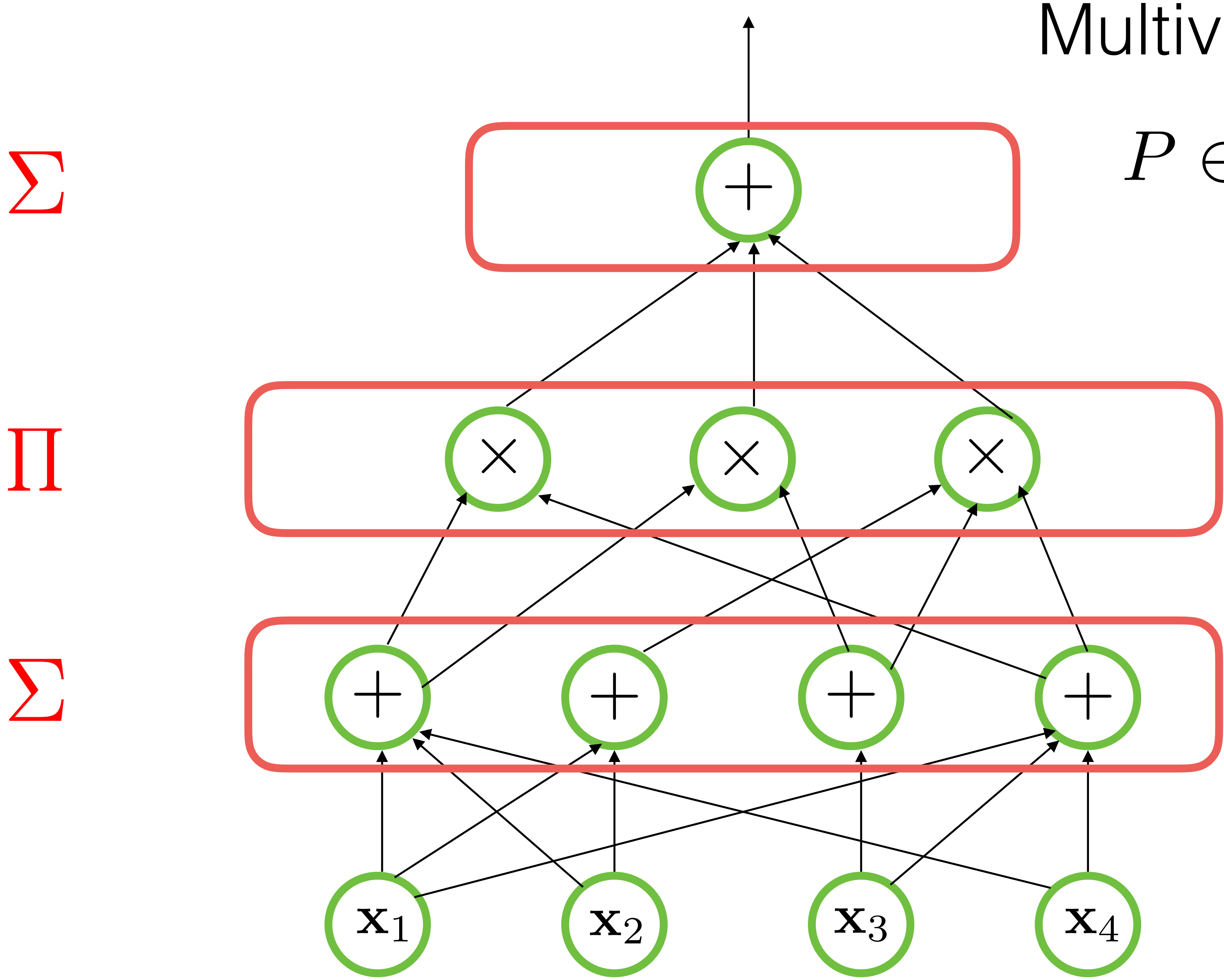

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*Assume $\mathbb{F} = \mathbb{Q}$

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- Many more such as VF, VBP, VNP…

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Application:

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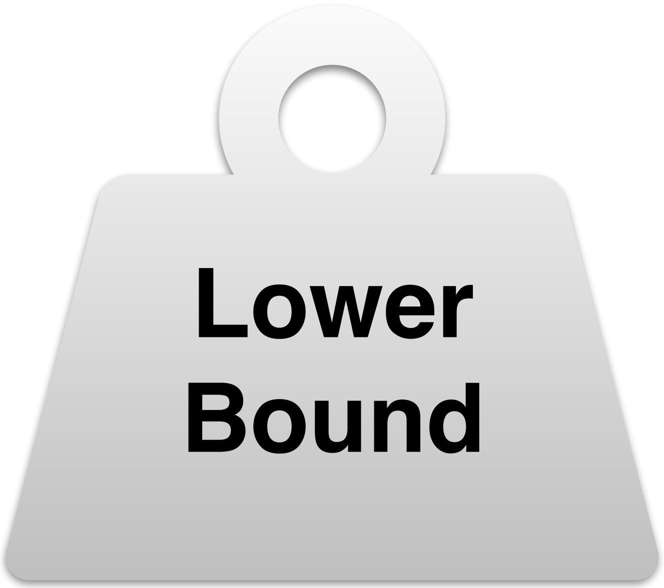

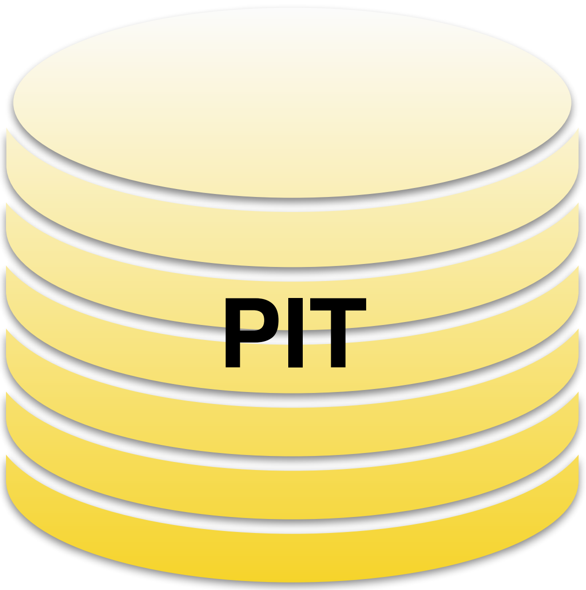

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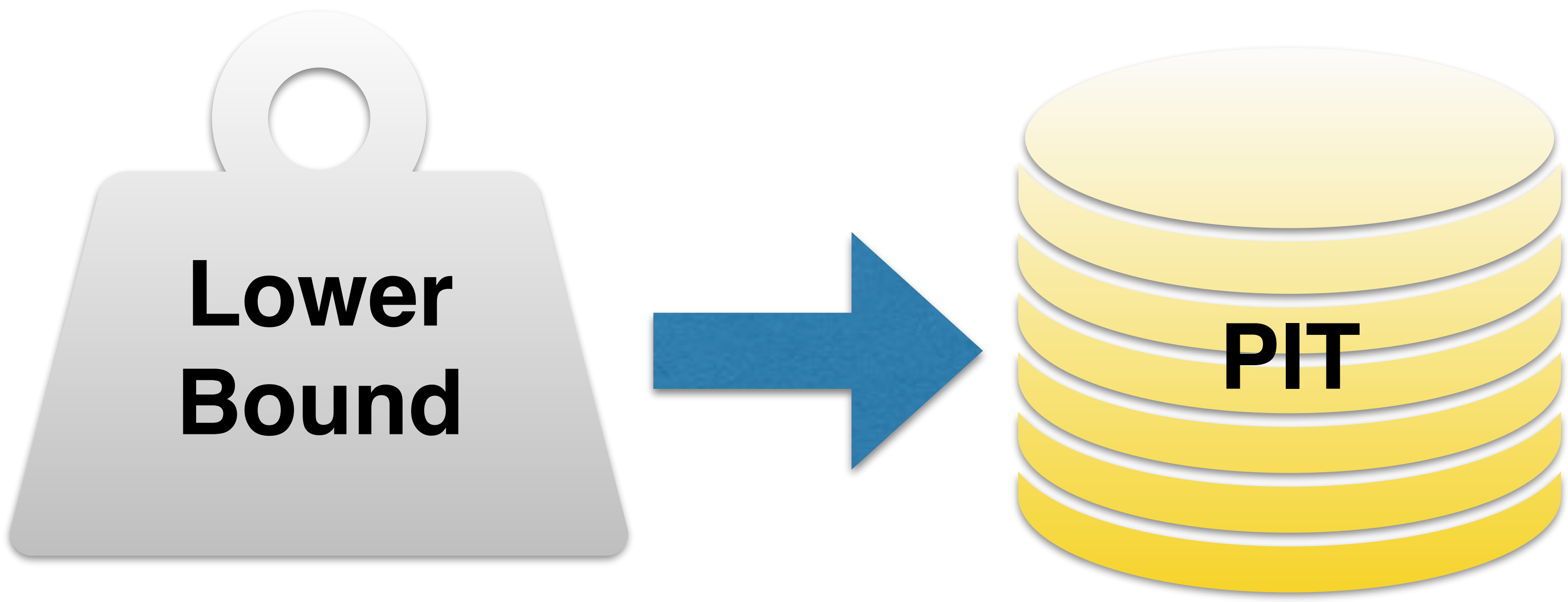

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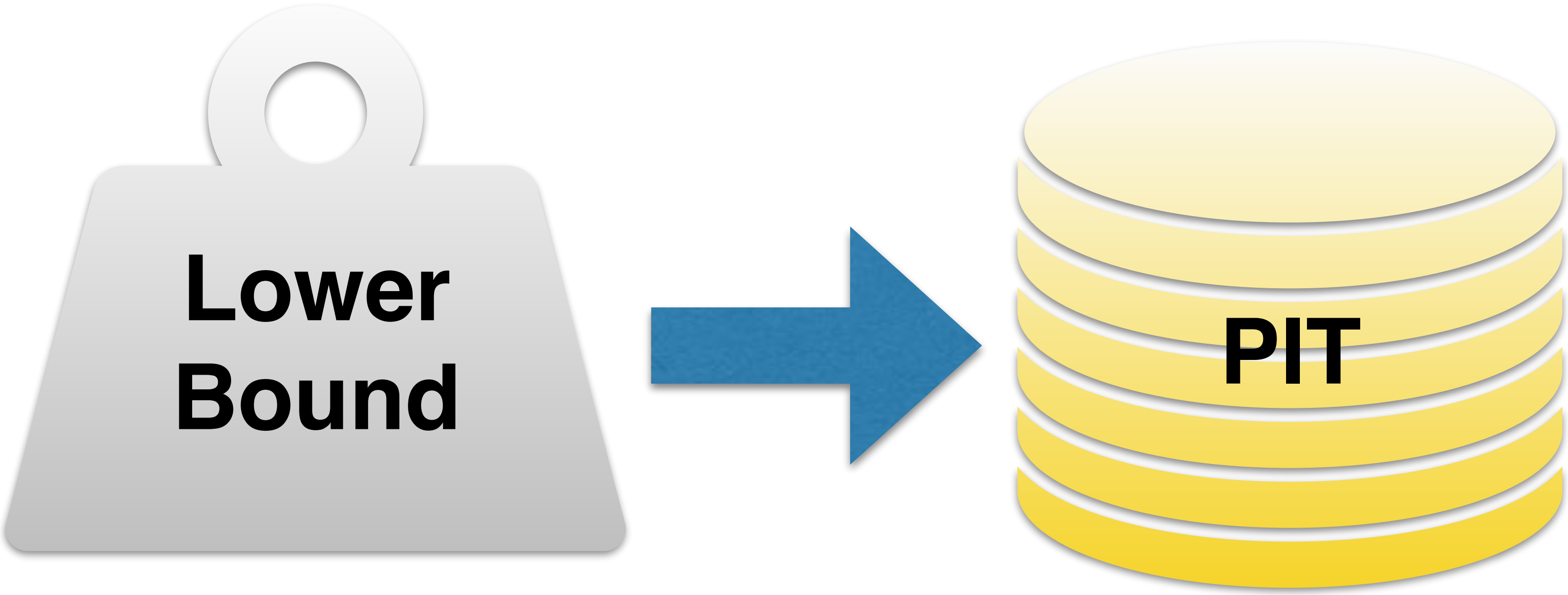

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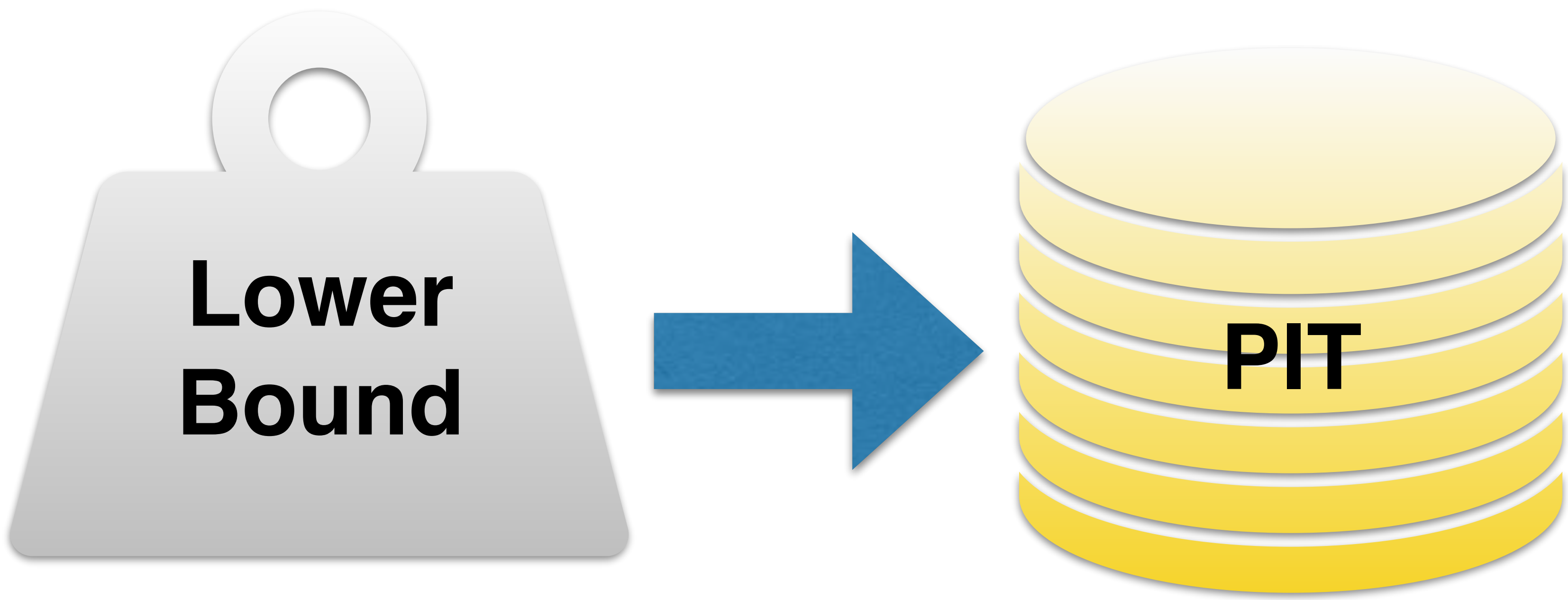

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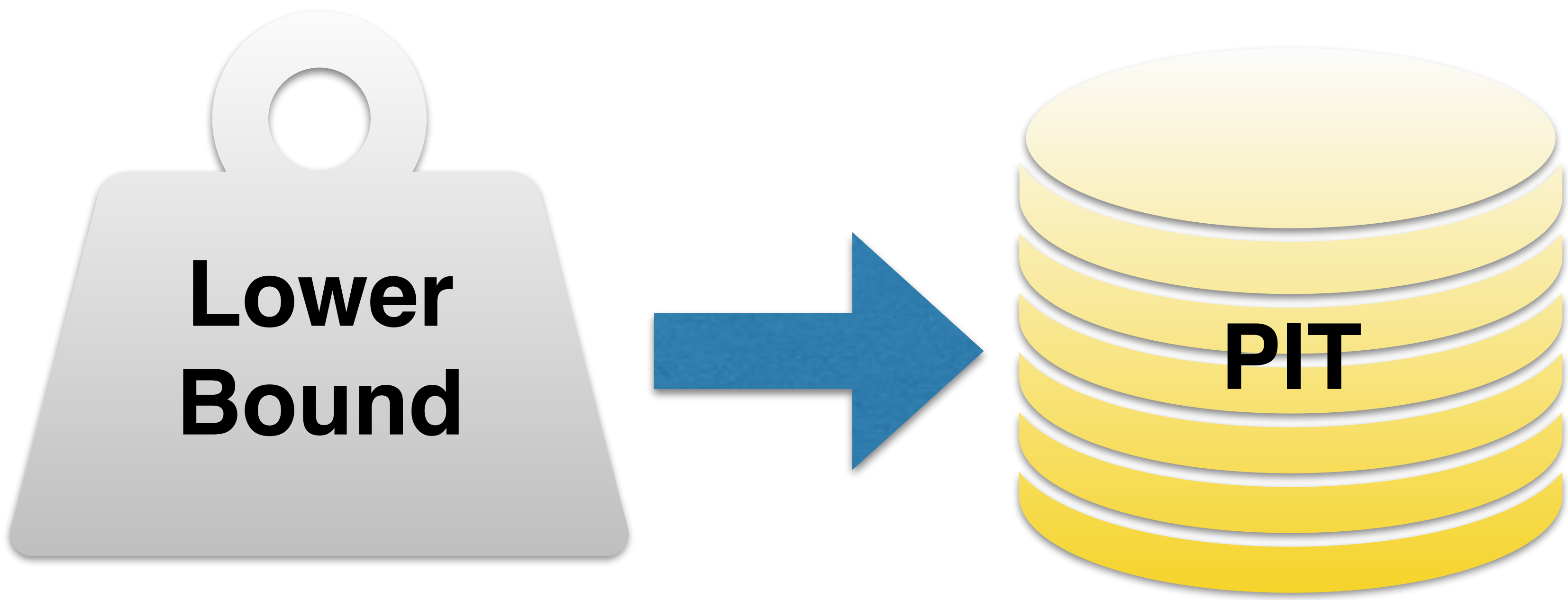

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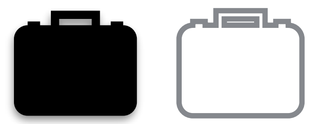

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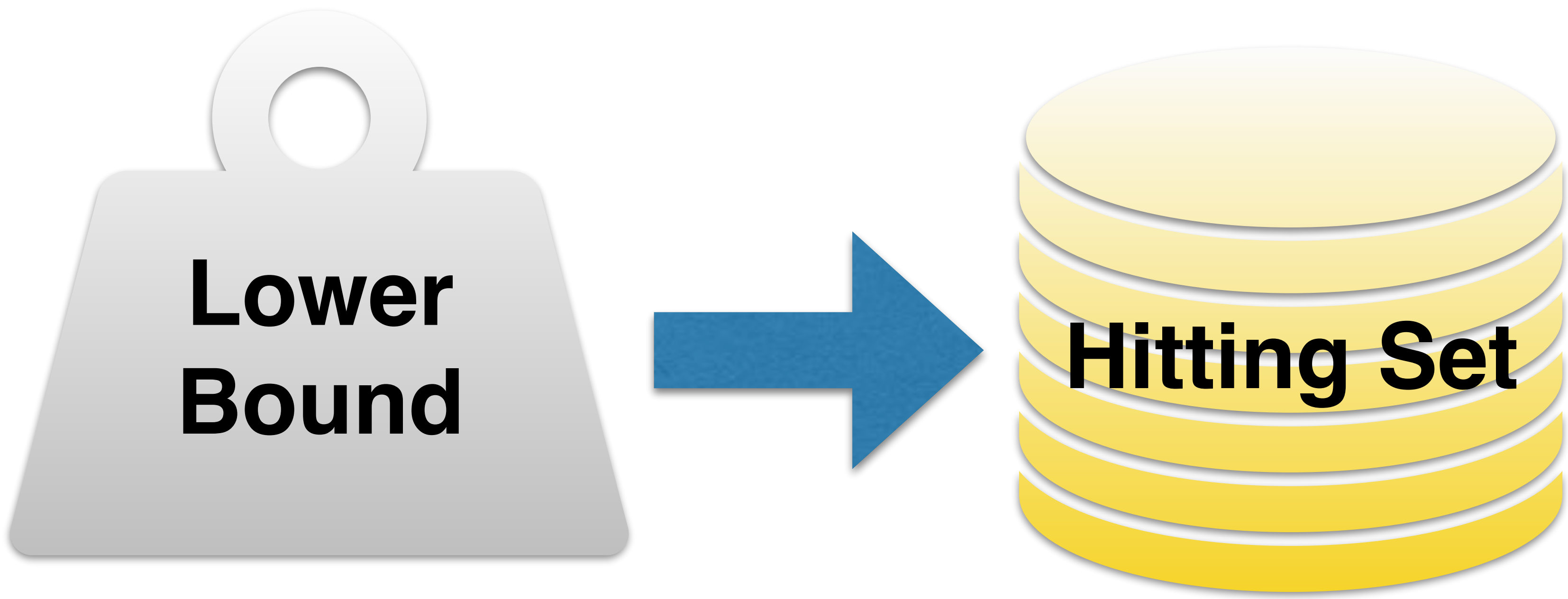
Hardness versus Randomness framework [KI04, DSY09, **C**KS18]



