Near-optimal Bootstrapping of Hitting Sets

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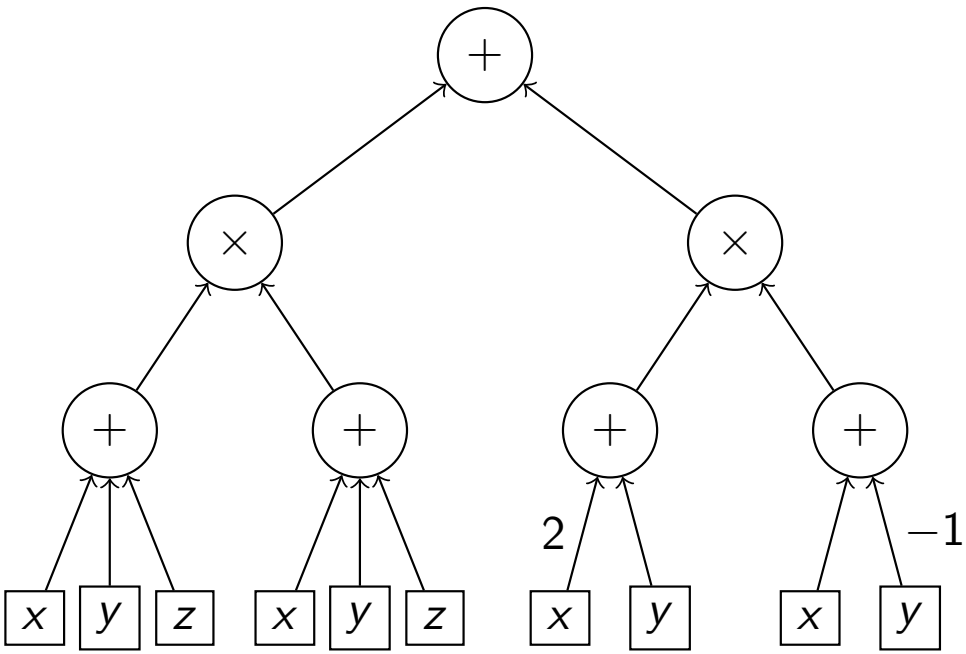
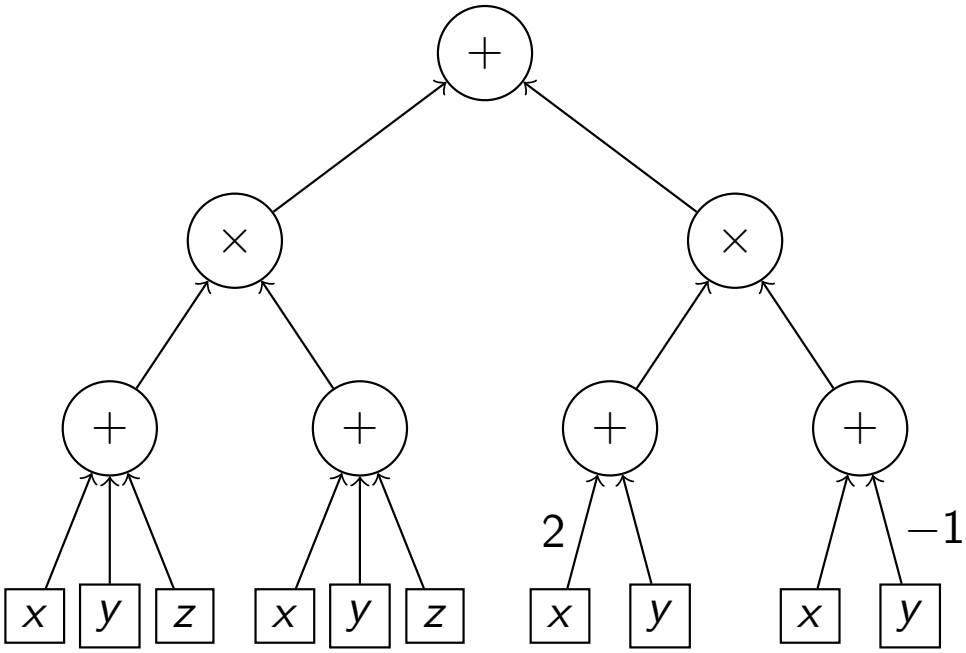
Algebraic Models



Algebraic Formula

Algebraic Models



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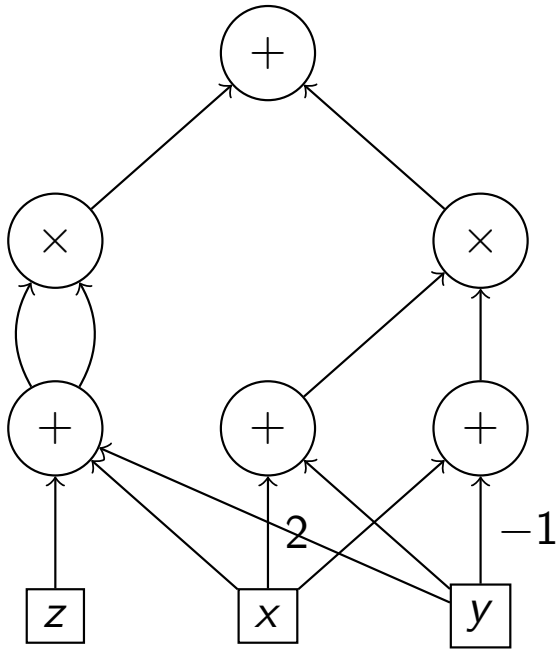

Algebraic Circuit

The Hardness Question

$$f = \mathrm{Perm}\left(\begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix}\right)$$

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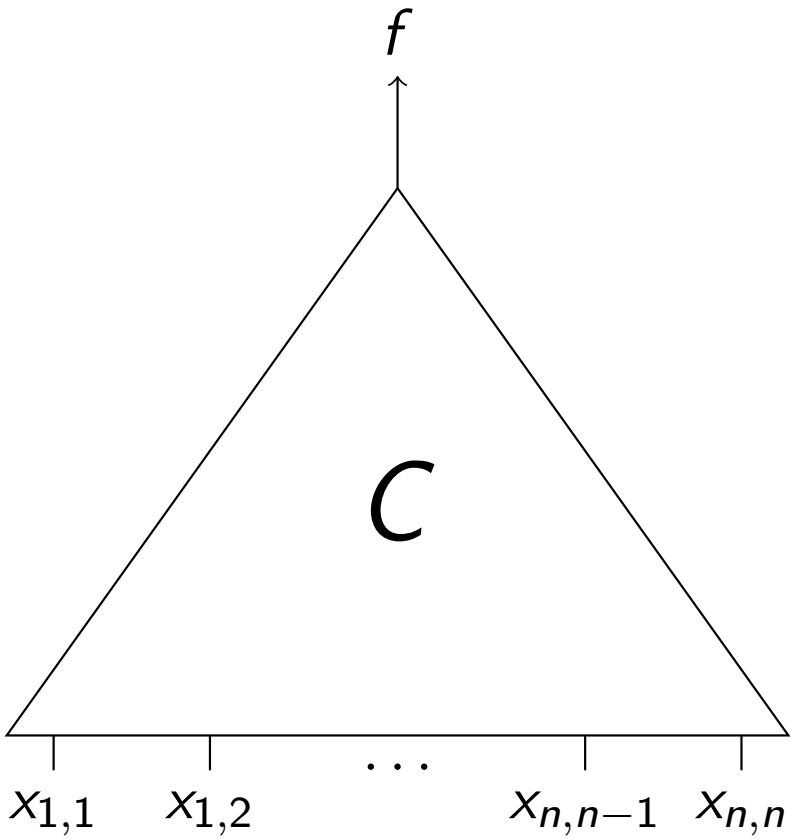

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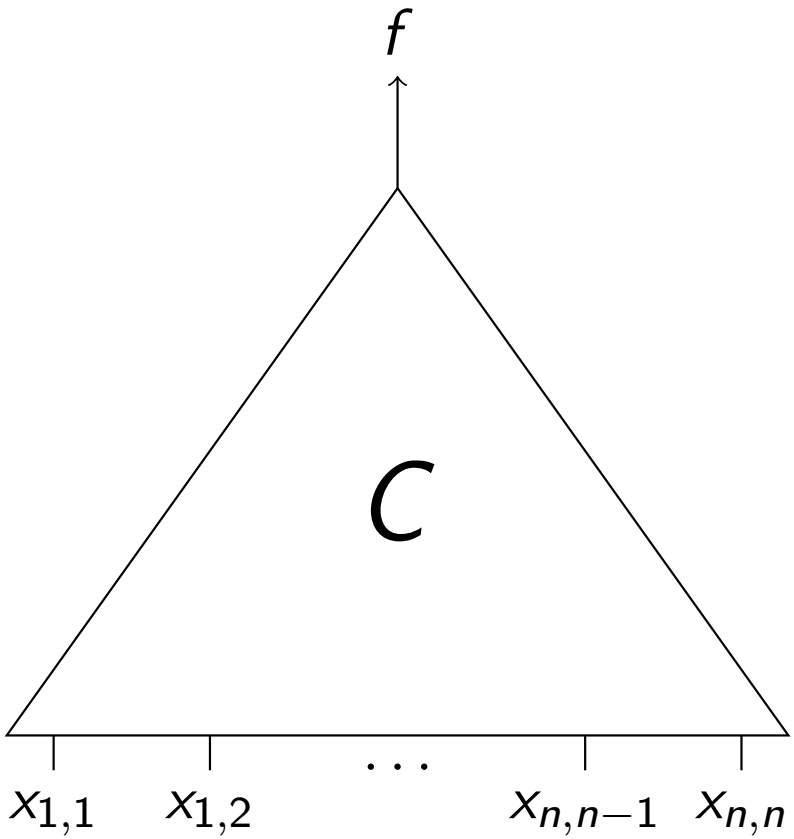