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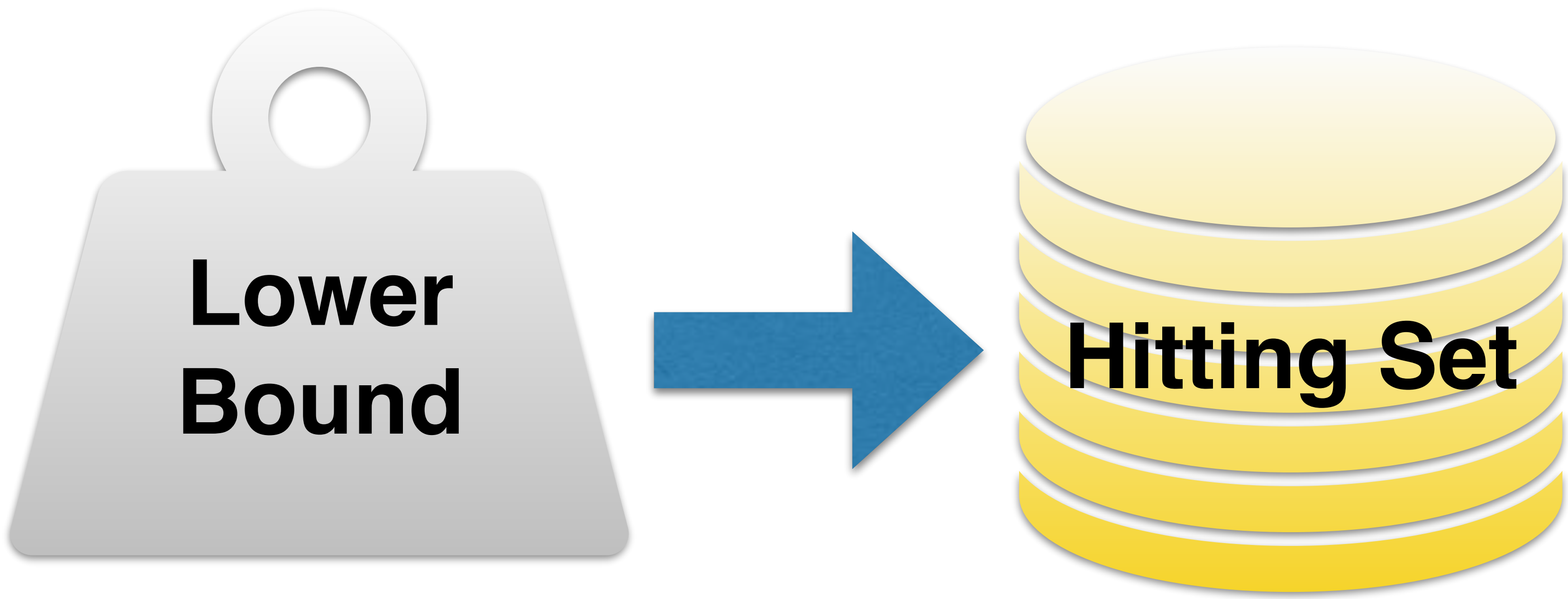

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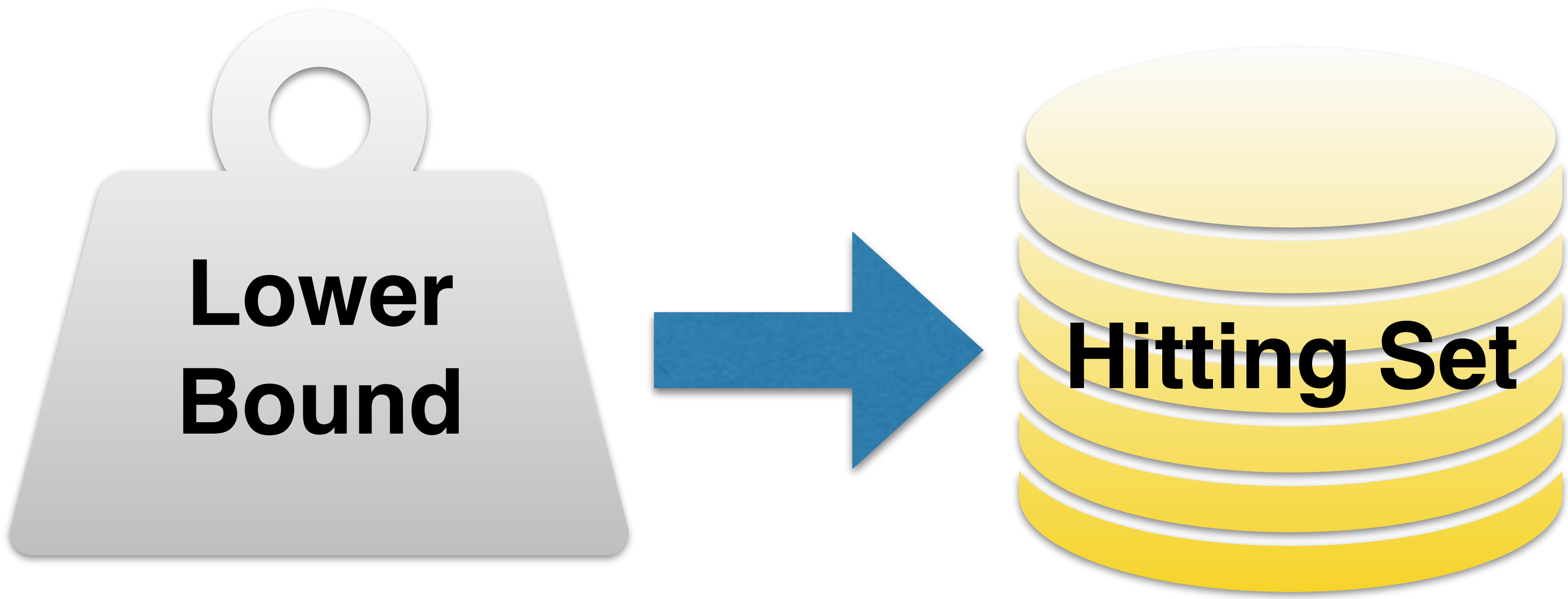

Nisan-Wigderson generator

Schwartz-Zippel lemma

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Brute-force to find hitting set in time $d^{O(\ell)}$

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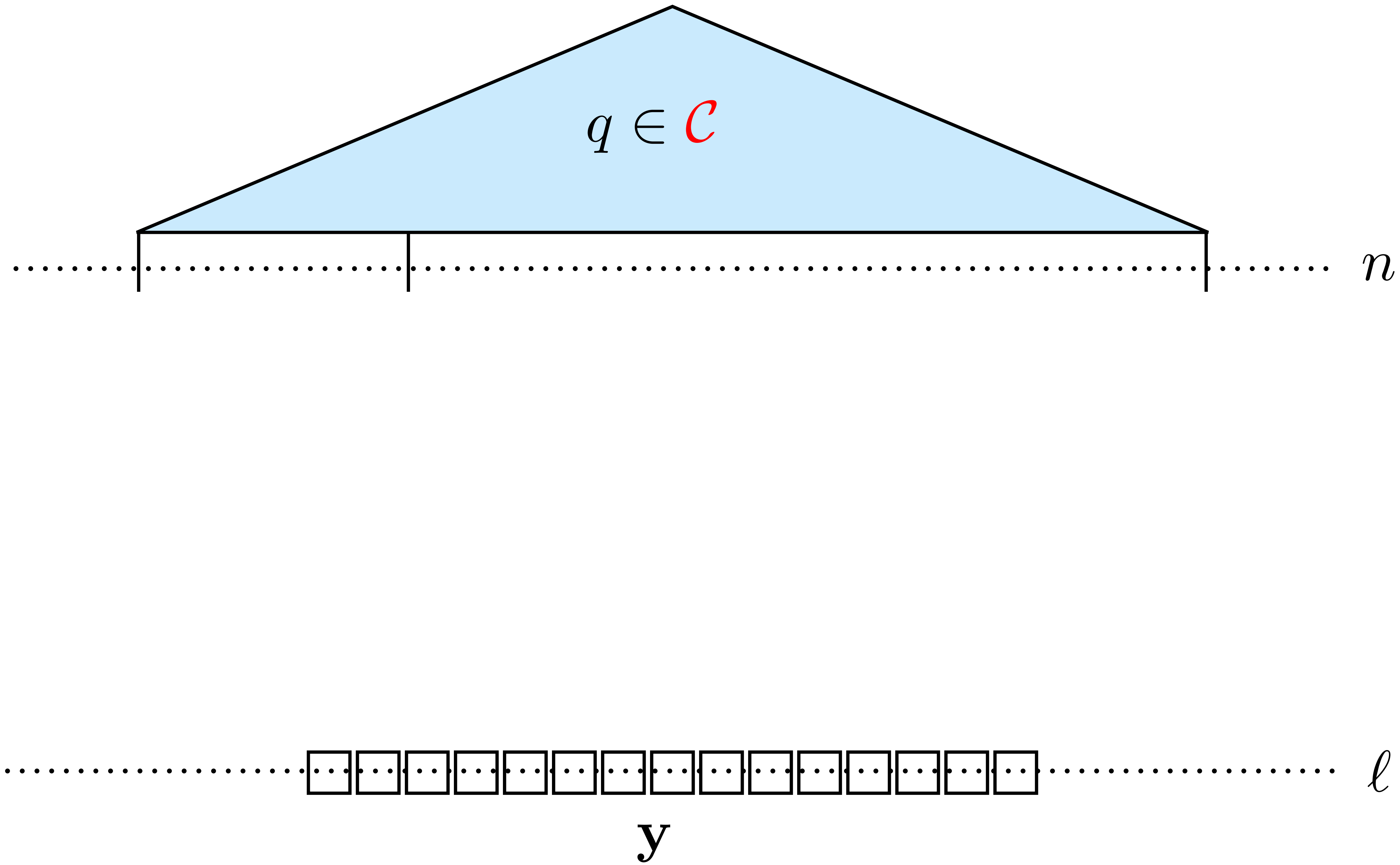

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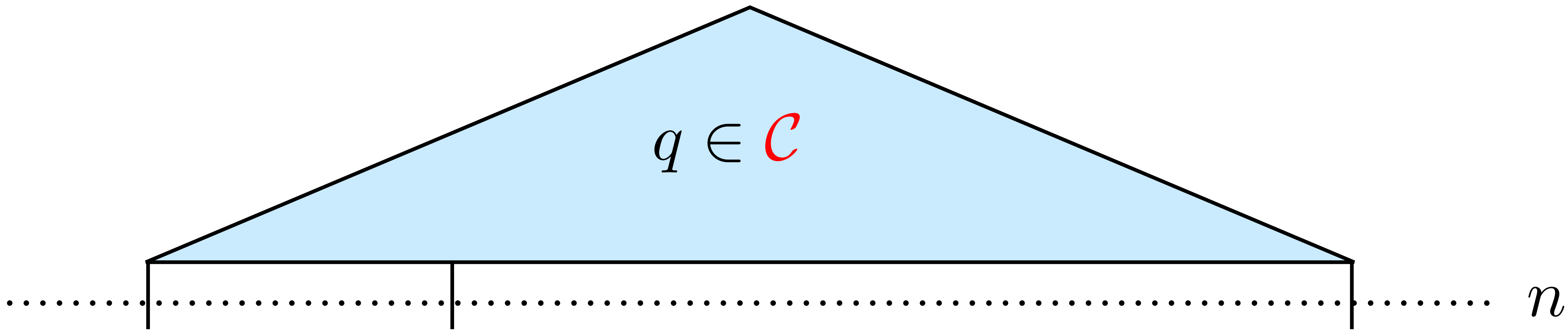

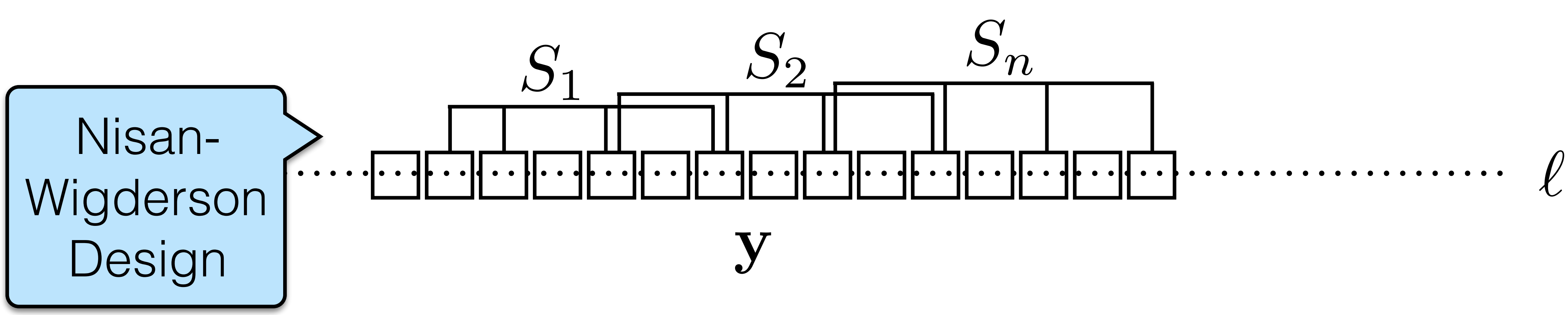

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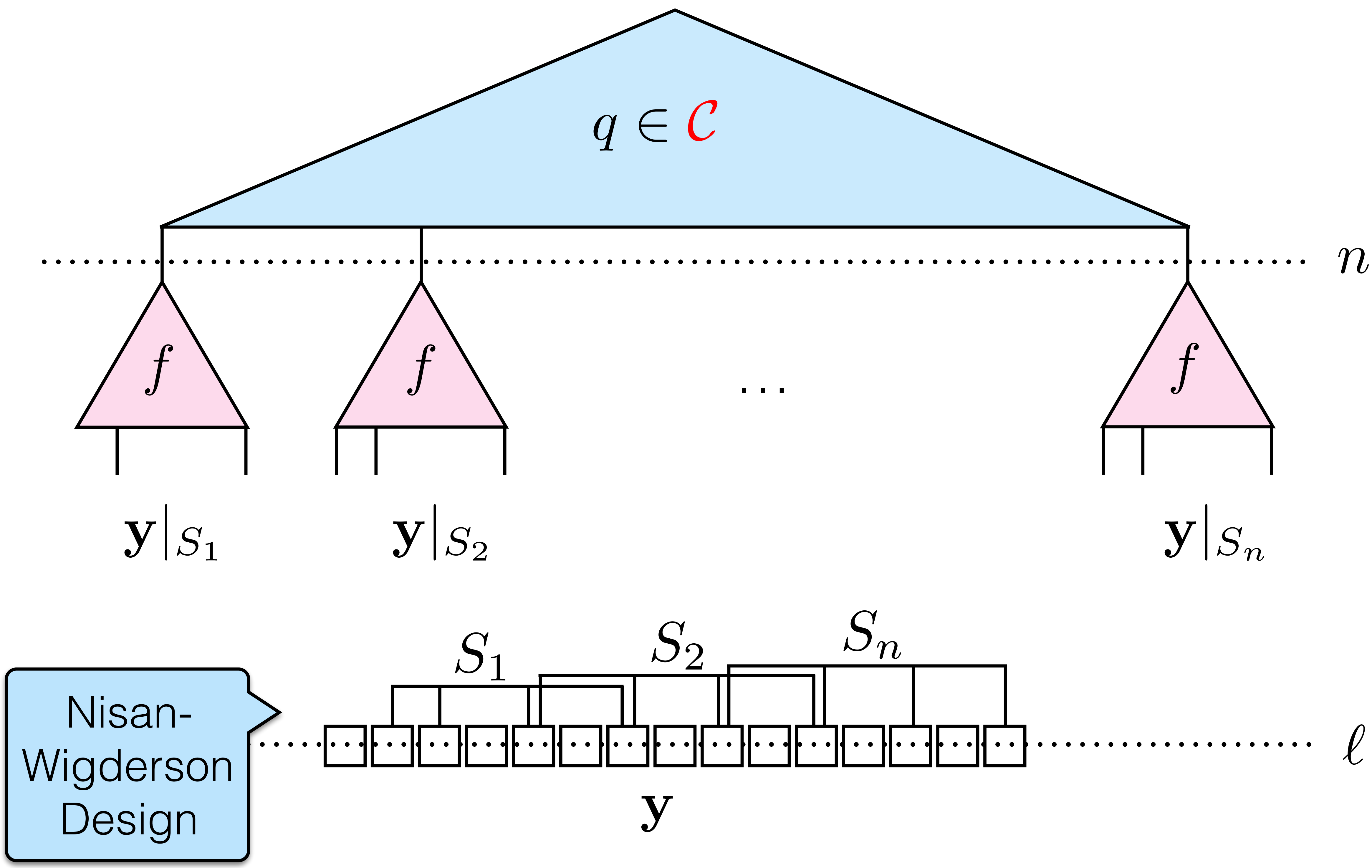

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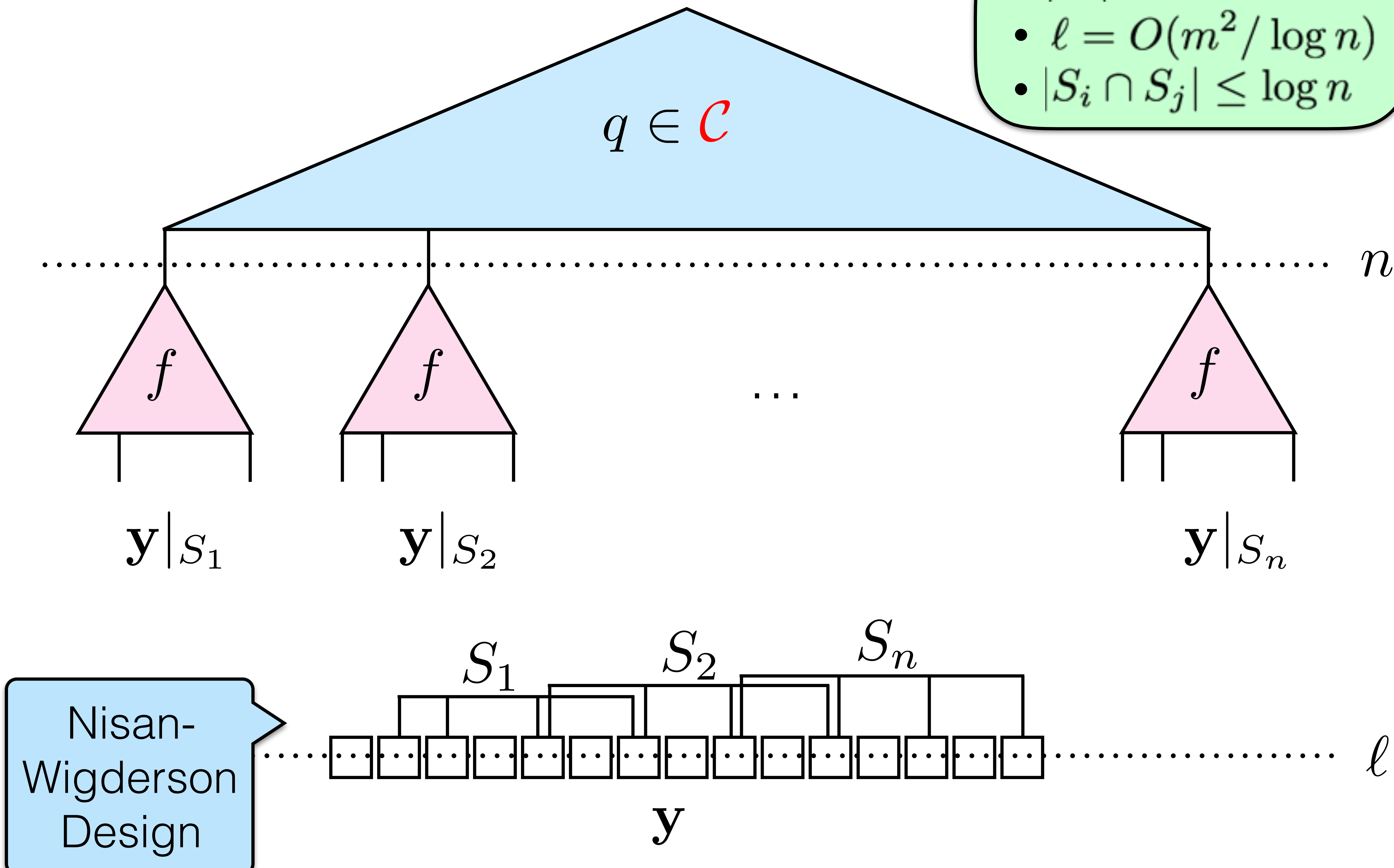