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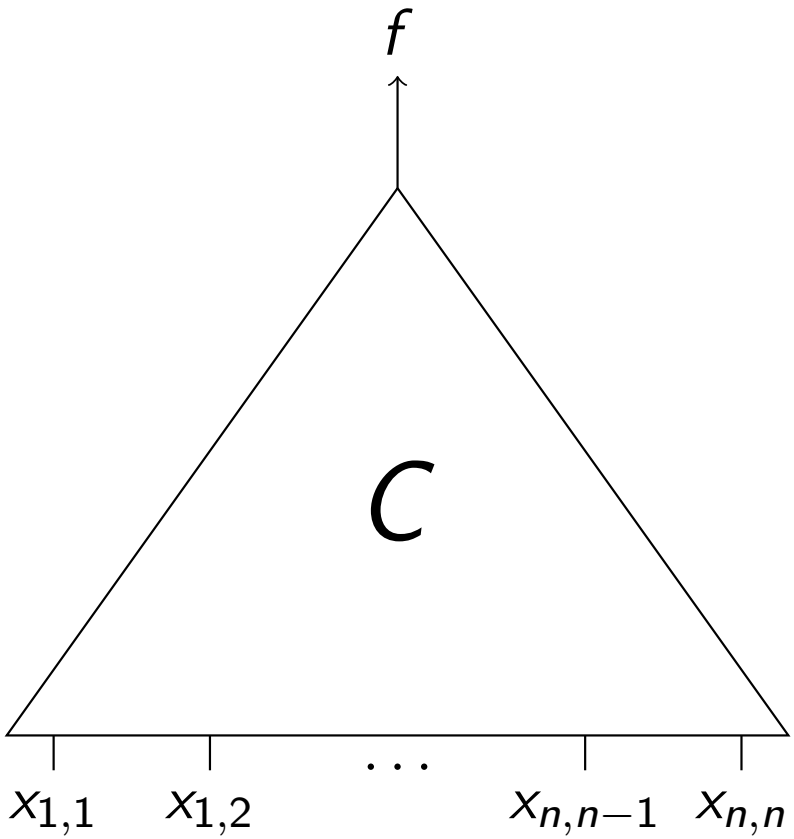

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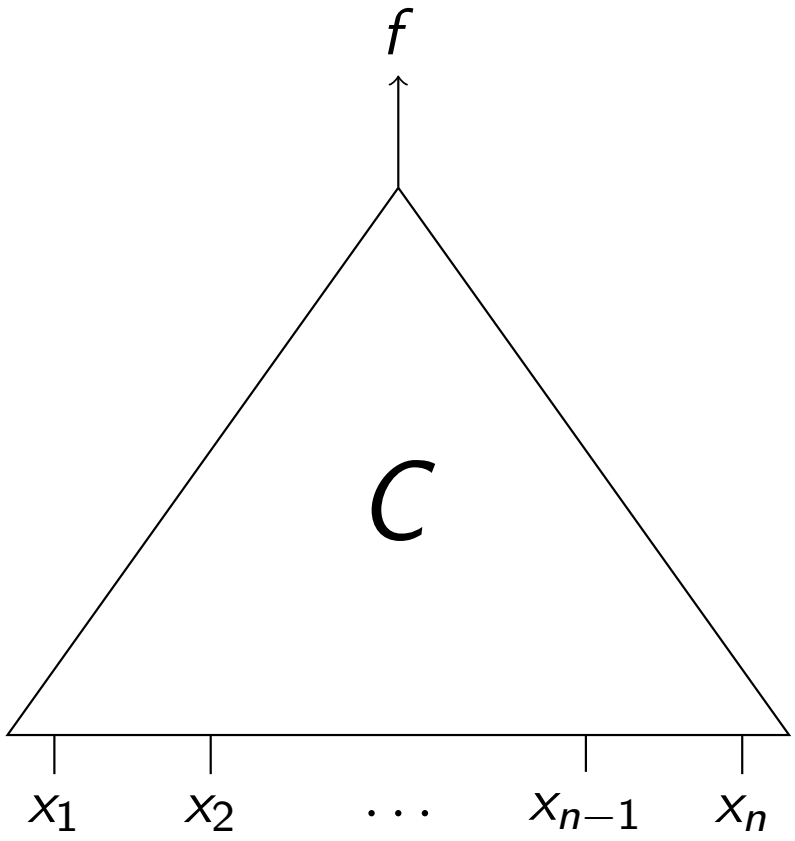
Find an "explicit" n-variate $f(\mathbf{x})$ that requires $n^{\omega(1)}$ sized circuits?

Identity Testing



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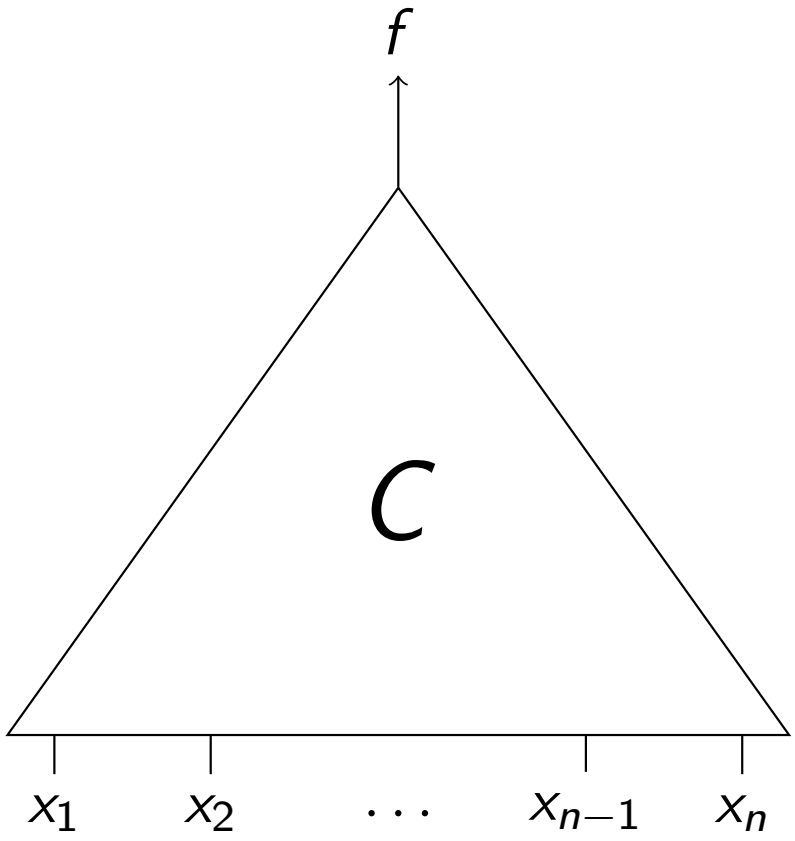

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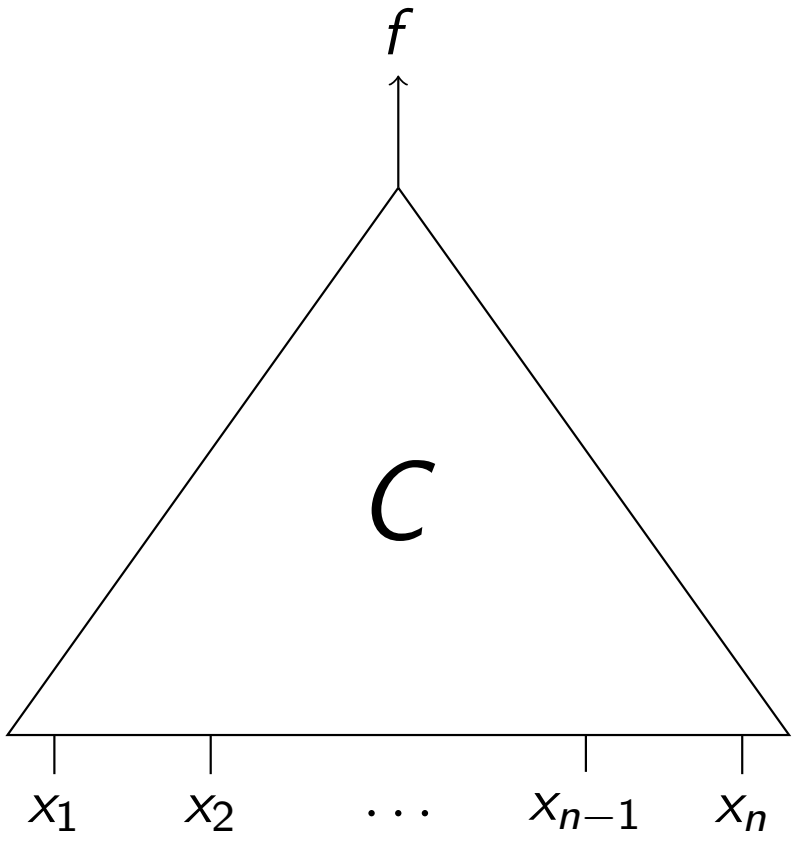

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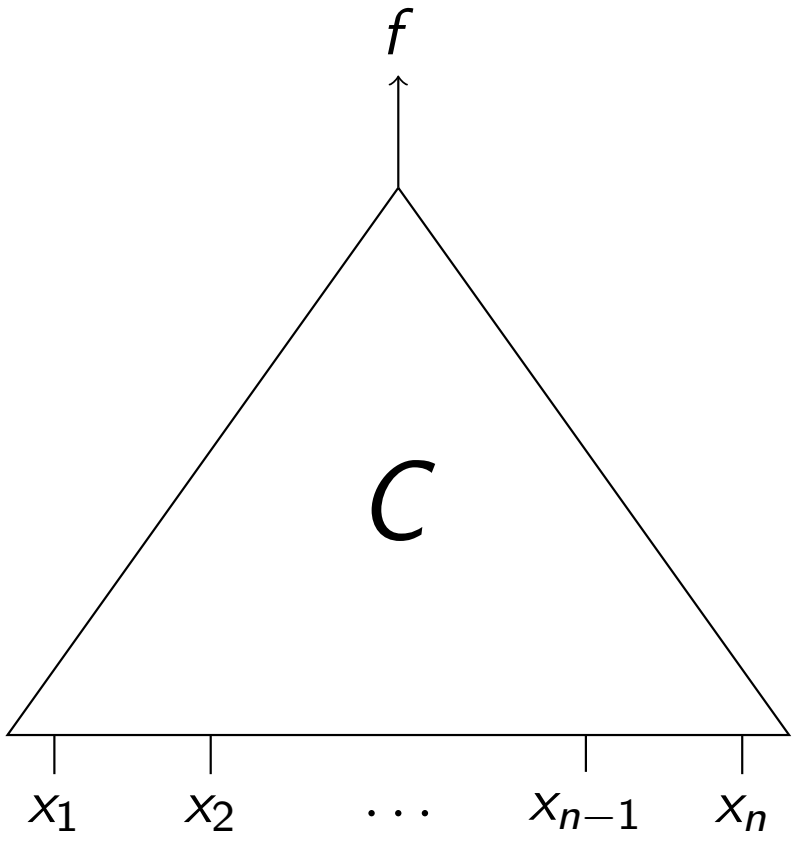

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Whitebox: Does the given circuit compute 0?