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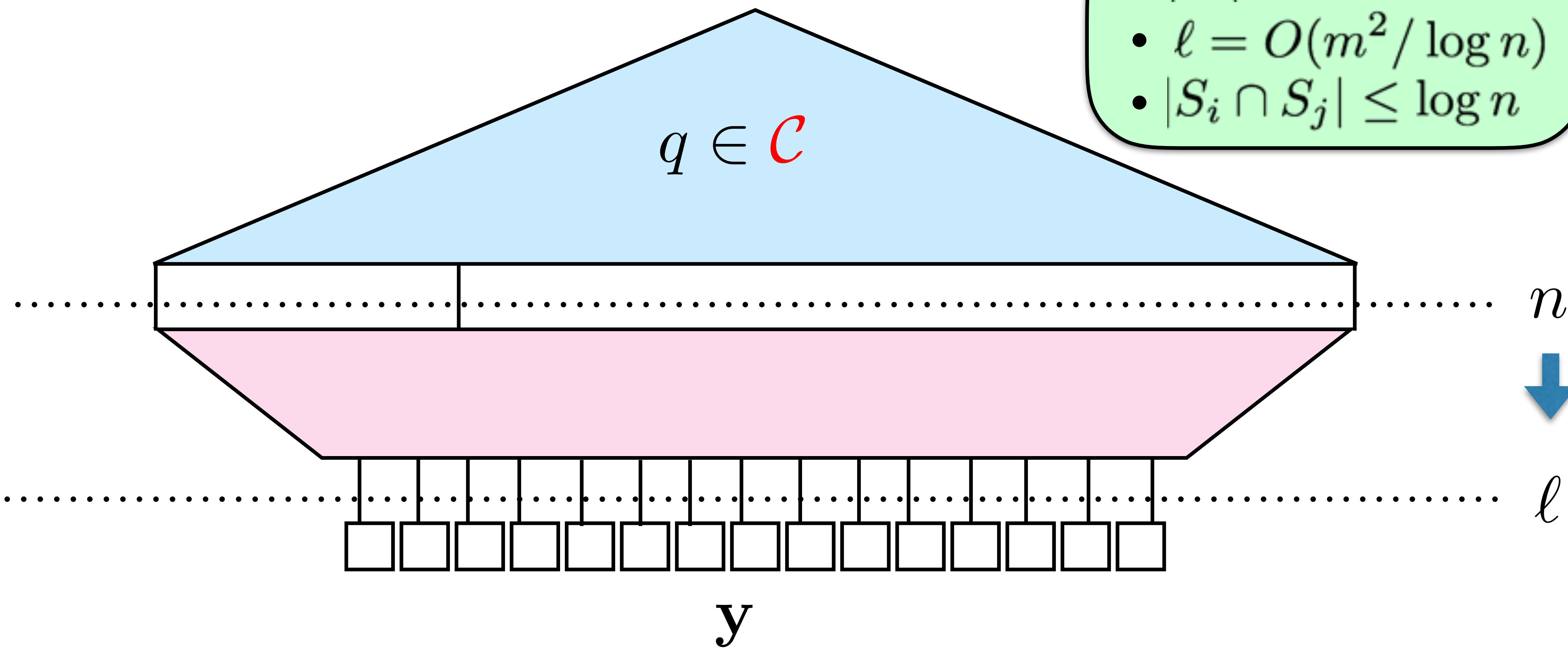

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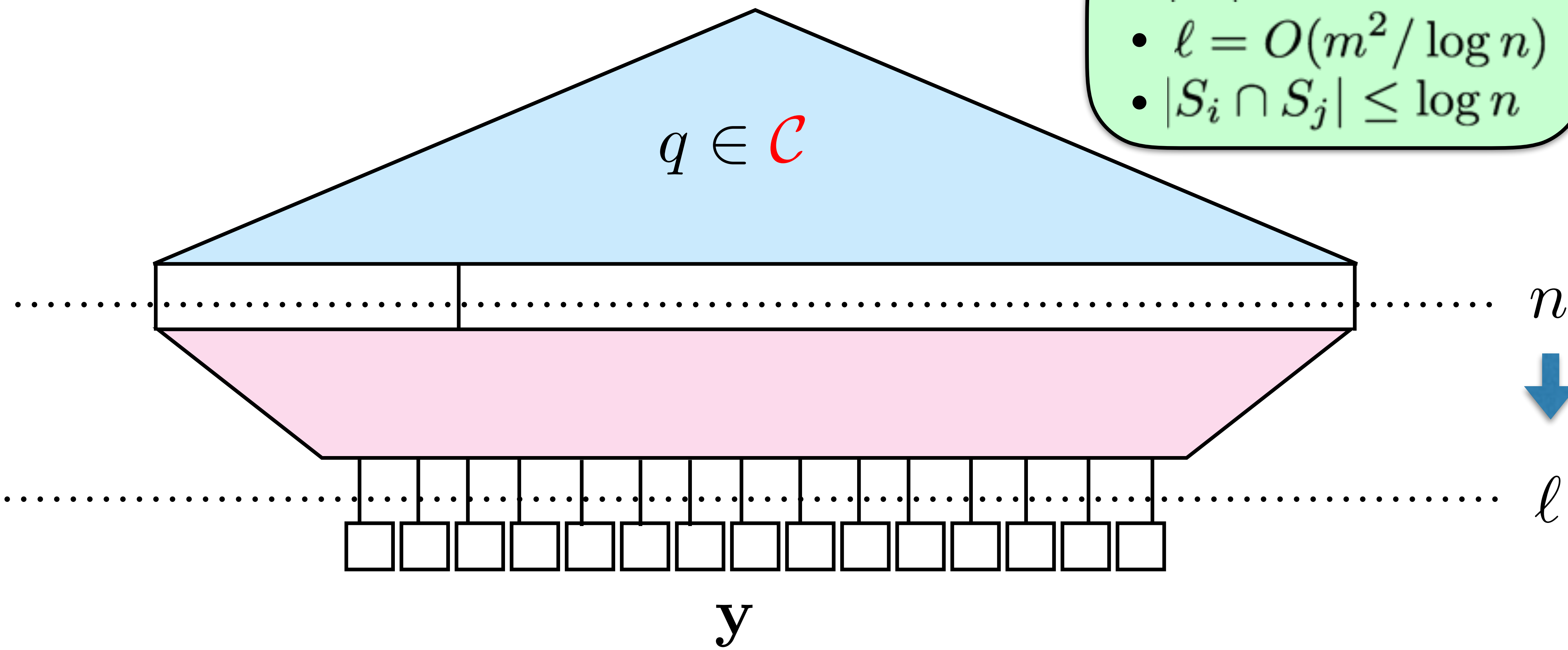

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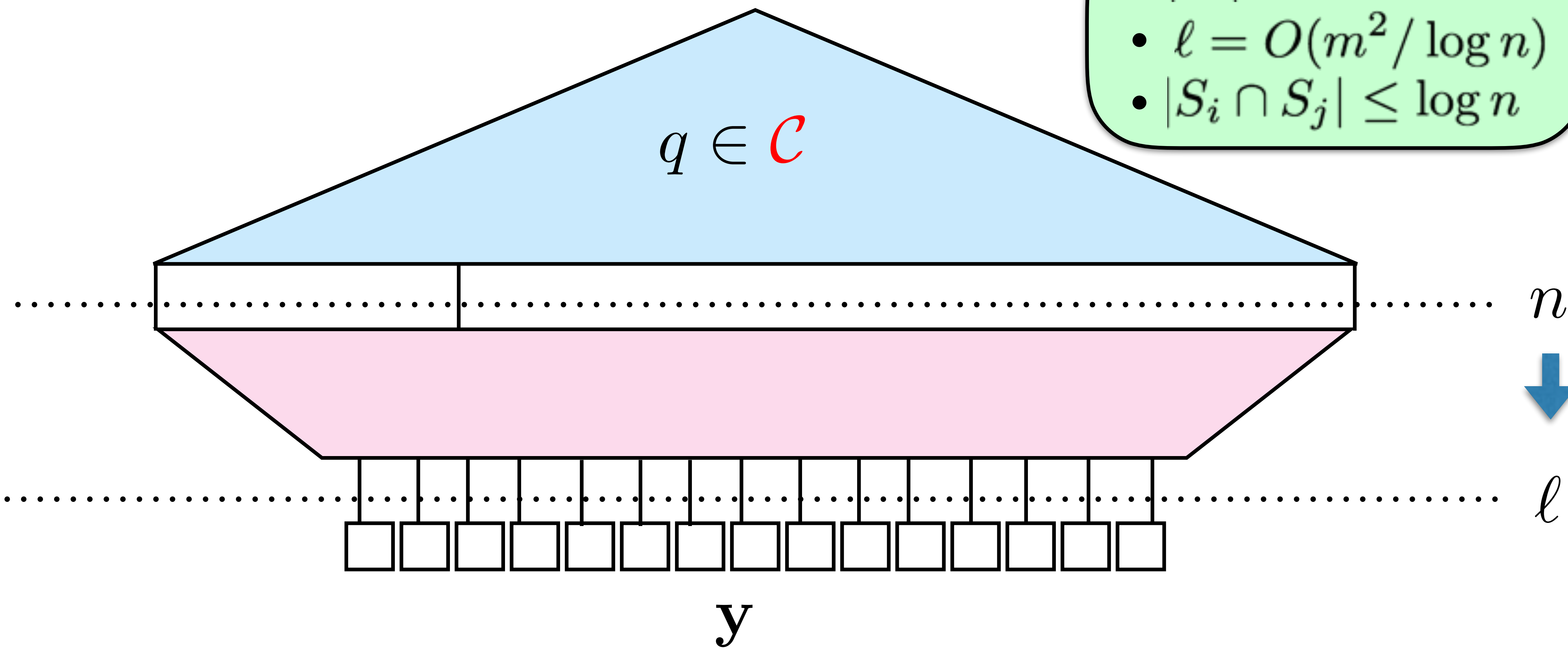

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Lemma [Schwartz, Zippel]

There exists hitting set of size $d^{O(\ell)}$ for degree d ℓ-variate polynomials.

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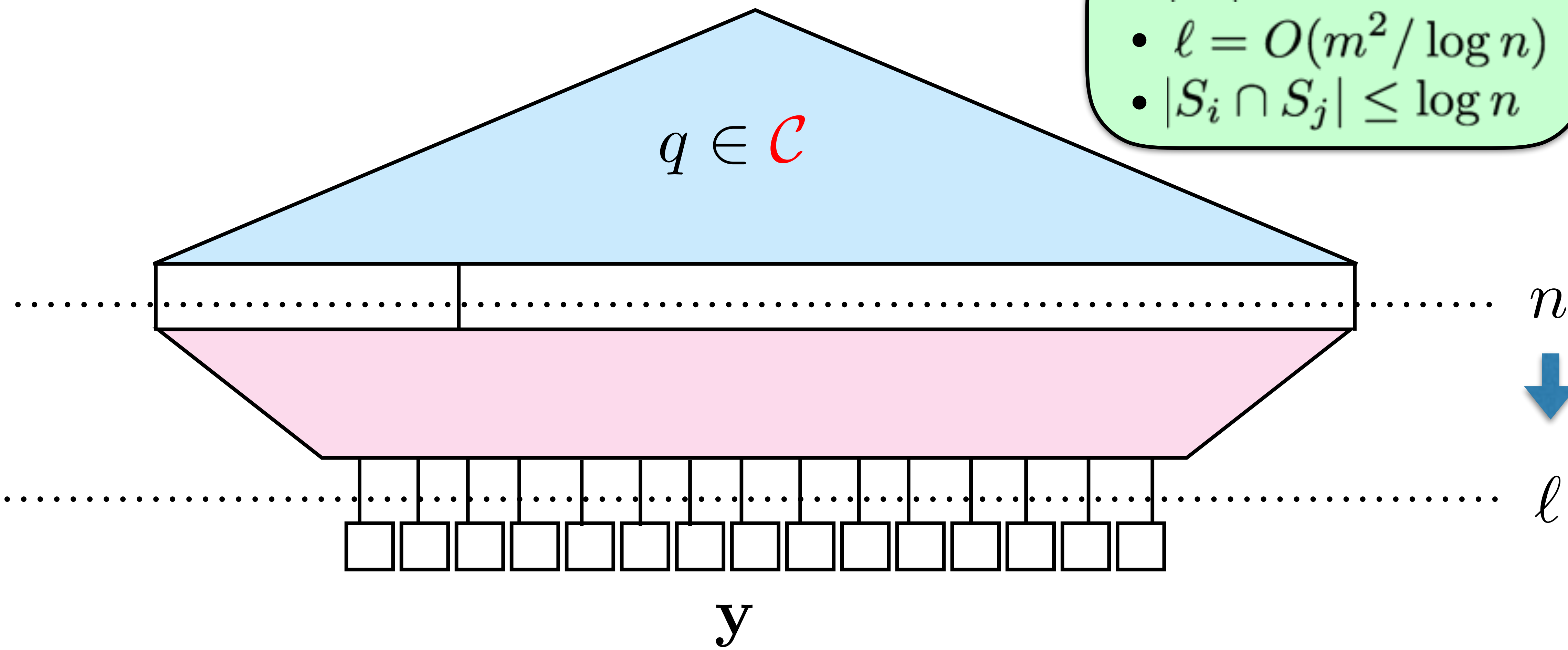

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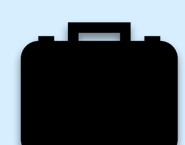
$\ell = o(n) \quad \Rightarrow \quad 2^{o(n)}$ time  PIT

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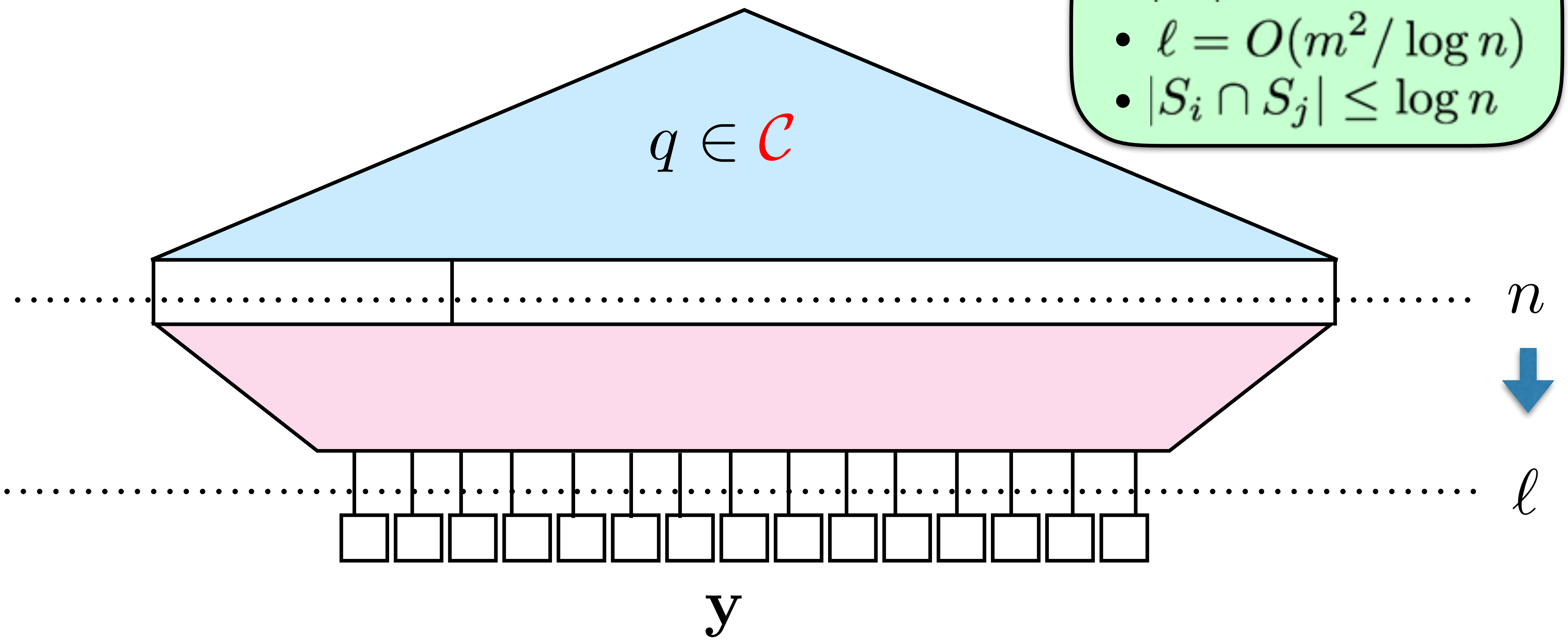
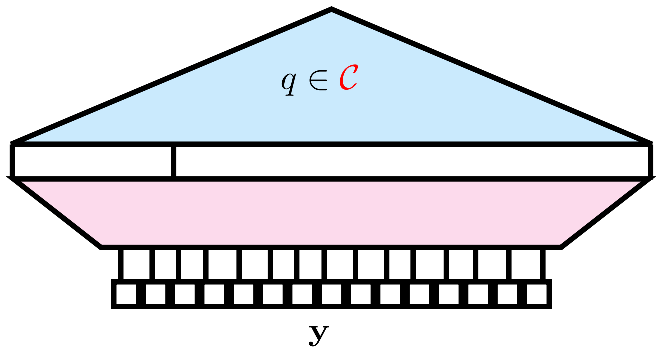


$$Q(\mathbf{y}) = q\big(f(\mathbf{y}|_{S_1}), f(\mathbf{y}|_{S_2}), \ldots, f(\mathbf{y}|_{S_n})\big)$$

Want: If $q \neq 0$, then $Q \neq 0$.

Key Lemma



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Goal: If $q \not\equiv 0$, then $Q \not\equiv 0$.

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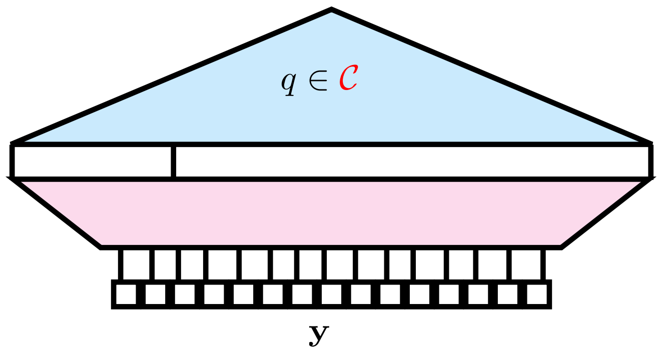

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Suppose $Q \equiv 0$, then f can be computed by a small circuit.

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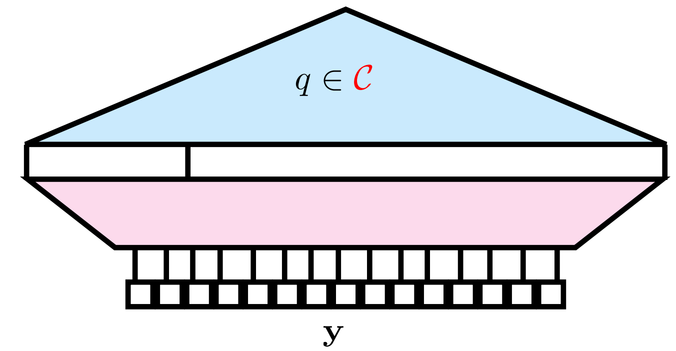

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Suppose $Q \equiv 0$, then f can be computed by a small circuit.

To show f having a small circuit, we need **polynomial factorization**!