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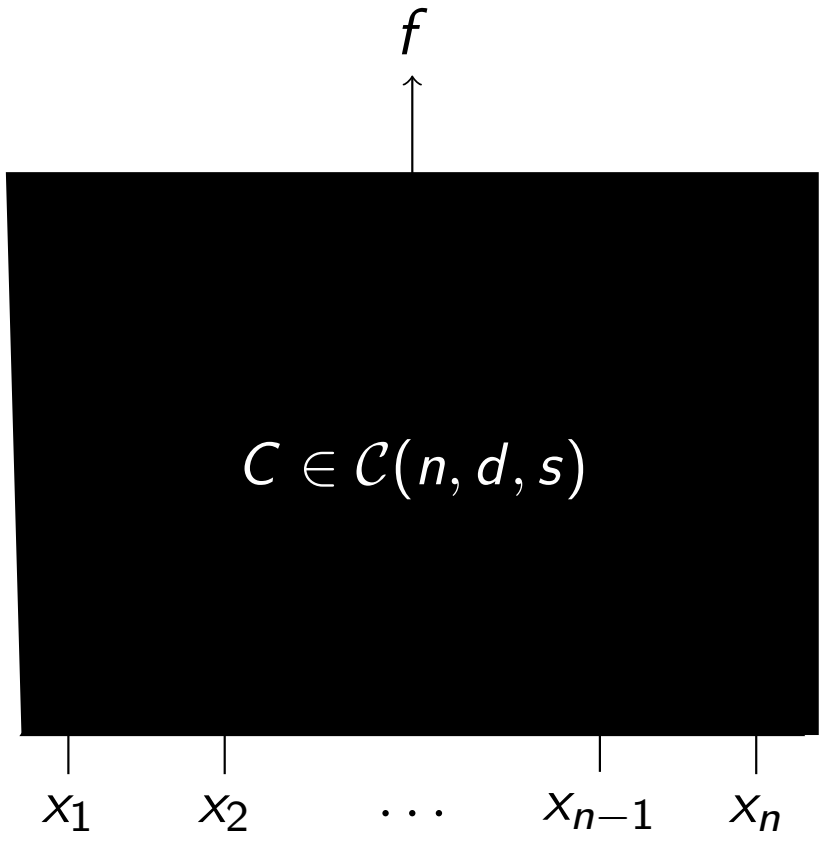

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Blackbox: Evaluate C on some points to tell if $C = 0$.

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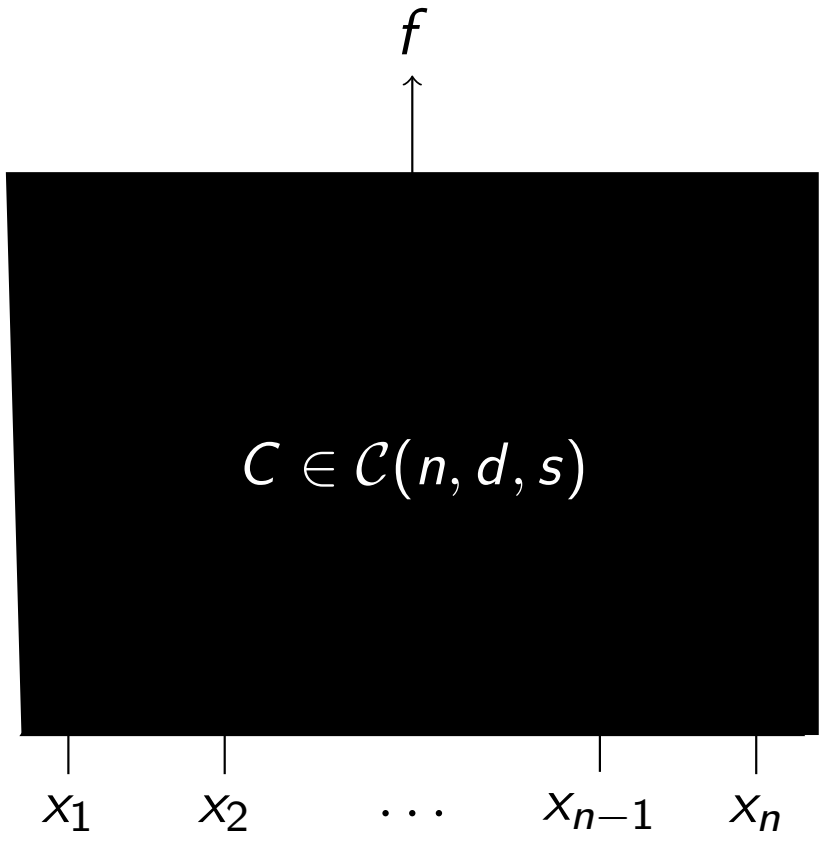

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Is $f = 0$?

Hitting Set: Find $H_{\mathcal{C}}$ such that $C = 0$ **iff** it is 0 on every $h \in H_{\mathcal{C}}$.

Hitting Sets

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Counting Argument: There is a non-explicit poly(n, d, s) sized hitting set for the class of all n-variate, degree-d circuits of size s, $\mathcal{C}(n, d, s)$.

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OPEN: Find an explicit hitting set of size $d^{o(n)}$ for $\mathcal{C}(n, d, s)$.

Improving slightly non-trivial Hitting Sets

Theorem [Agrawal, Ghosh, Saxena 2018]
Suppose for a large constant n and all $s \geq n$, there is an explicit hitting set of size

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$\text{tiny}(s) = \exp(\exp(O(\log^* s)))$

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What is Bootstrapping?

Hitting Set

Hard Polynomial

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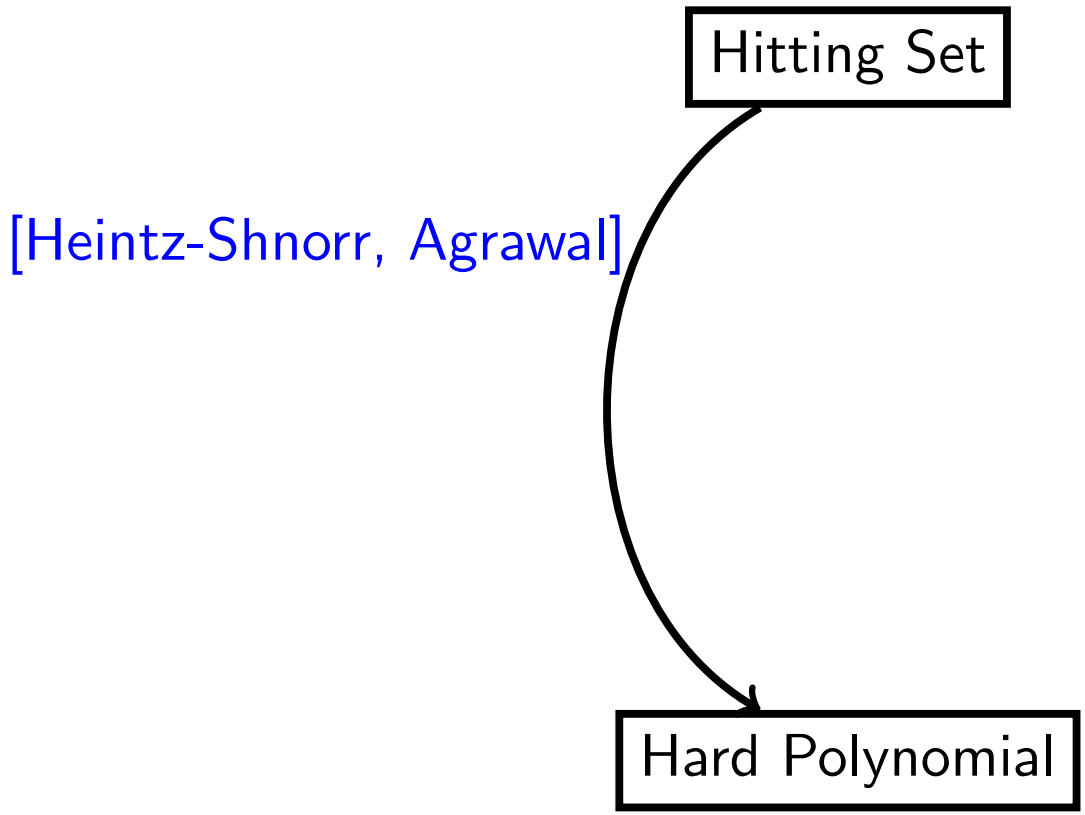