Proof Sketch of the Key Lemma

$\exists q \in \mathcal{C},\ Q \equiv 0$

$\Downarrow$

f has a small circuit

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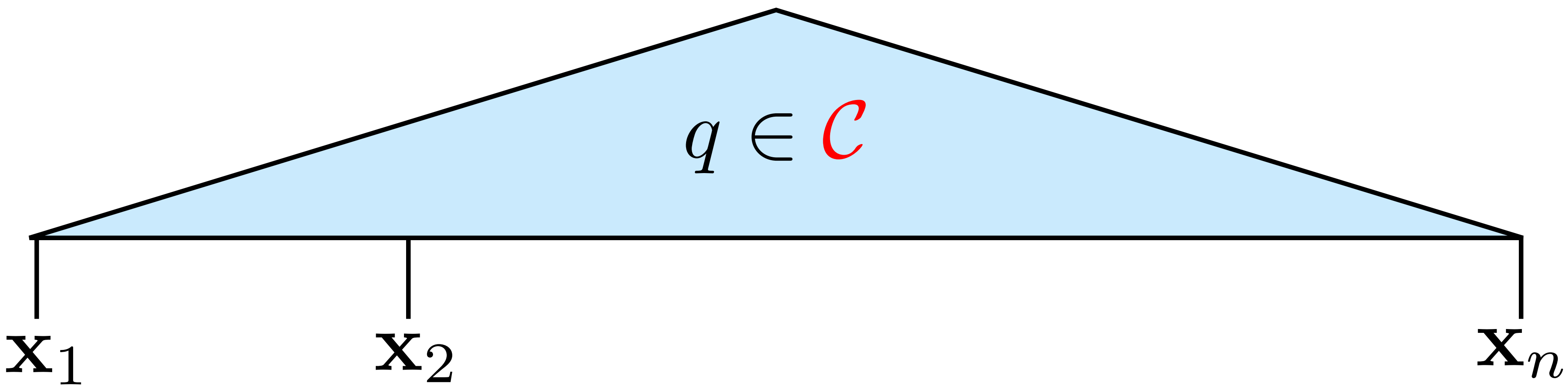

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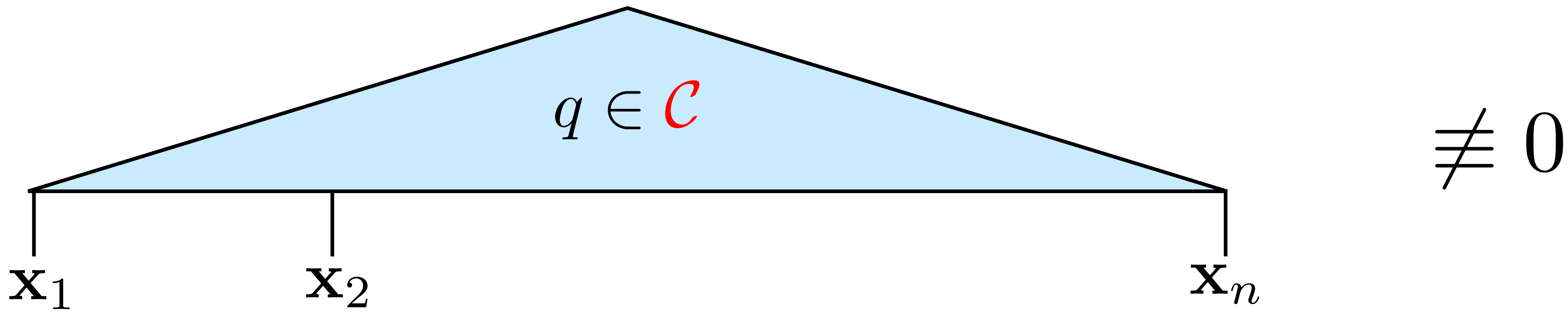

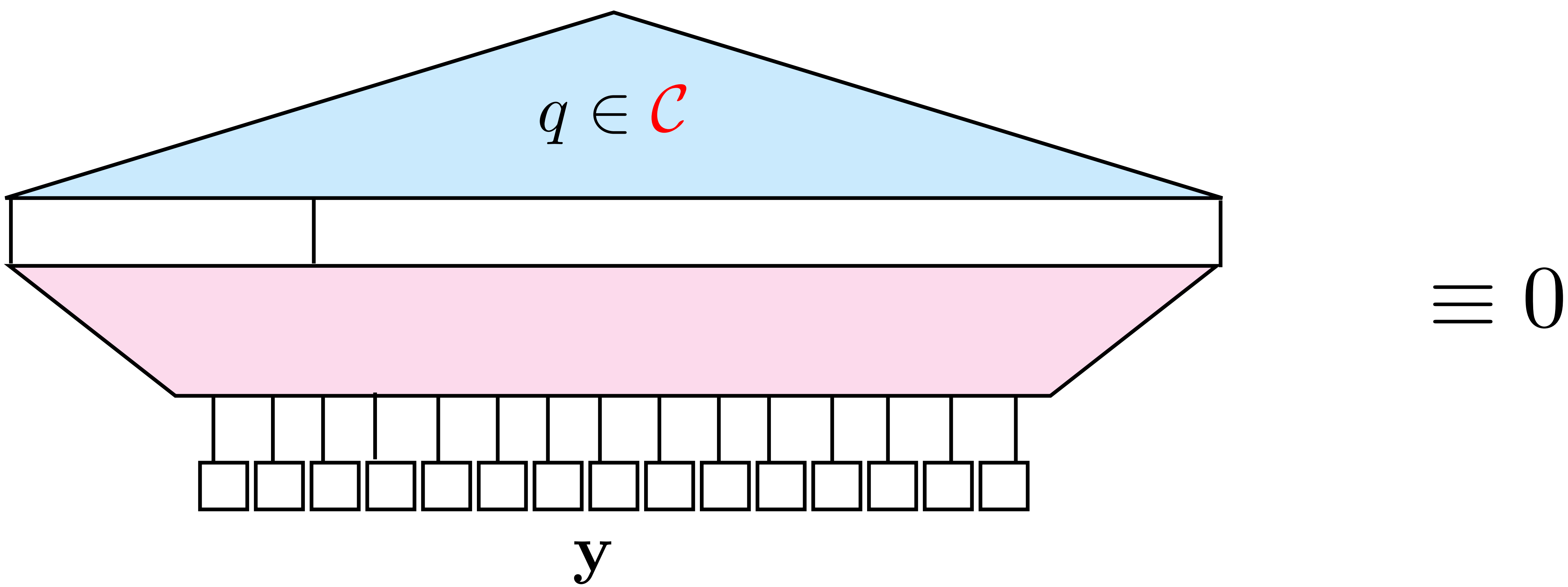

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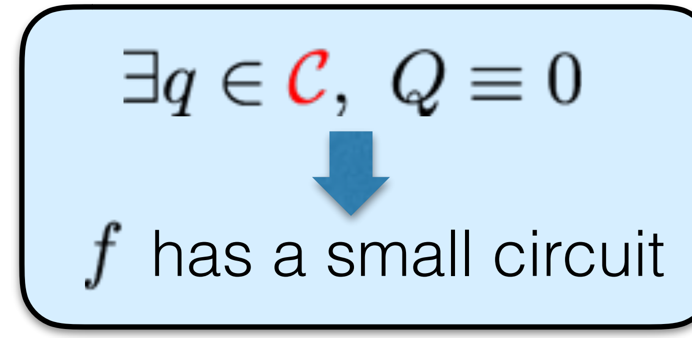

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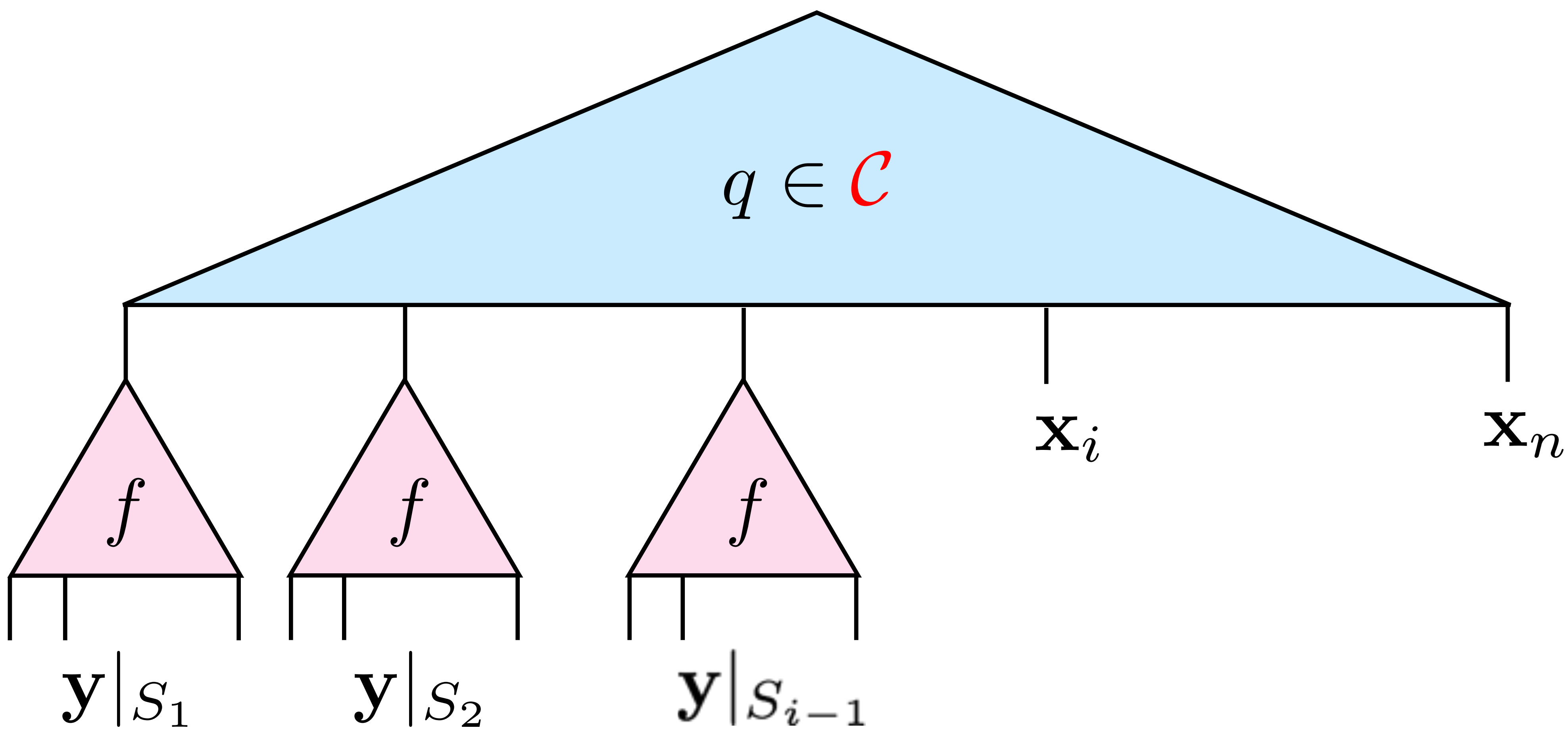

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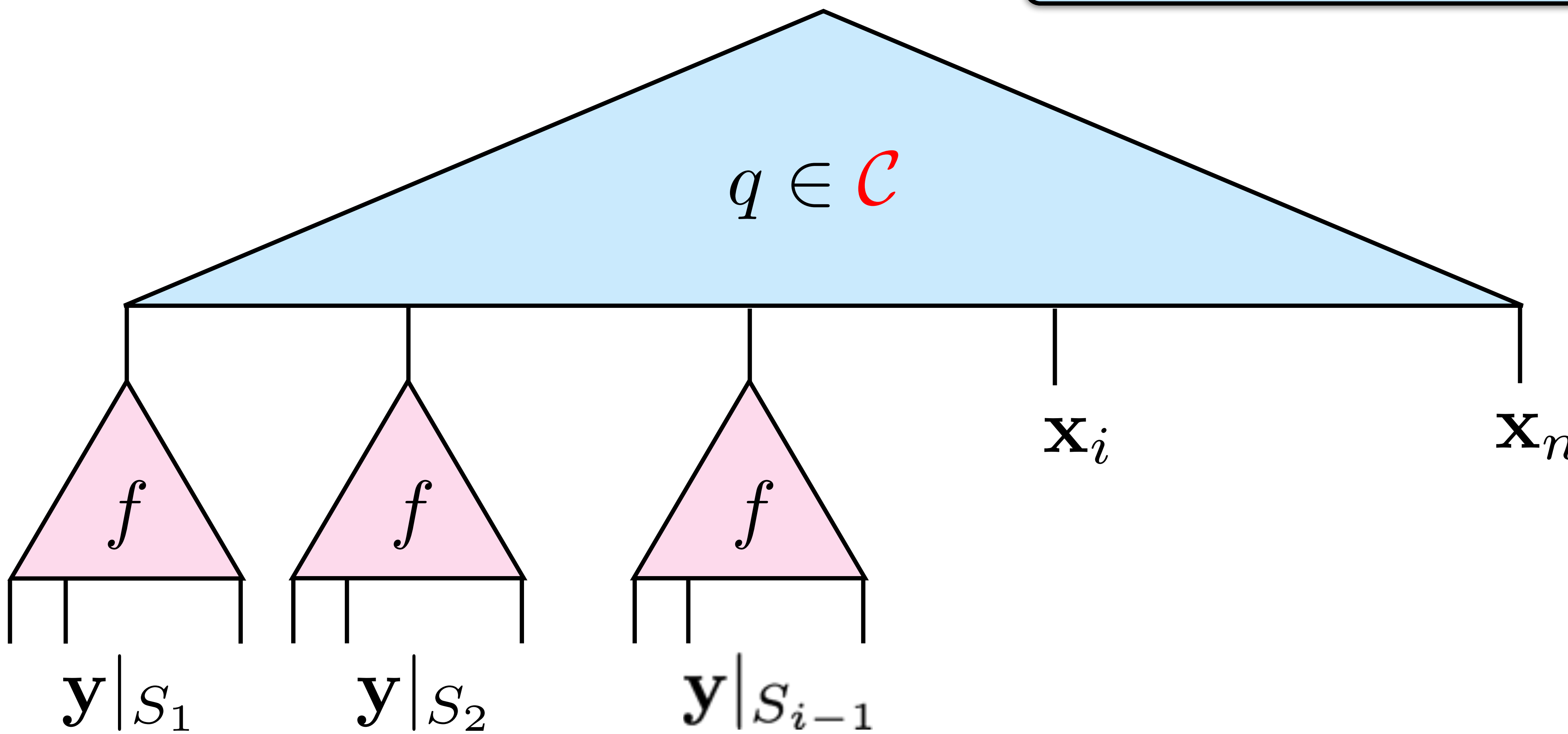

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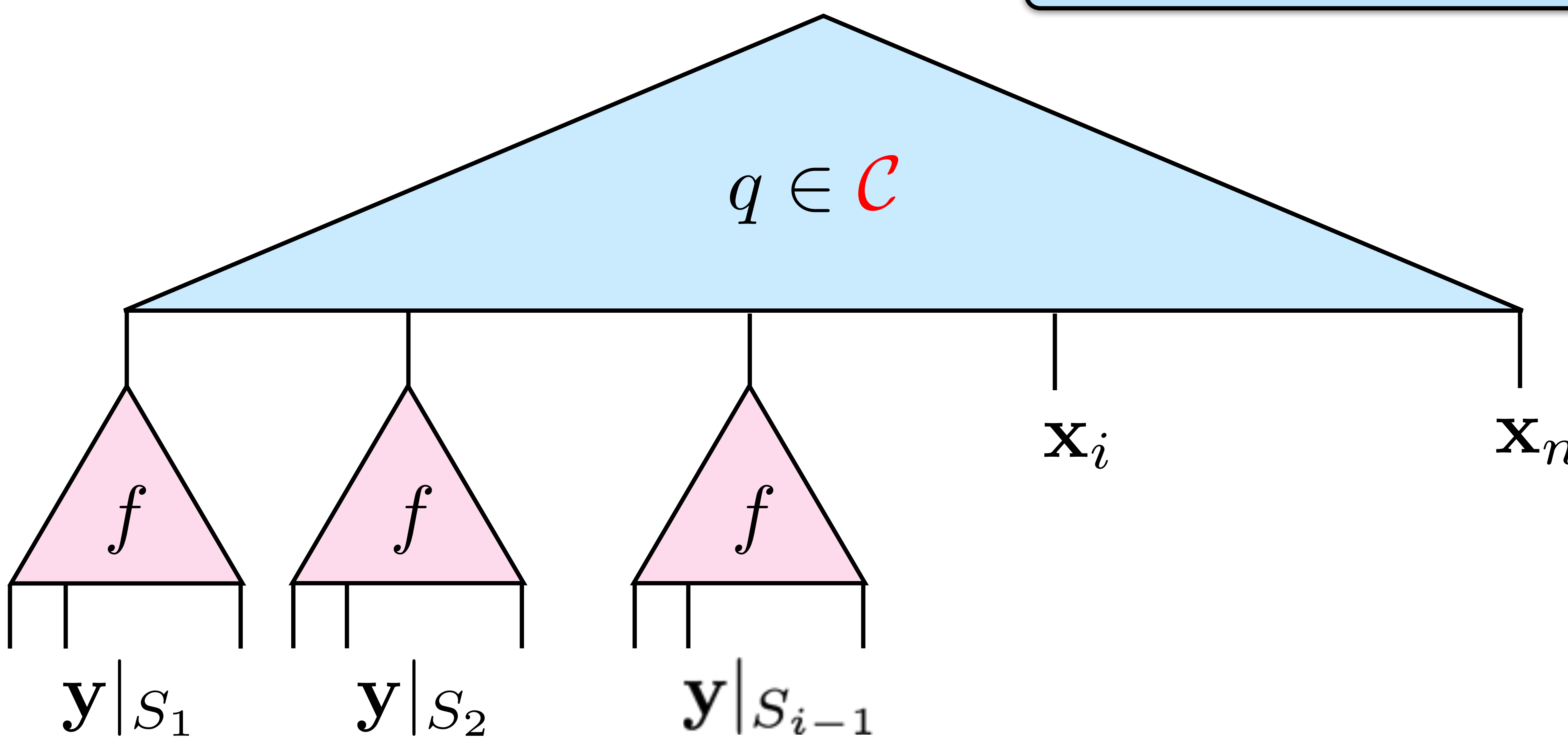

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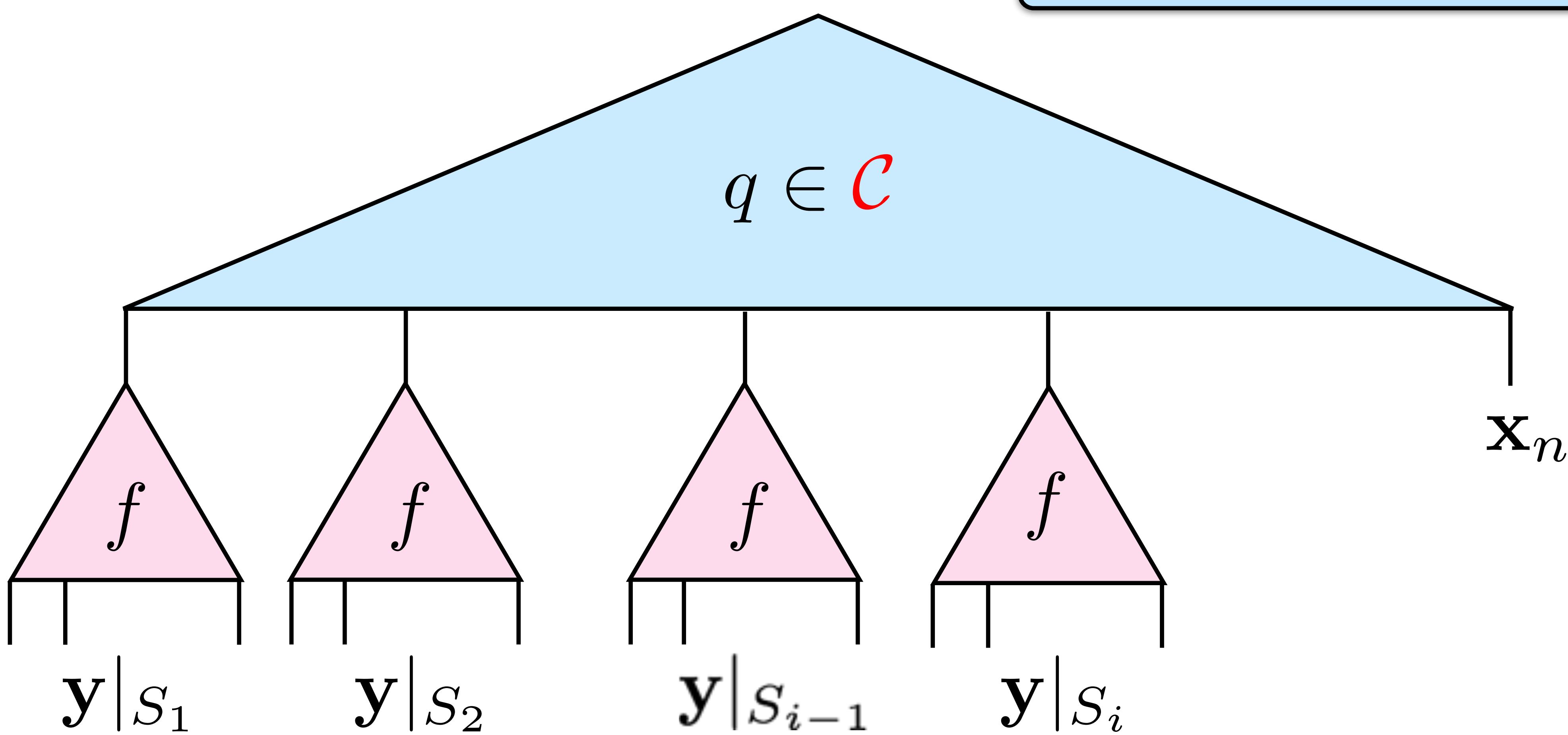