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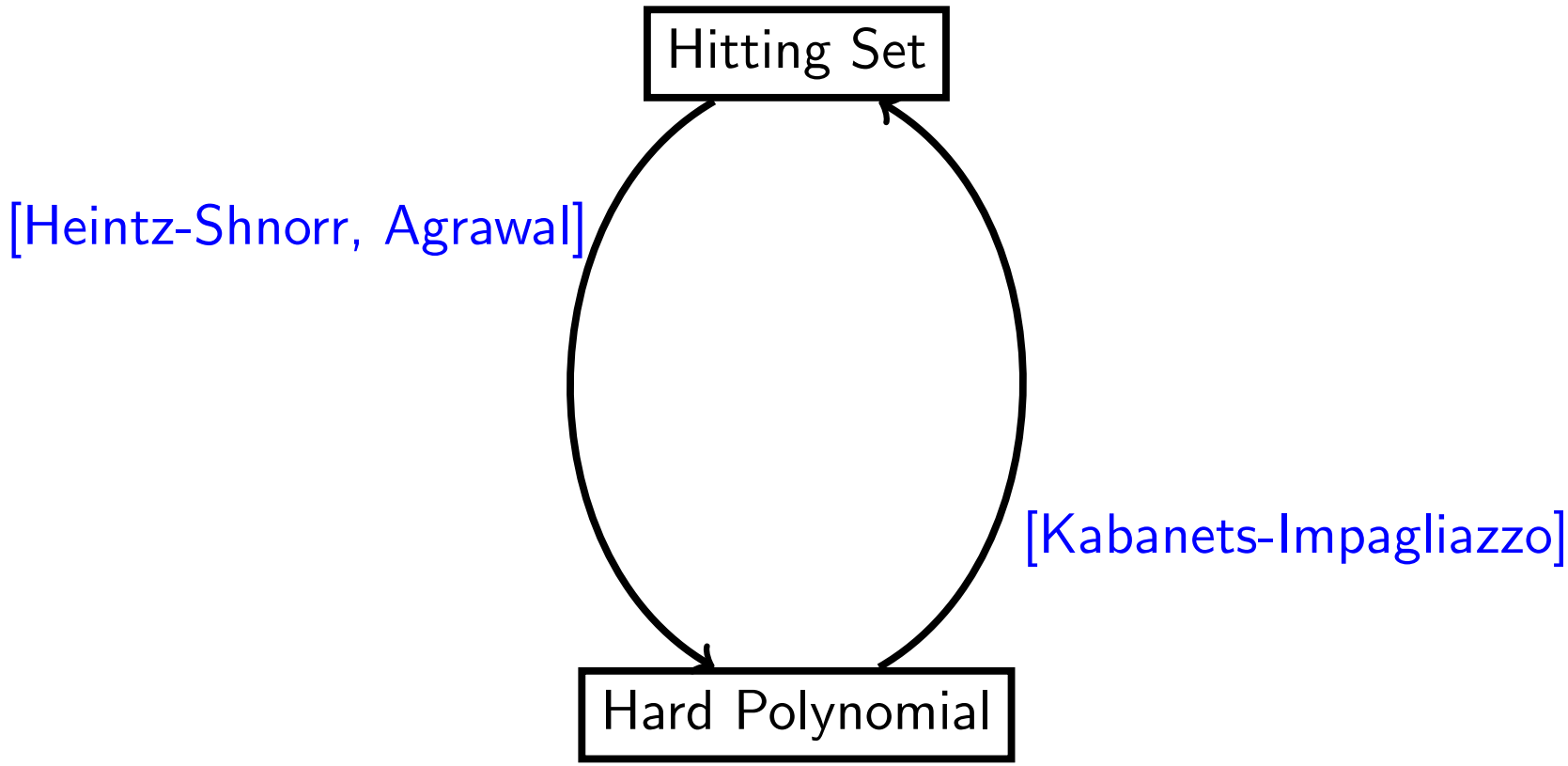

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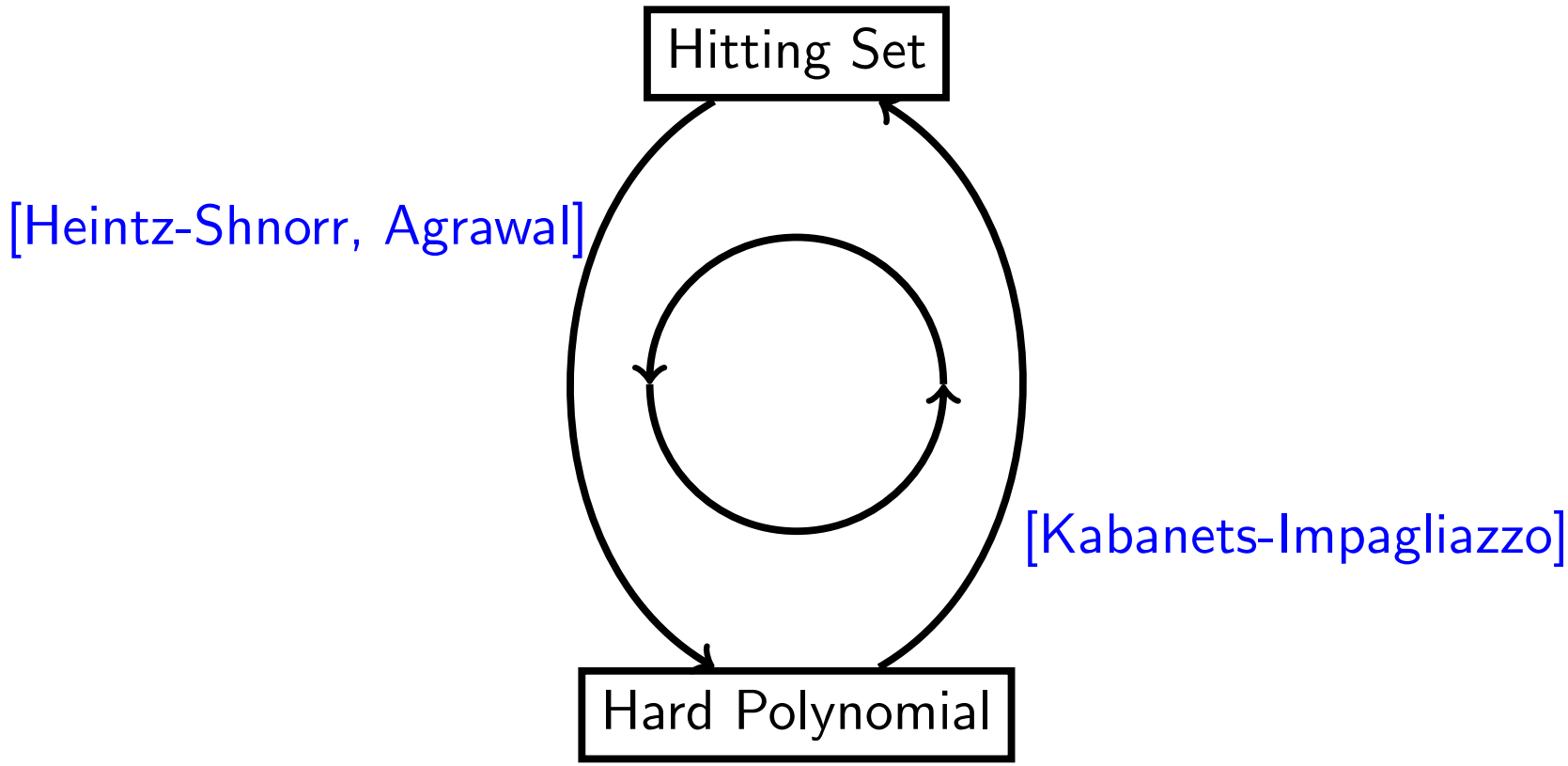