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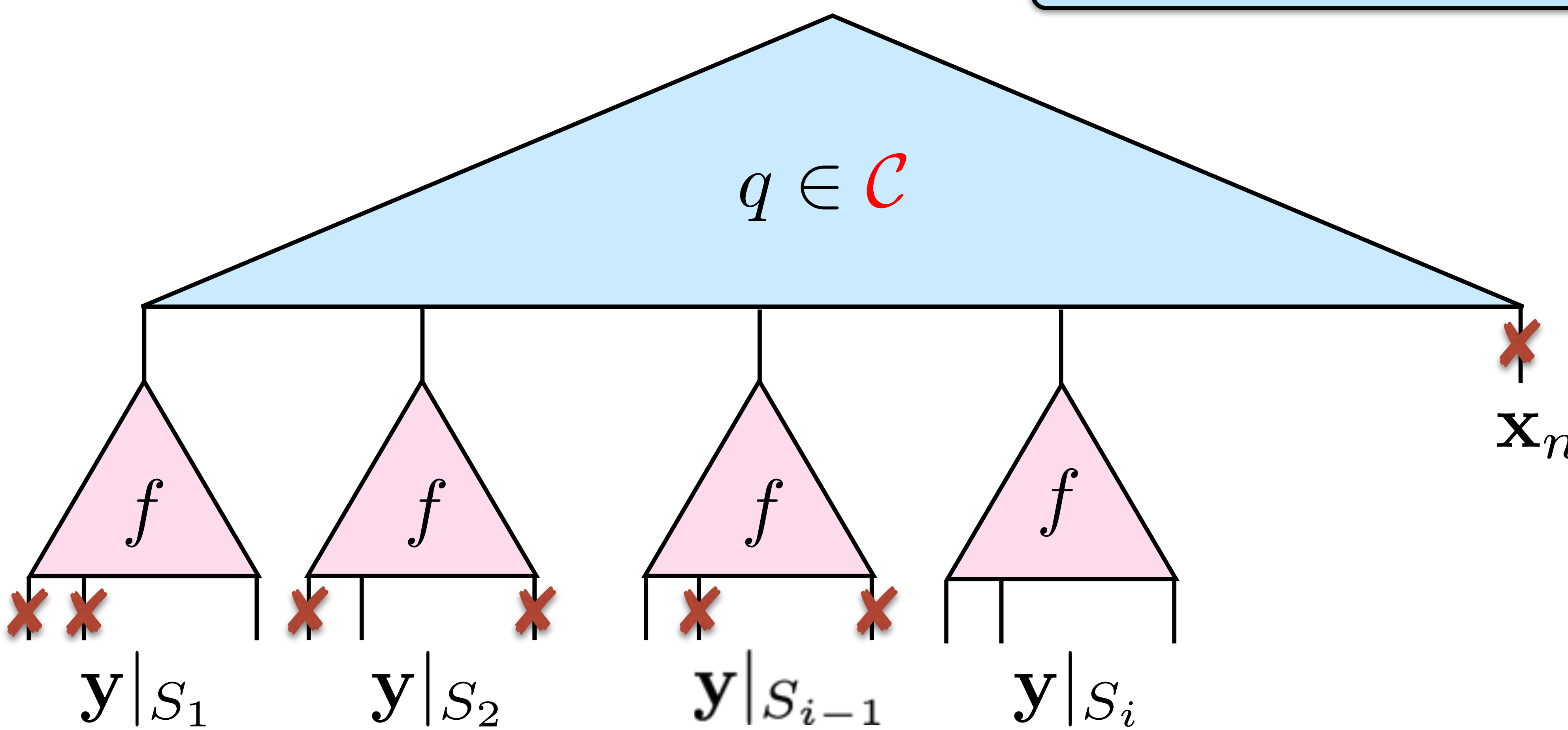

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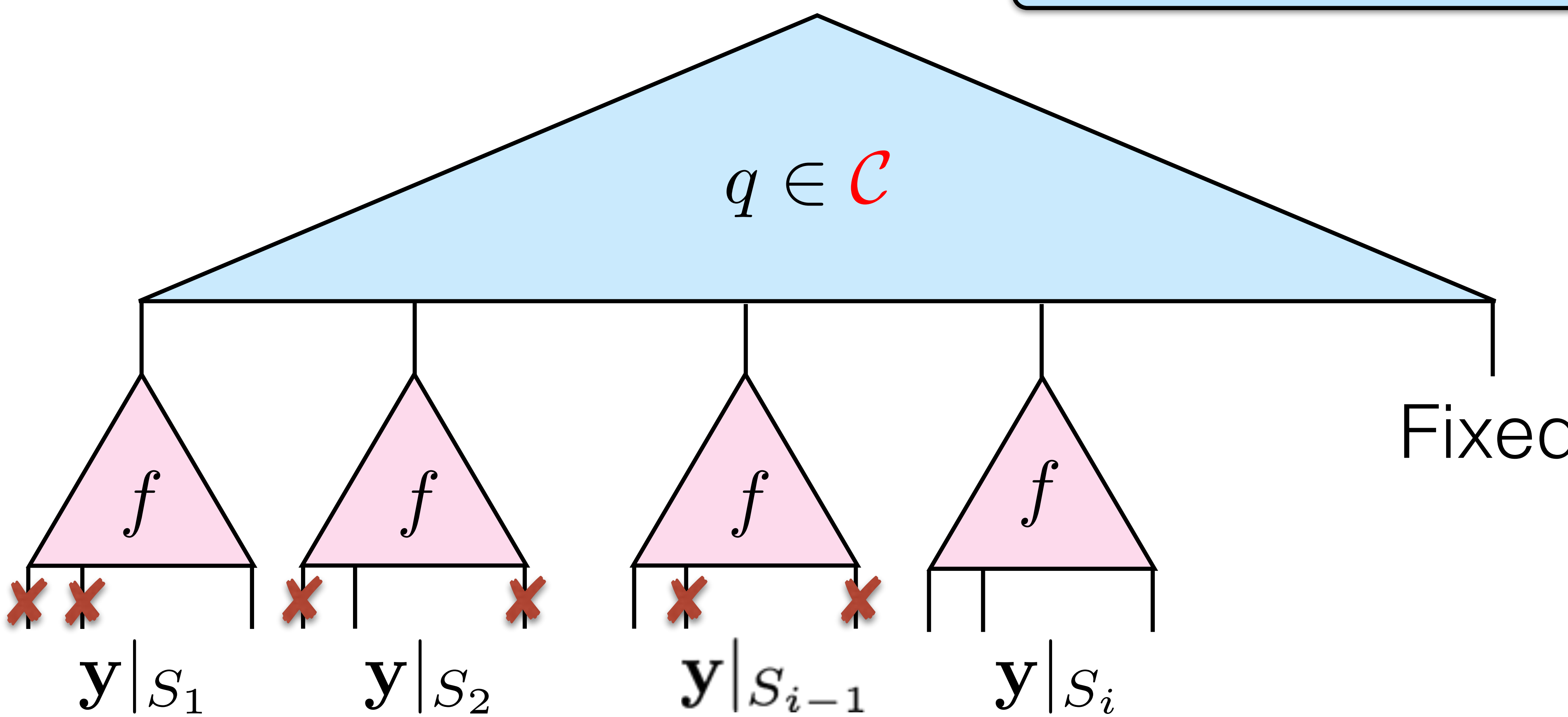

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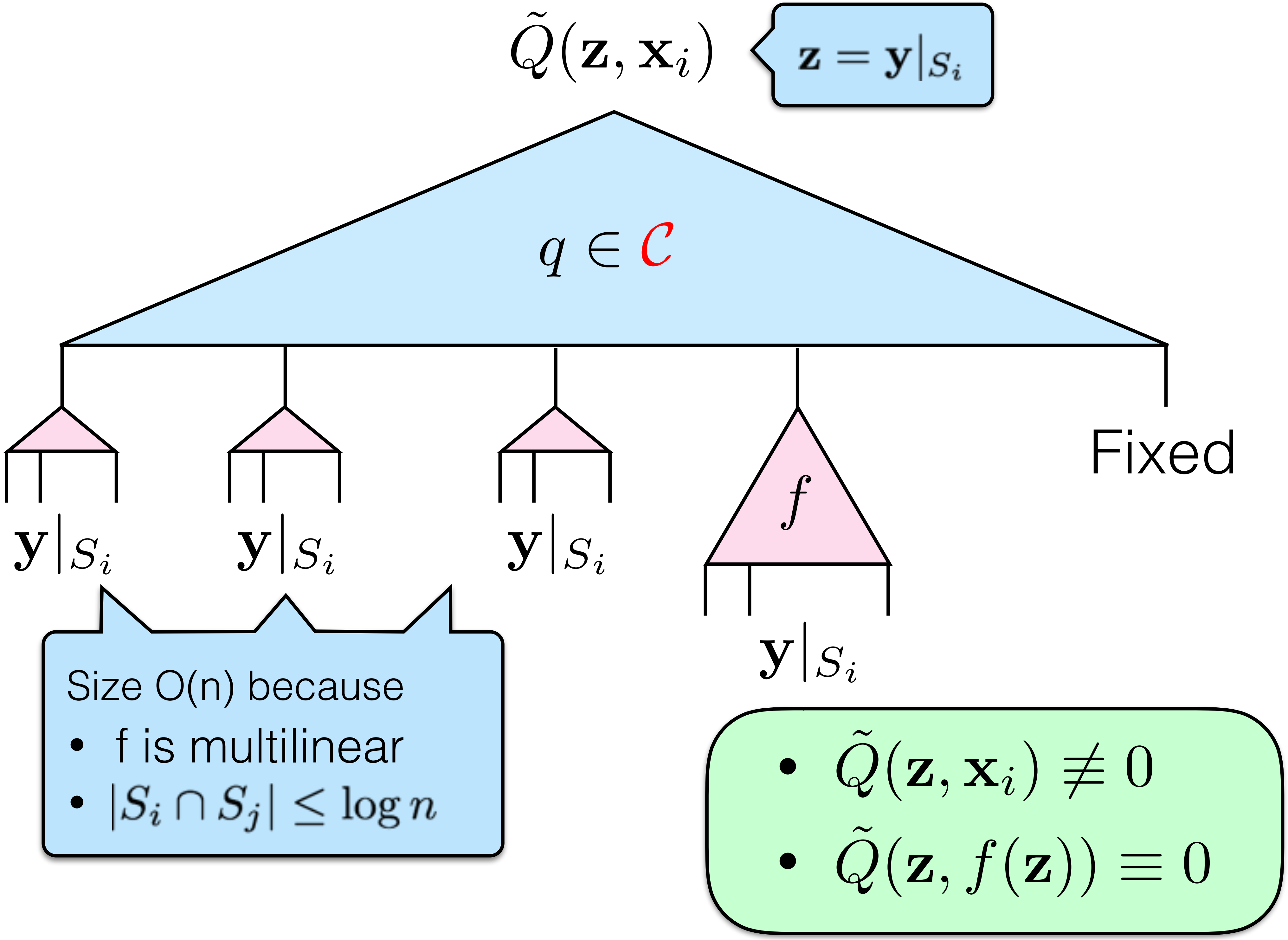

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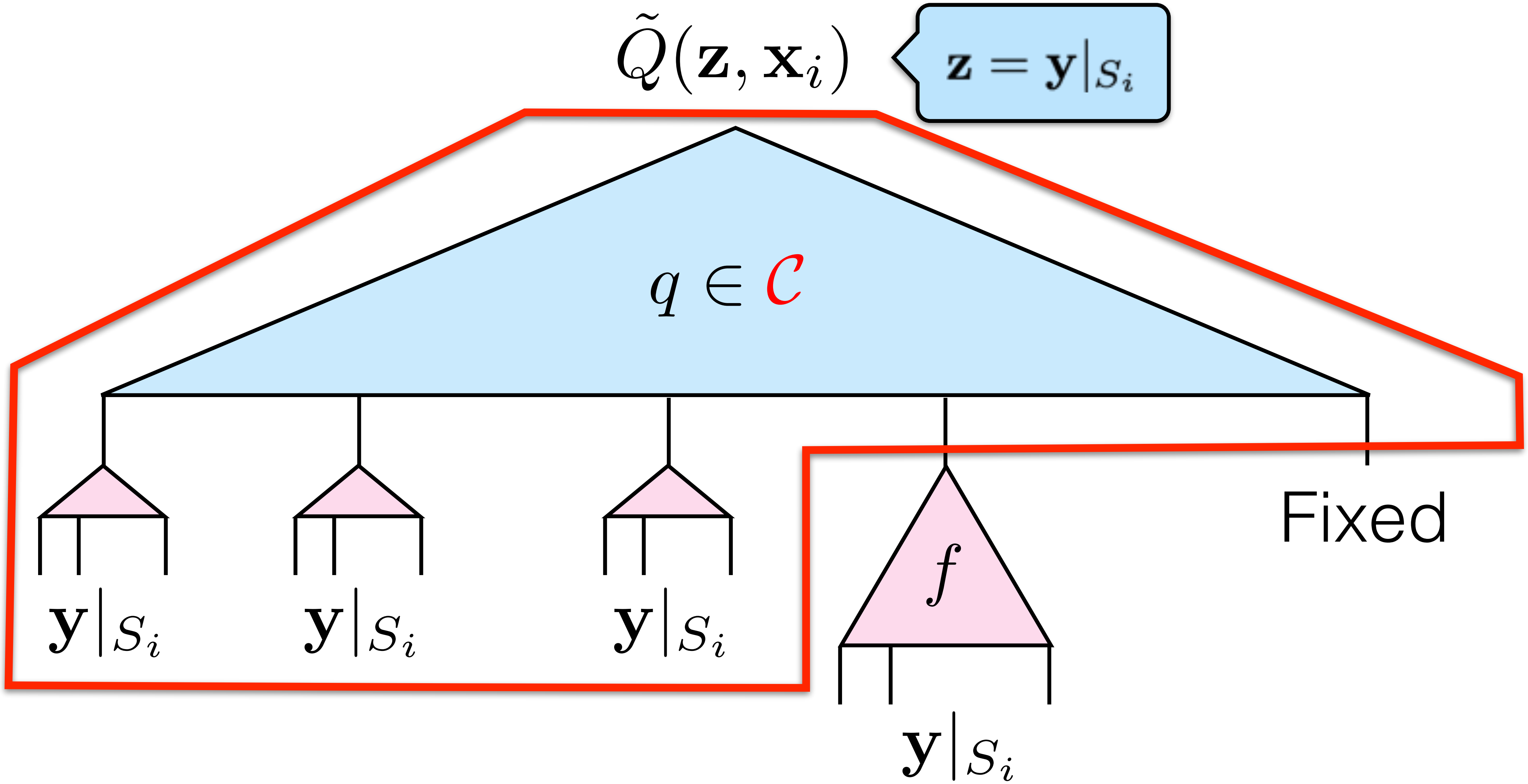

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Reducing to **polynomial factorization**!

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Polynomial Factorization

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Goal: For any $P(\mathbf{z}, y) \in \mathcal{C}$ such that $P(\mathbf{z}, f(\mathbf{z})) = 0$.
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Induction hypothesis:
Maintain $h_i \in \mathrm{VP}$ for $i = 0, 1, \ldots, d = \deg(f)$ such that

$$\mathcal{H}_{\leq i}[h_i] = \mathcal{H}_{\leq i}[f].$$

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Want $h_i - h_{i-1} = \mathcal{H}_i[f - h_{i-1}]$.

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In the following,
we ignore $\mathbf{z}$ for simplicity.

Lemma: $\mathcal{H}_i[f - h_{i-1}] = -\dfrac{\mathcal{H}_i[P(\mathbf{z}, h_{i-1}(\mathbf{z}))]}{\delta}$.

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&= P(h_{i-1}) + \frac{\partial}{\partial y} P(h_{i-1}) \cdot (f - h_{i-1}) \\
&+ \frac{\partial^2}{\partial y^2} P(h_{i-1}) \cdot (f - h_{i-1})^2 + \cdots
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Lemma: $\mathcal{H}_i[f - h_{i-1}] = -\dfrac{\mathcal{H}_i[P(\mathbf{z}, h_{i-1}(\mathbf{z}))]}{\delta}.$

By induction $\deg(f - h_{i-1}) > i - 1.$

Pf: By Taylor's expansion.

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$$\frac{\partial P(\mathbf{0}, f(\mathbf{0}))}{\partial y} = \delta \neq 0.$$

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Extend to Other Circuit Classes?

	$\mathcal{C}$	$\mathcal{C}'$
[Kal89]	VP	VP
[DSY09]	$\text{Depth-}\Delta$ with bounded individual degree	$\text{Depth-}\Delta + 3$
[DSS18, **C**KS18]	$\mathrm{VF}(n^{\log n})$ $(\text{resp. } \mathrm{VBP}(n^{\log n}), \mathrm{VNP}(n^{\log n}))$	$\mathrm{VF}(n^{\log n})$ $(\text{resp. } \mathrm{VBP}(n^{\log n}), \mathrm{VNP}(n^{\log n}))$
[**C**KS18]	$\text{Depth-}\Delta$ with degree $O(\log^2 n / \log^2 \log n)$	$\text{Depth-}\Delta + 3$
[**C**KS18]	VNP	VNP

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Issue: How to efficiently implement the **update step**?

Recall that we have

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Key: Avoid **recursion**!

Generator Lemma [DSY09, **C**KS18]

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Individual degree

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$\{C_i\}$ can be **reused** and **efficiently extracted** from P!

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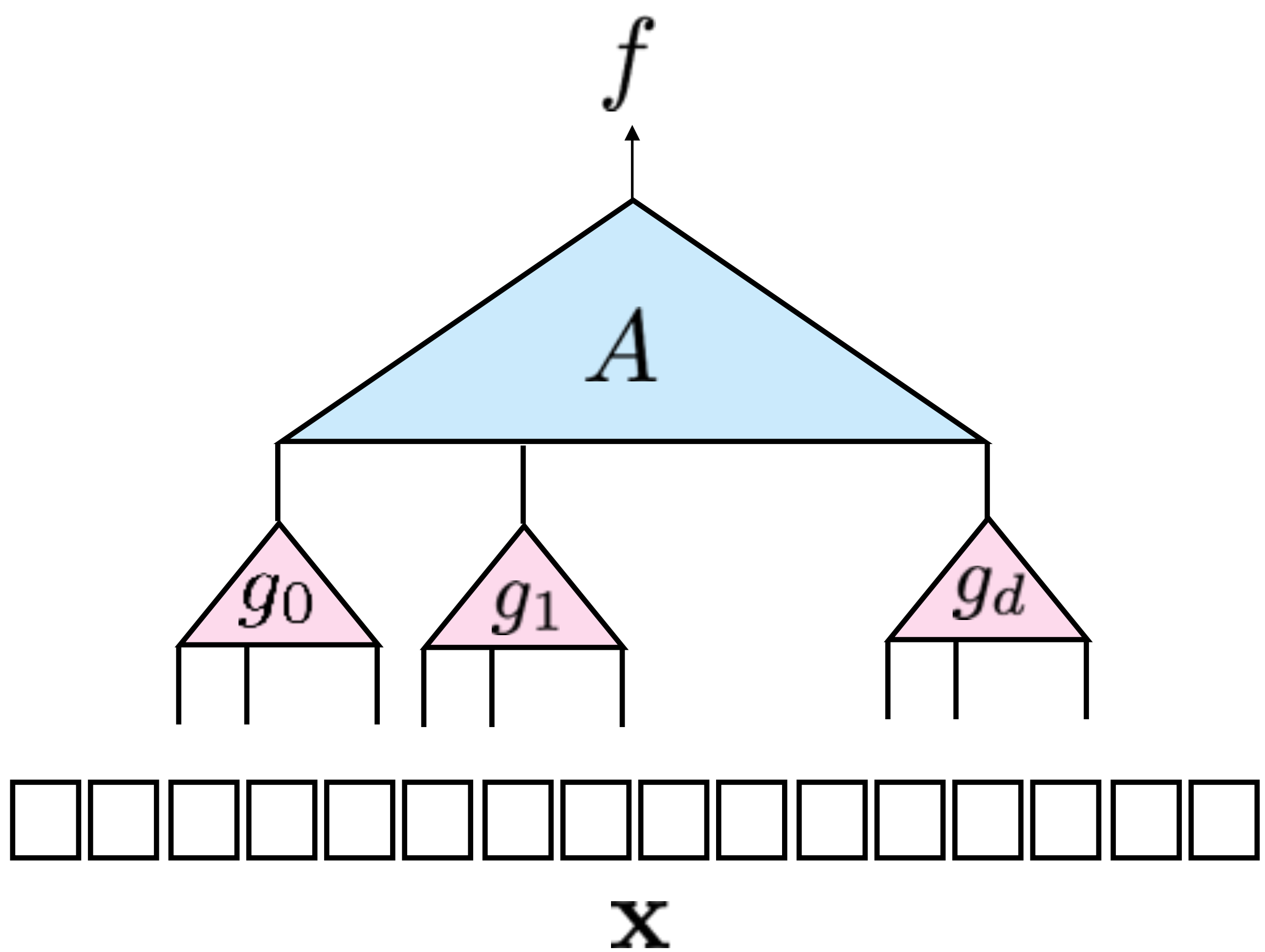

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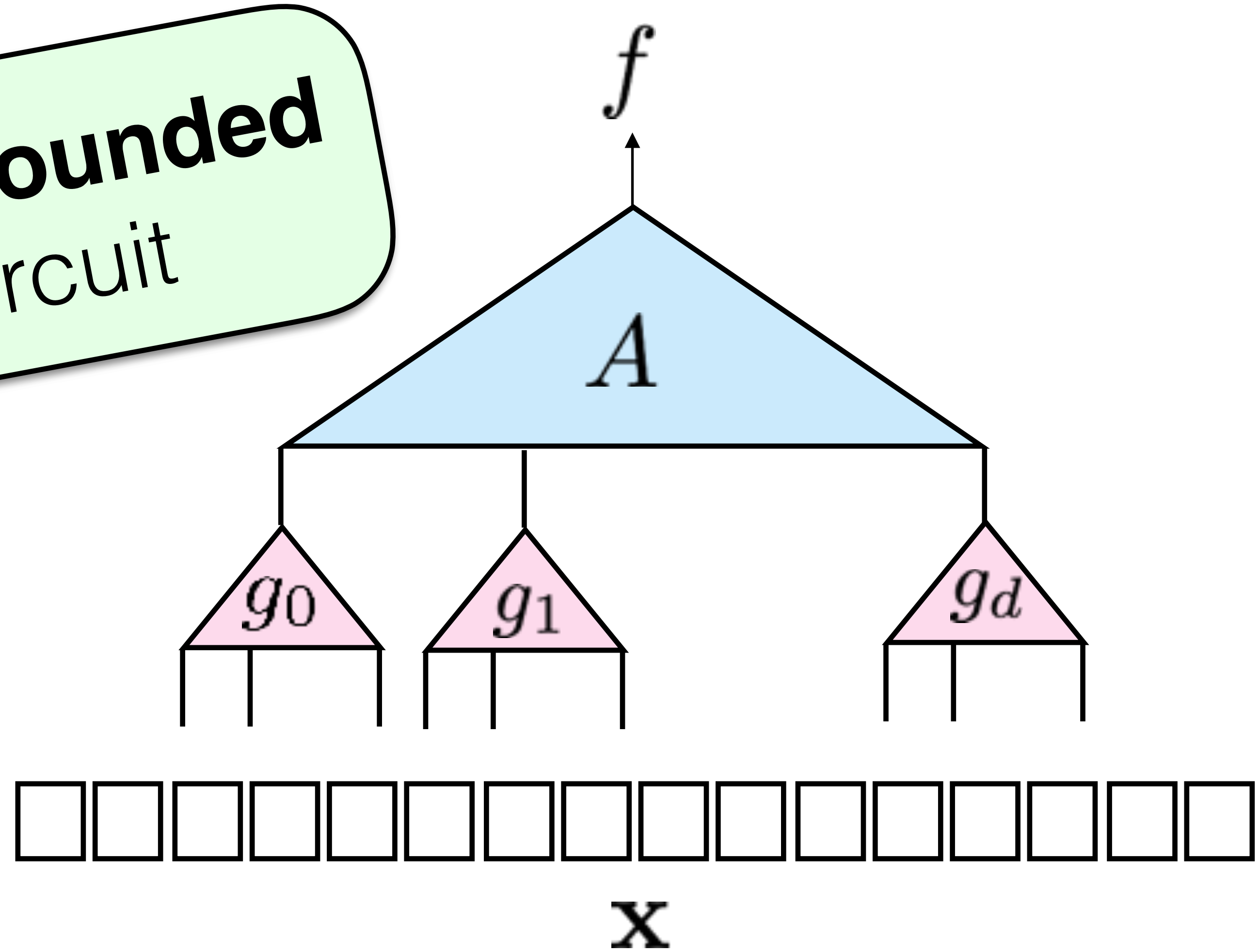
Closure for **bounded depth** circuit



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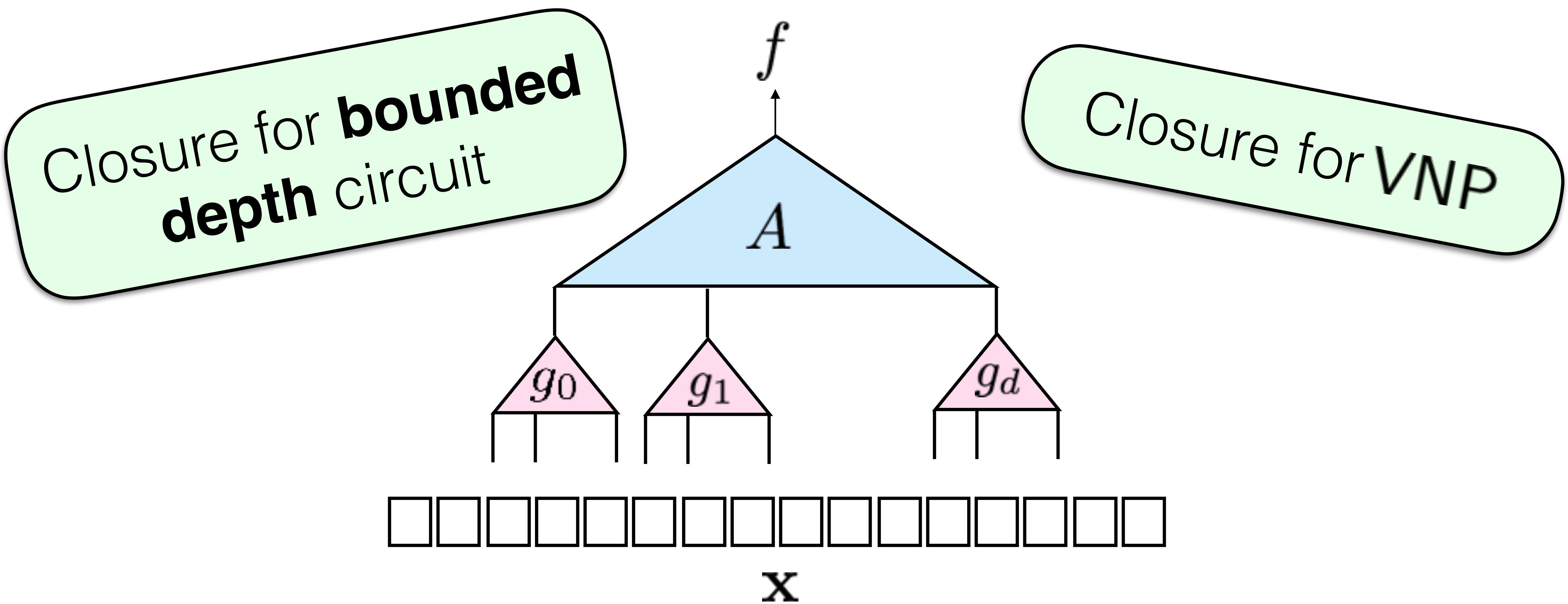
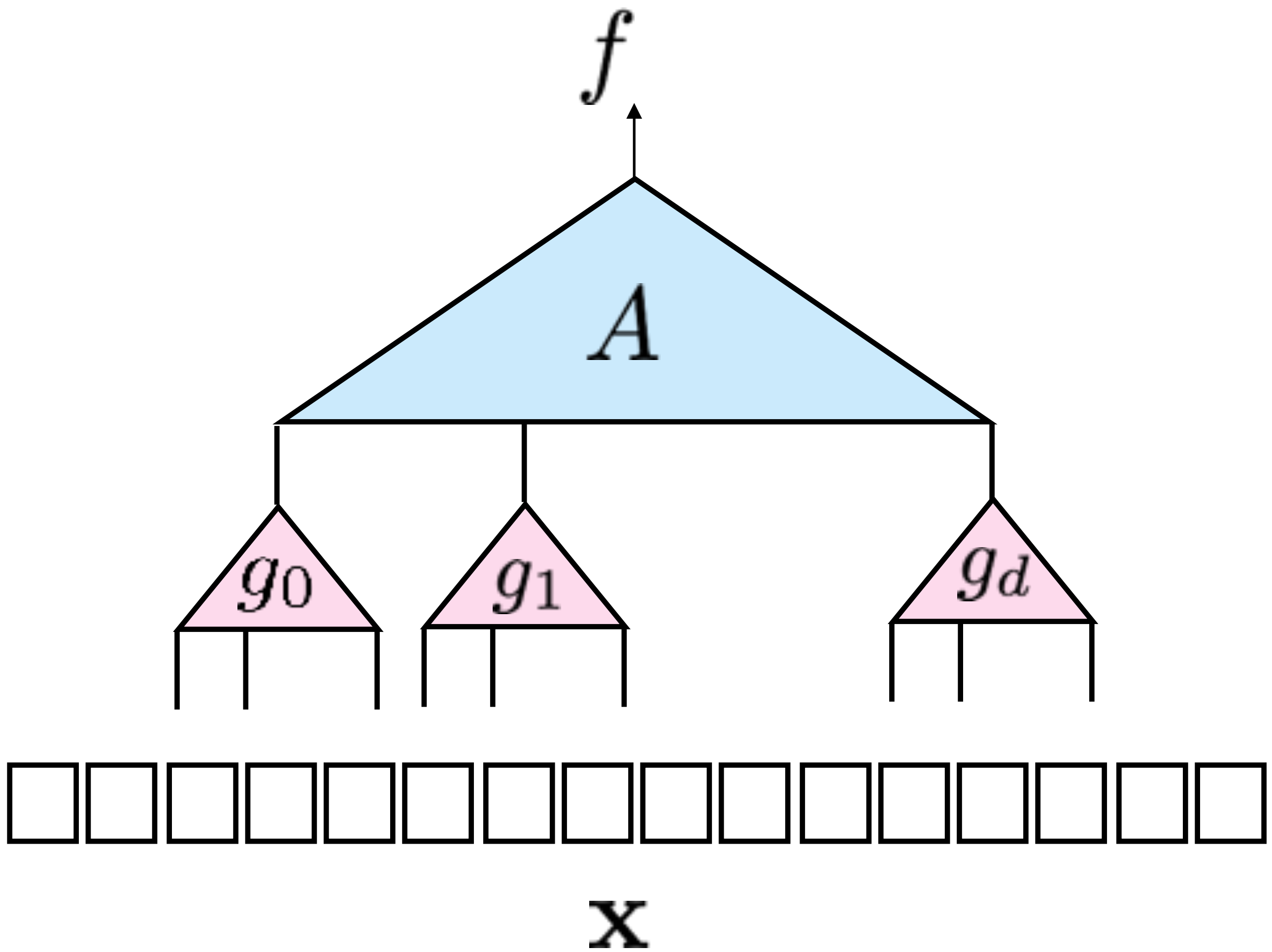


Closure for Bounded Depth Circuits [**C**KS18]



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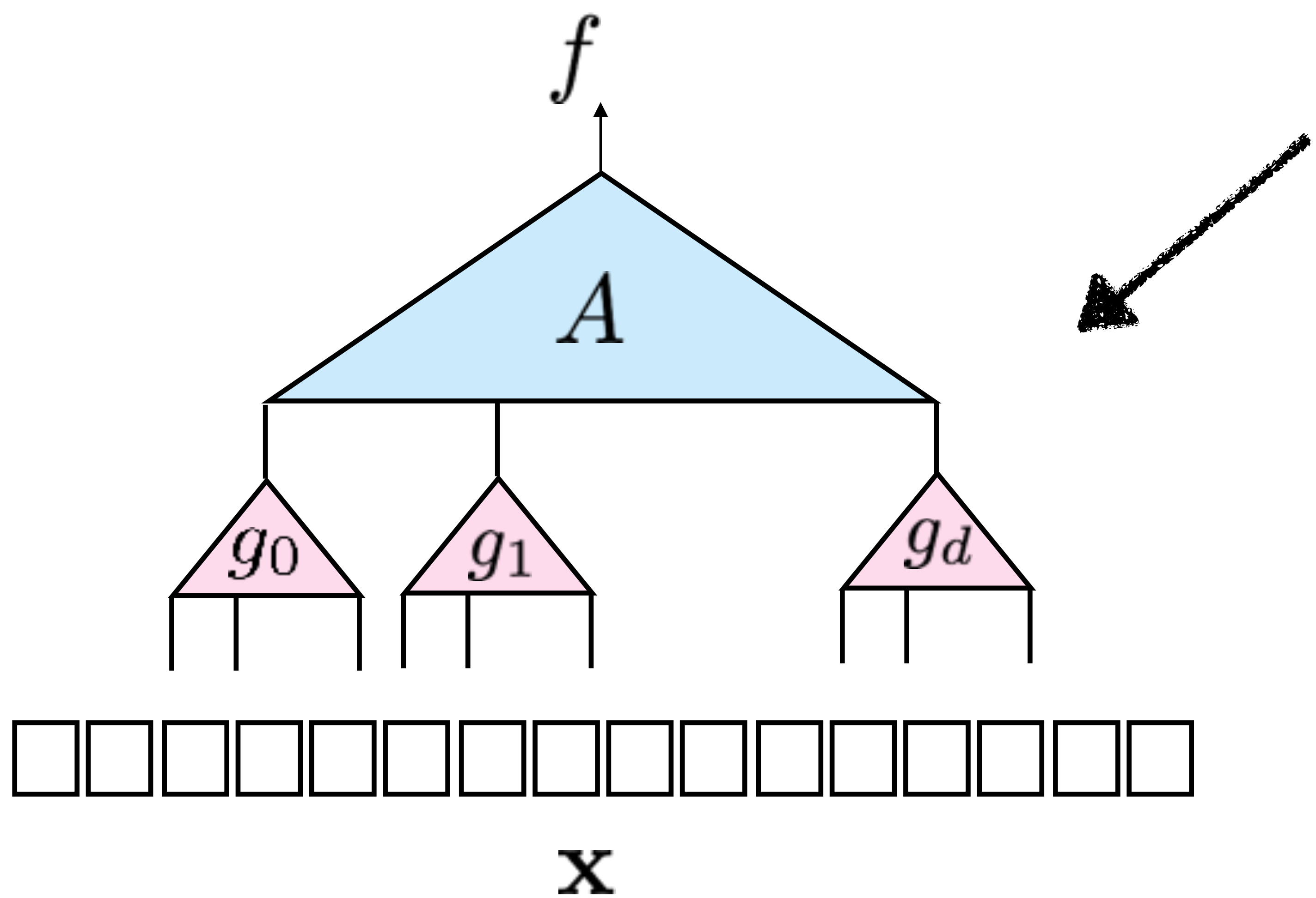
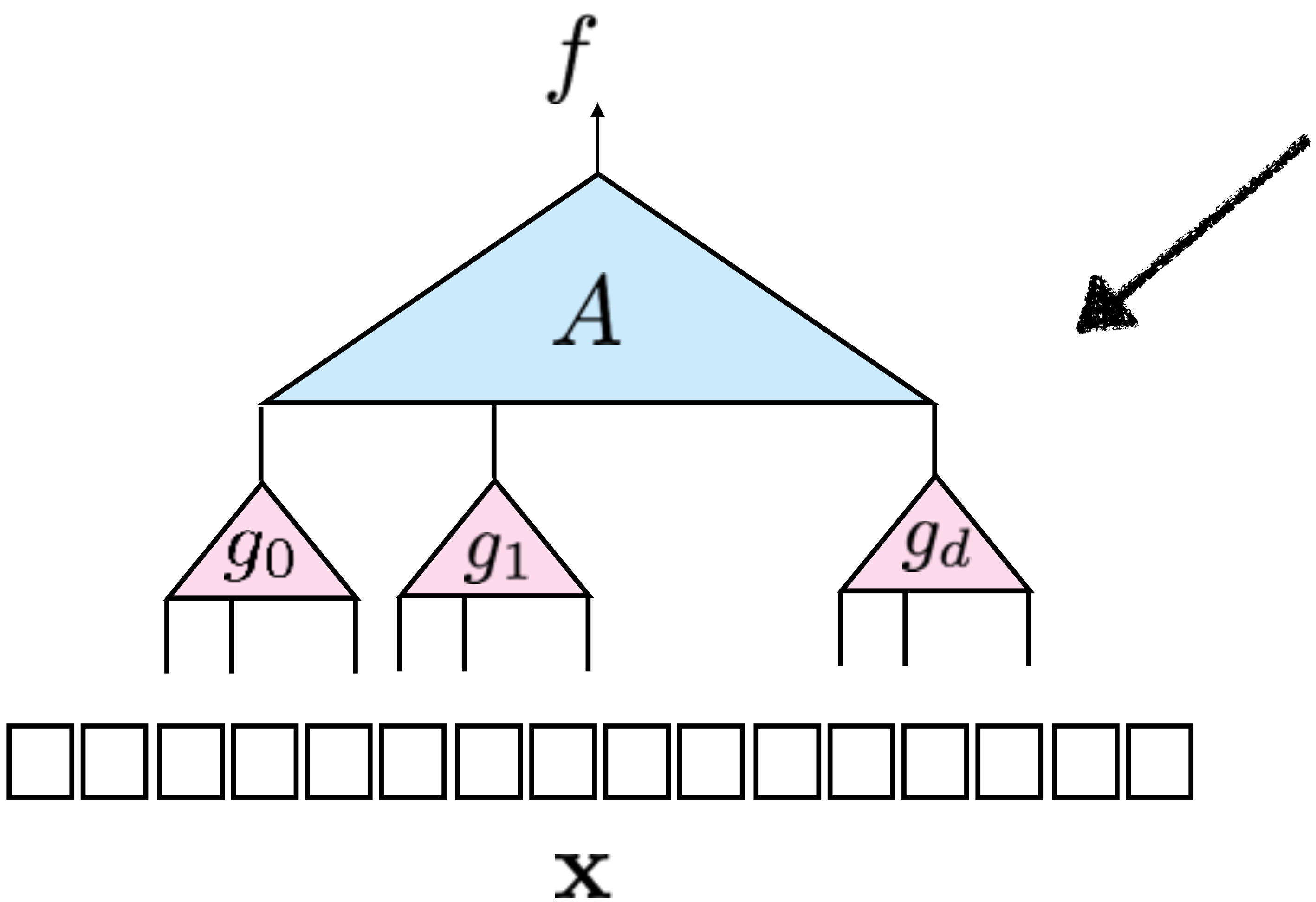


Depth reduction [GKKS13]

Closure for Bounded Depth Circuits [**C**KS18]



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- size s
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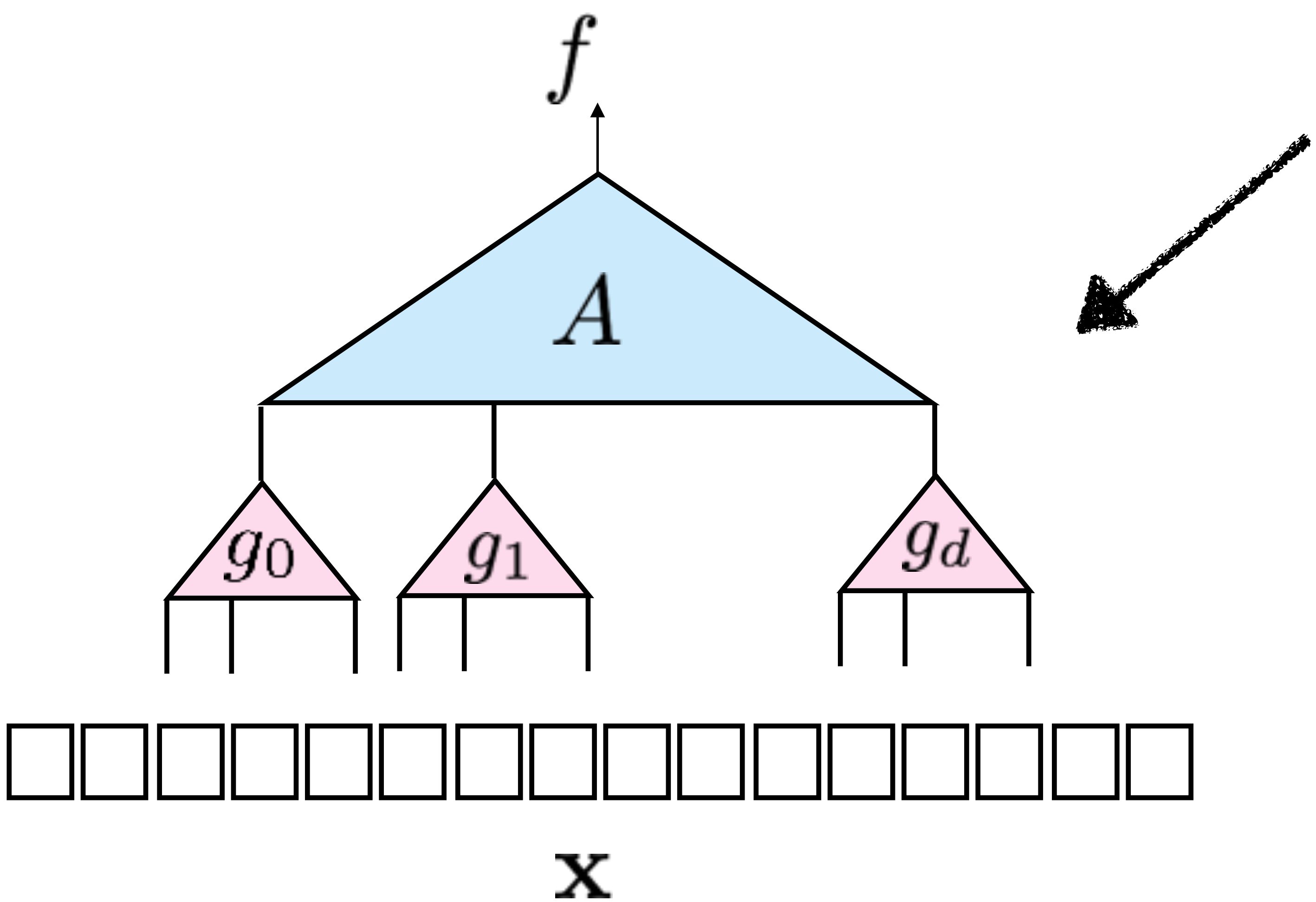
Closure for Bounded Depth Circuits [**C**KS18]



Depth reduction [GKKS13]

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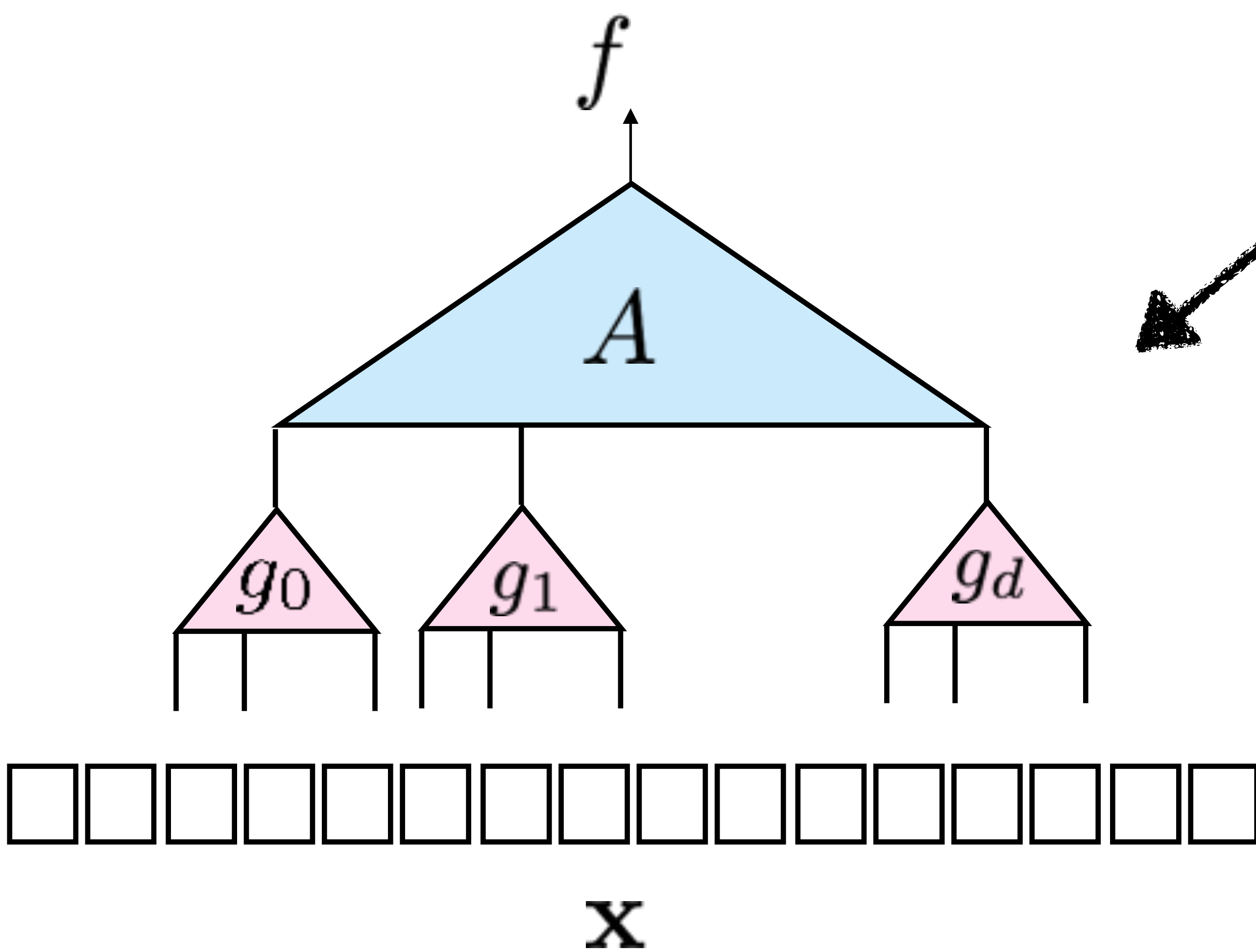
⬇

- size $(snd)^{O(\sqrt{d})}$
- depth 3, *i.e.*, $\Sigma\Pi\Sigma$

Closure for Bounded Depth Circuits [**C**KS18]



Depth reduction [AV08, Koi12, Tav15]

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↓

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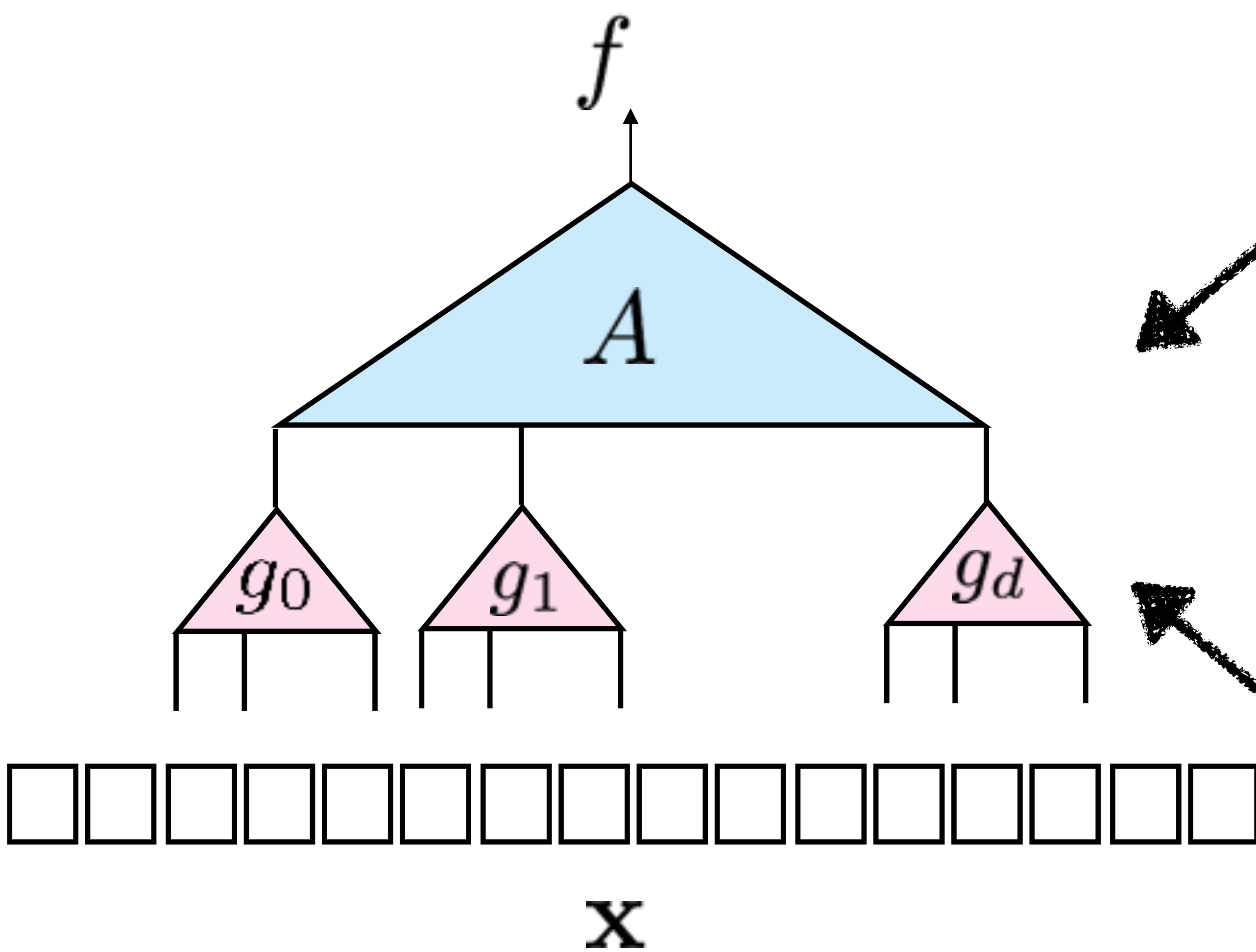
Closure for Bounded Depth Circuits [**C**KS18]



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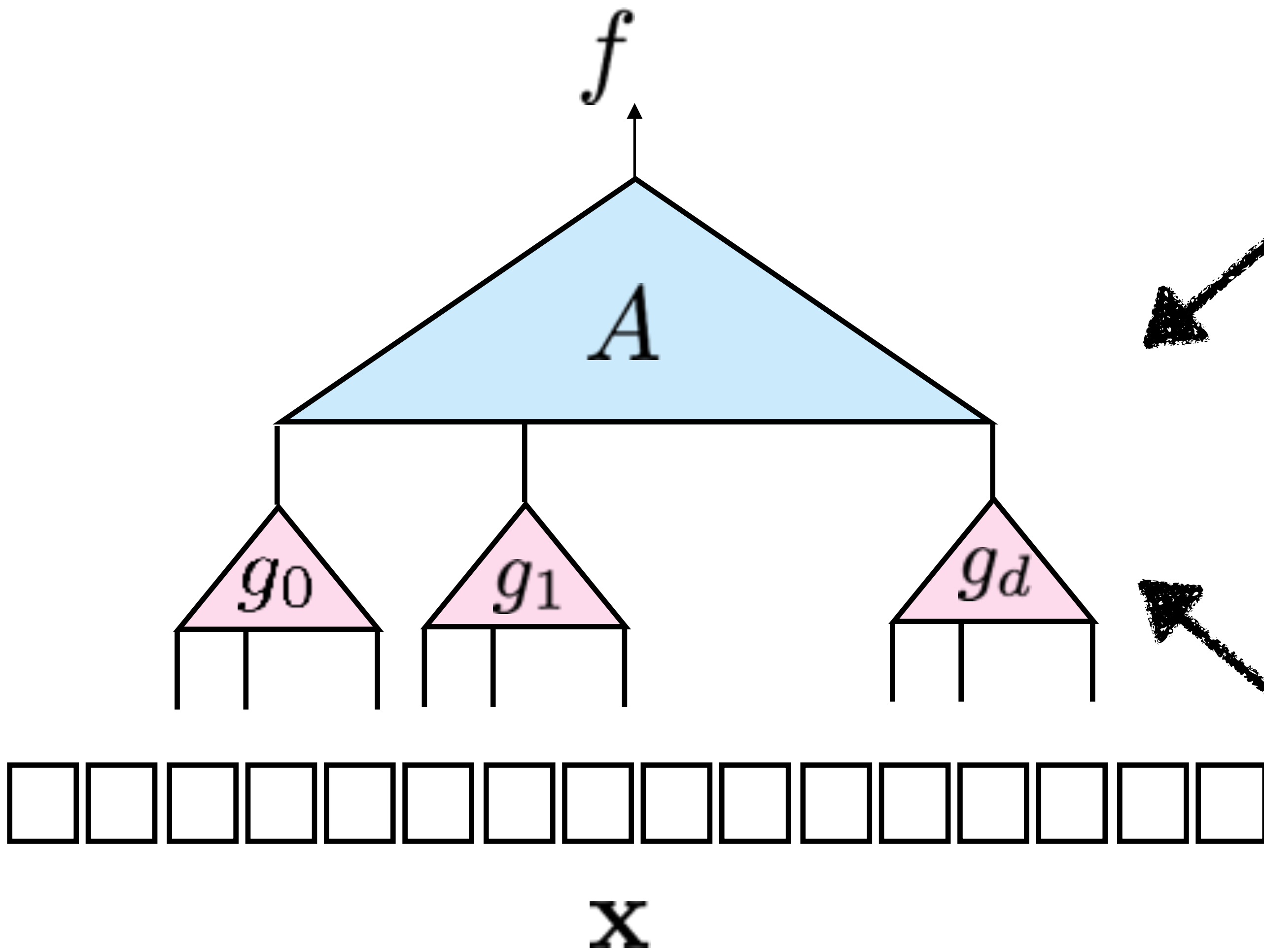

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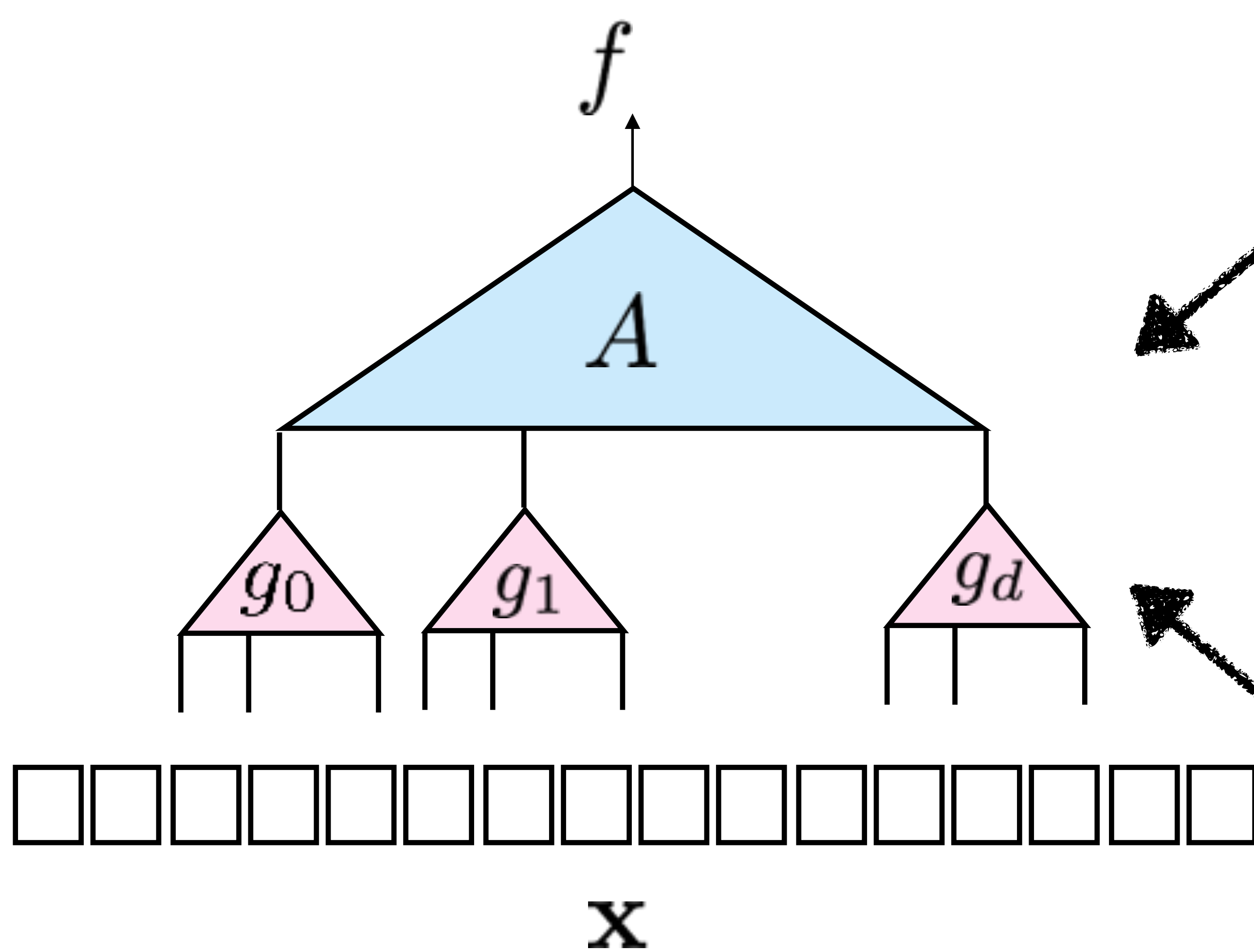

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The factor f has a small bounded depth circuit!

Closure for VNP [**C**KS18]

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Theorem (Valiant): For every f having circuit of size s and degree d, there's **formula** Q of size poly(s, d) s.t.

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Formula is useful for **composing** exponential sum!

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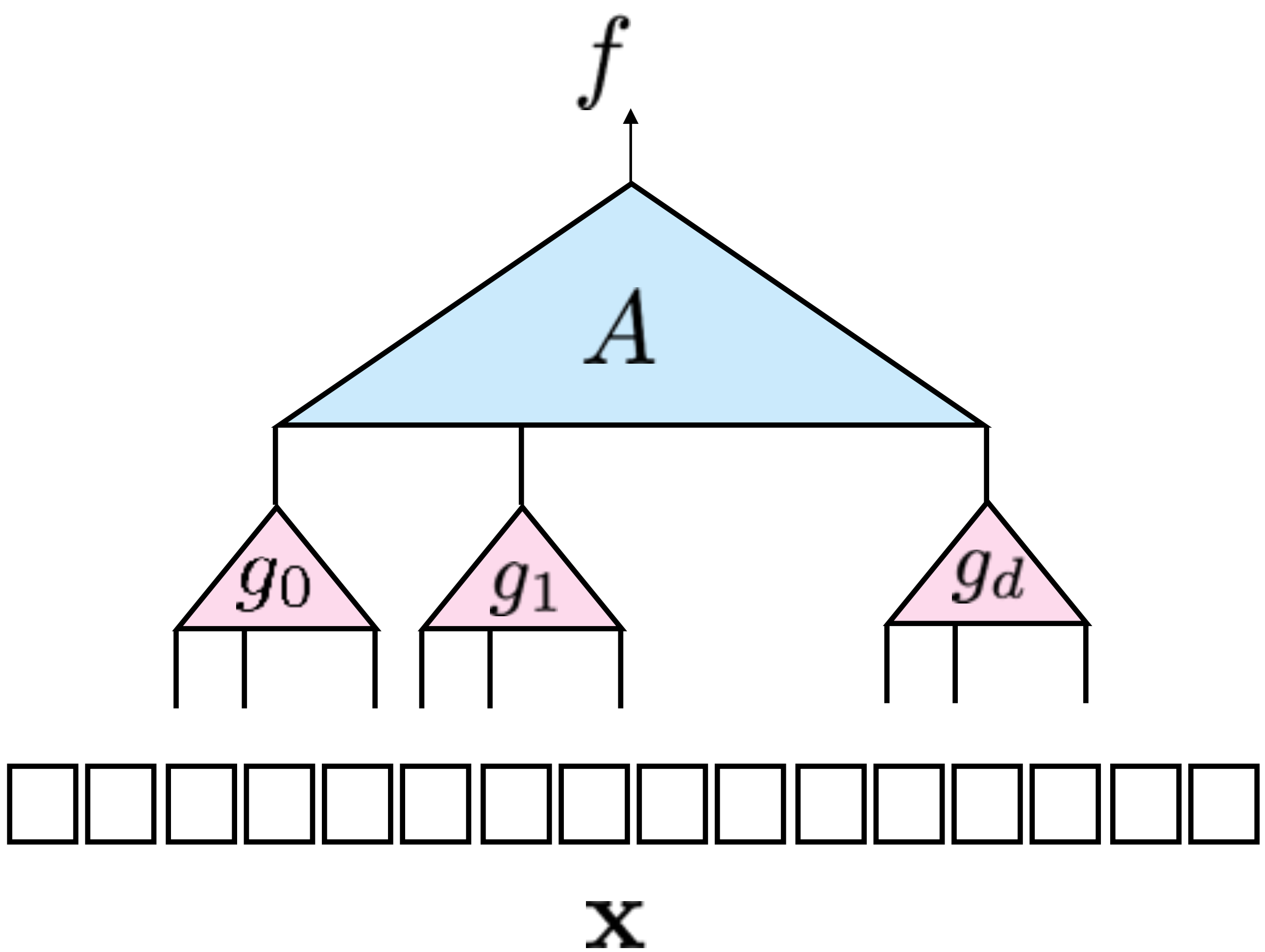

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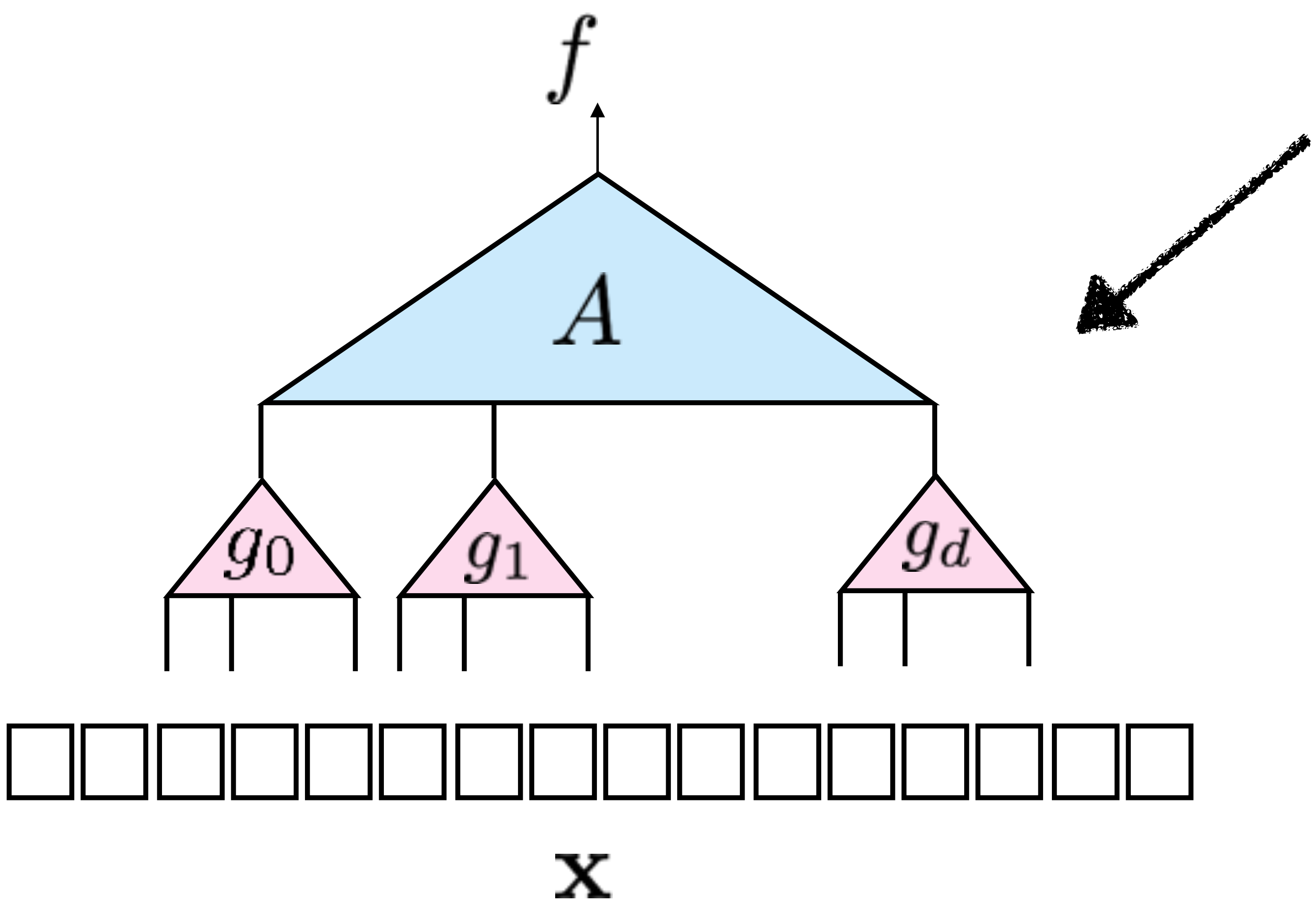
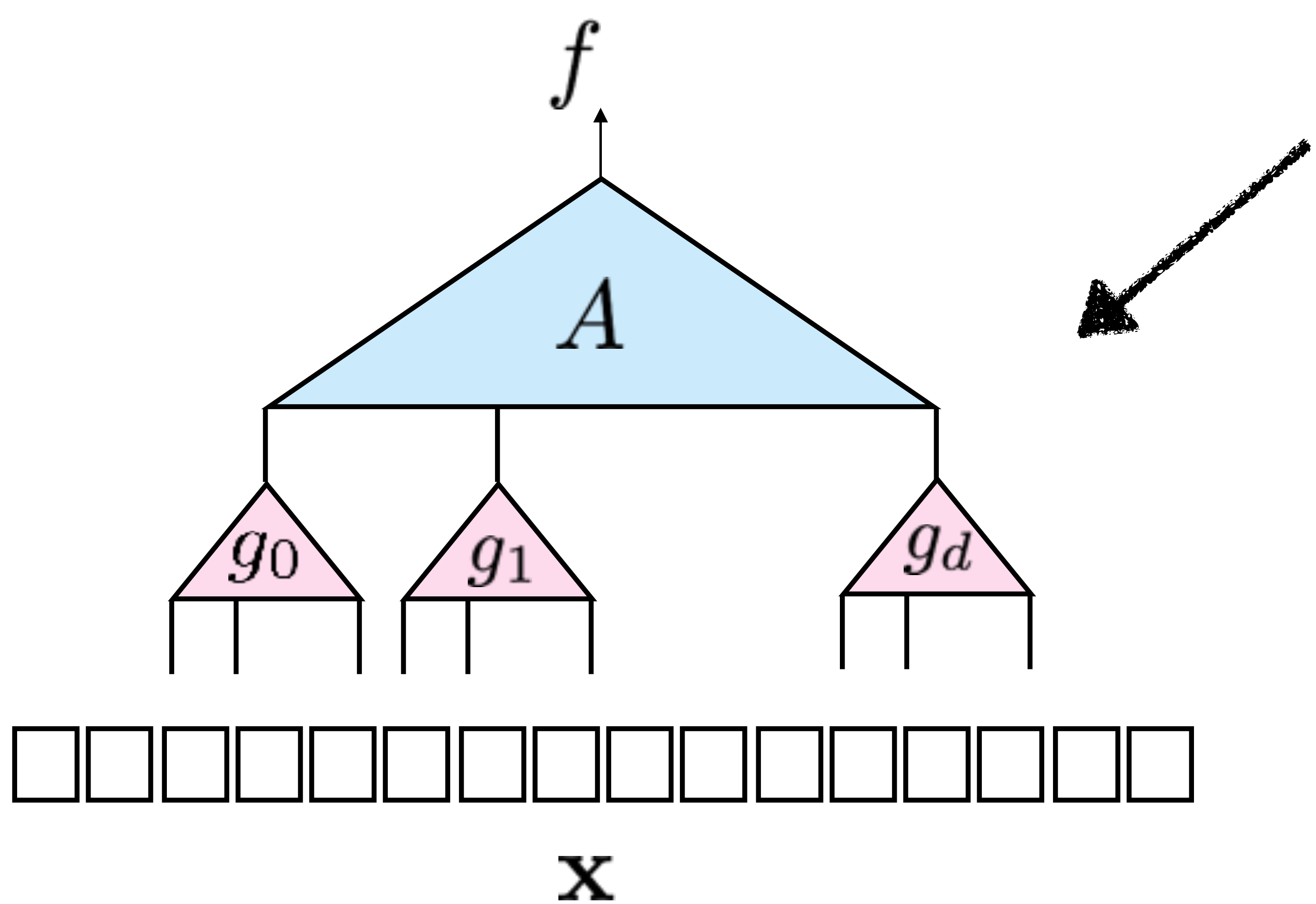


Valiant Theorem

Closure for VNP [**C**KS18]



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$A(\mathbf{z})$

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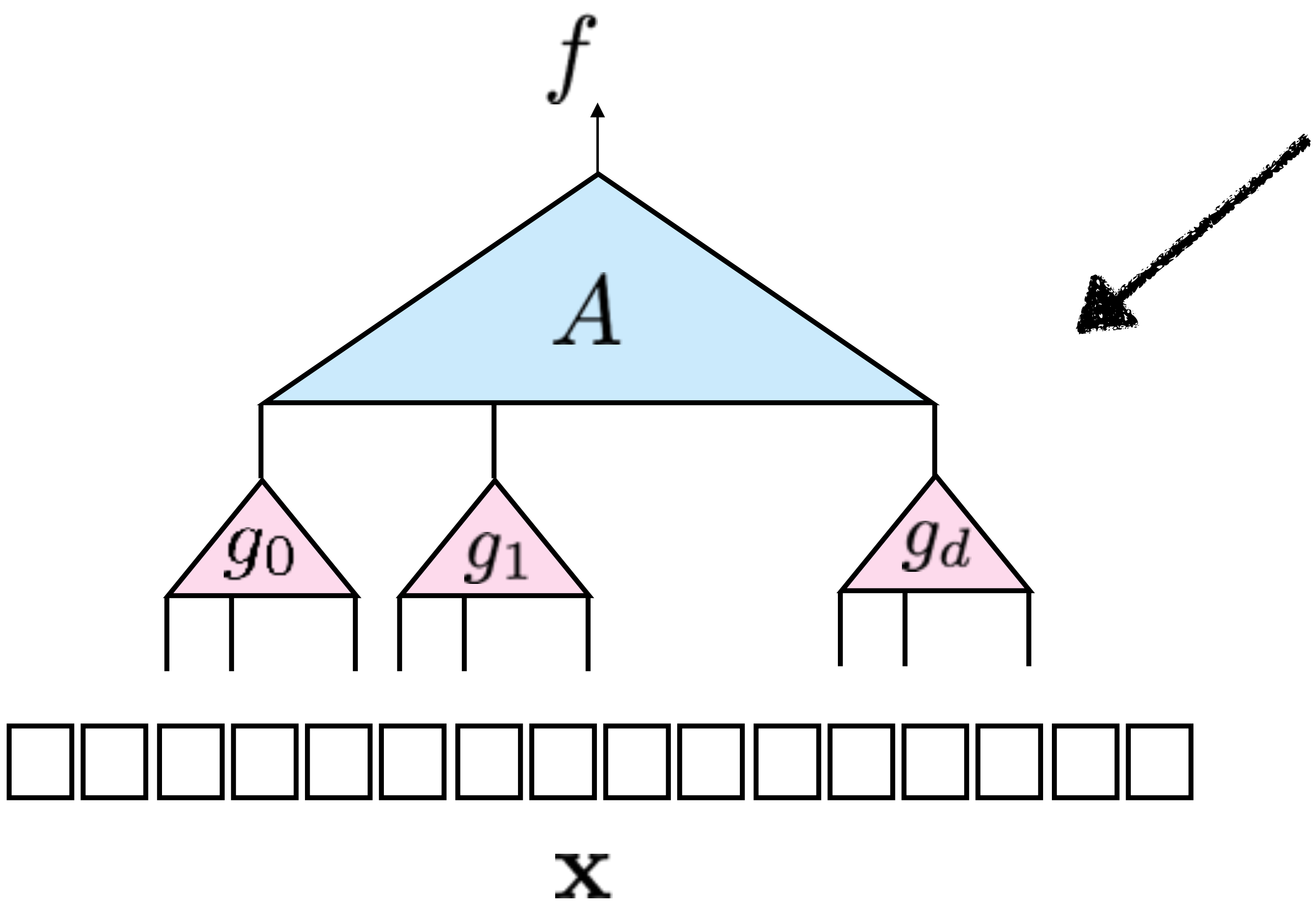

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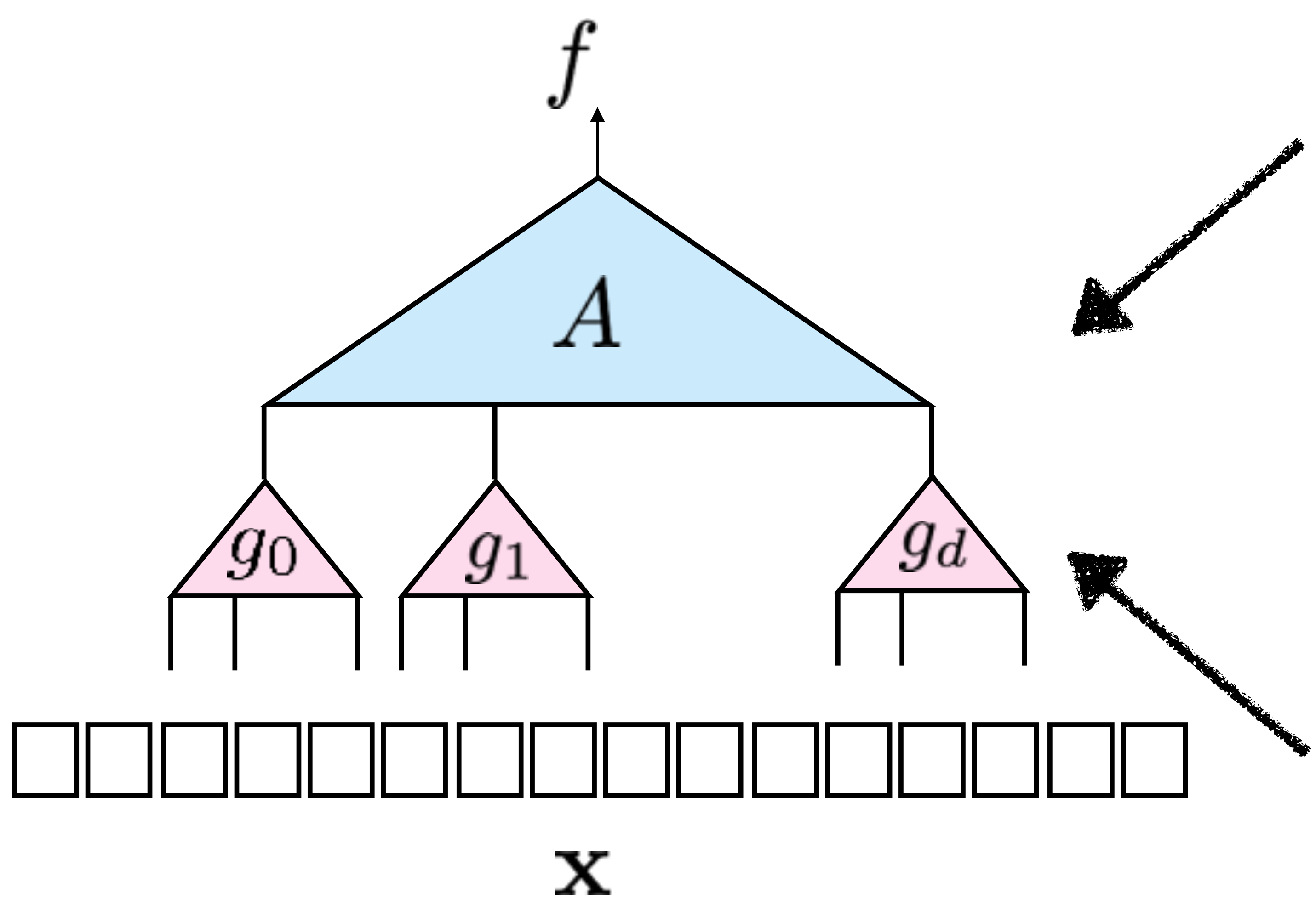

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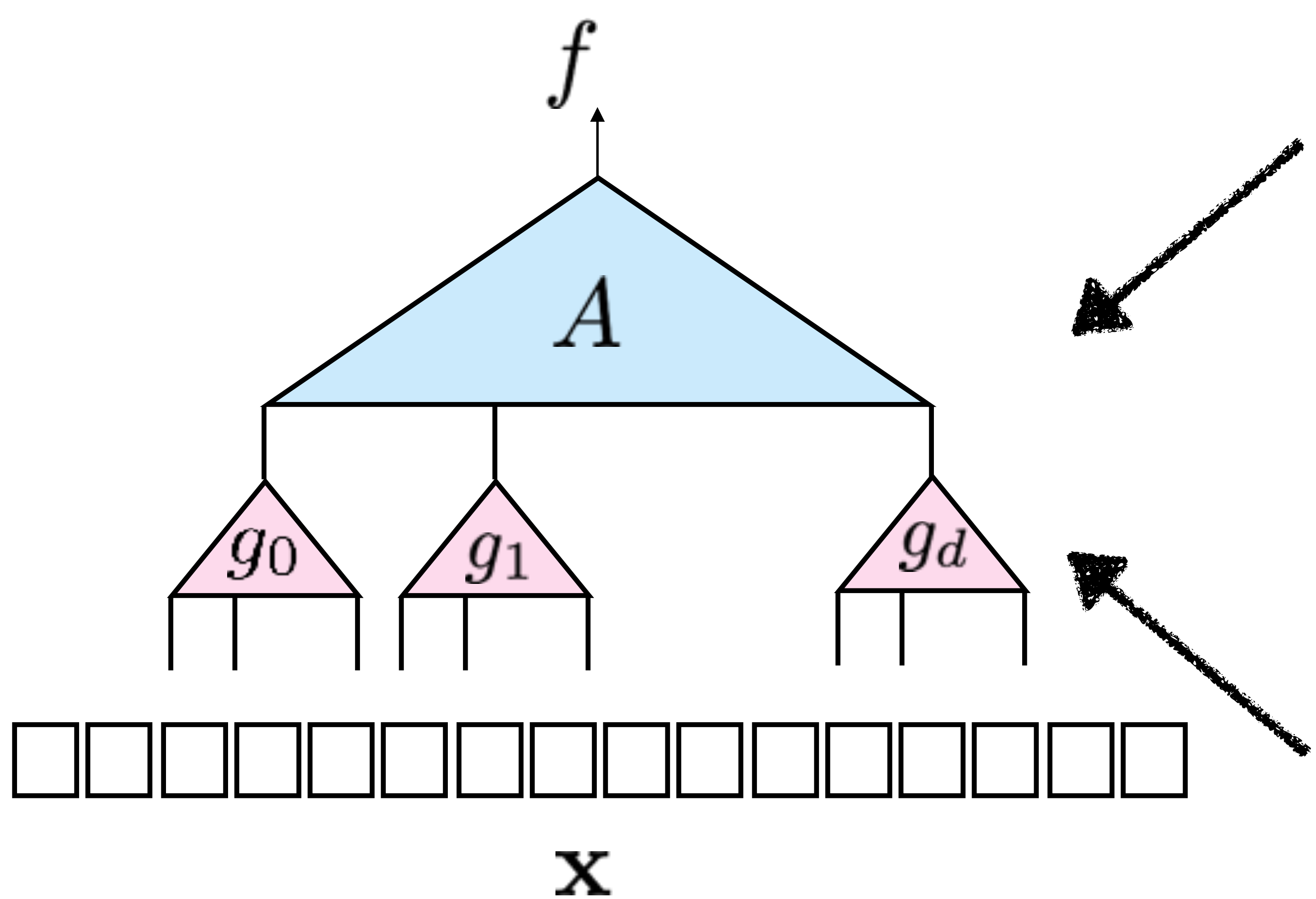

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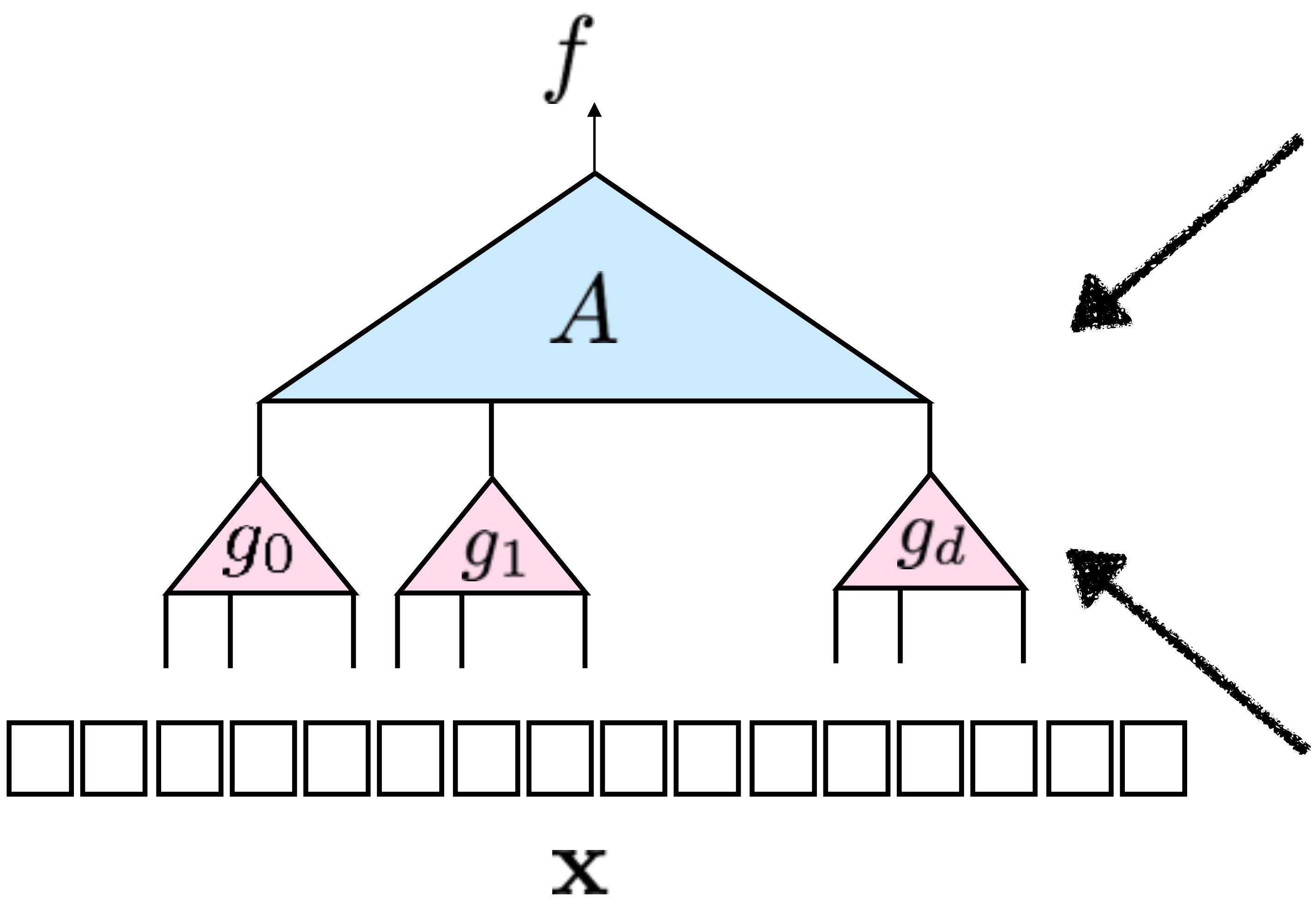

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Proof of Generator Lemma

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3. Thus, A_i can be written as circuit over $\{A_{i-1}, g_0, \ldots, g_i\}$ of size $O(d^2)$.

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$$\mathcal{H}_{\leq i}[P(h_{i-1})] = \mathcal{H}_{\leq i}\left[f_0 + \mathcal{H}_{\leq d}\left[\frac{\partial}{\partial y}P(f_0)\right]\tilde{h} + \cdots + \mathcal{H}_{\leq d}\left[\frac{\partial^i}{\partial y^i}P(f_0)\right]\tilde{h}^i\right] .$$

3. Thus, A_i can be written as circuit over $\{A_{i-1}, g_0, \dots, g_i\}$ of size $O(d^2)$.

4. As the size blow-up from A_{i-1} is additive, A_i has circuit of size $O(d^2 i)$.

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Conclusion

- **Closure for bounded depth poly-log degree circuit**
  - New hardness versus randomness connection.
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- **Key technique**
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Thank you!

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