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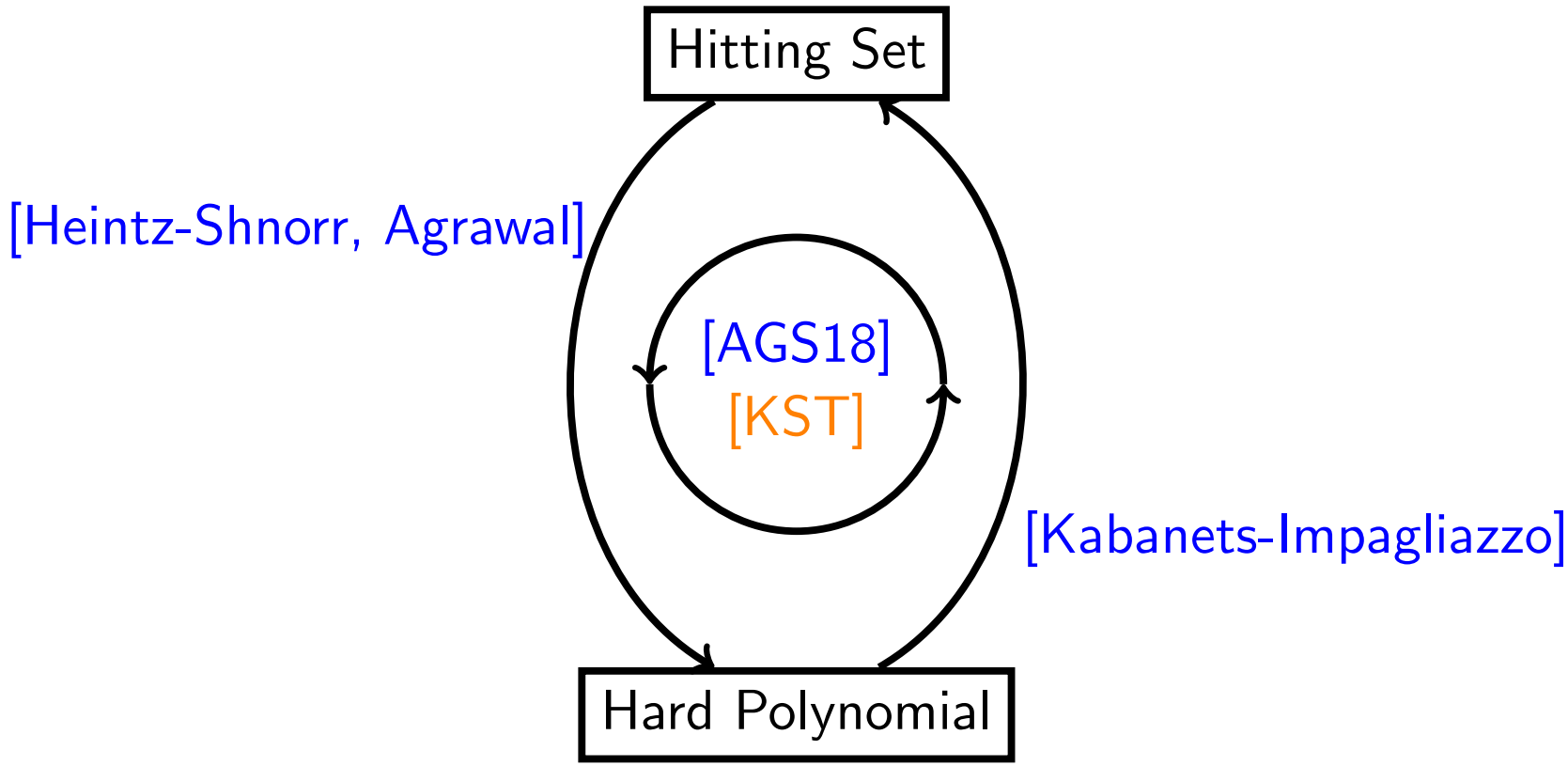

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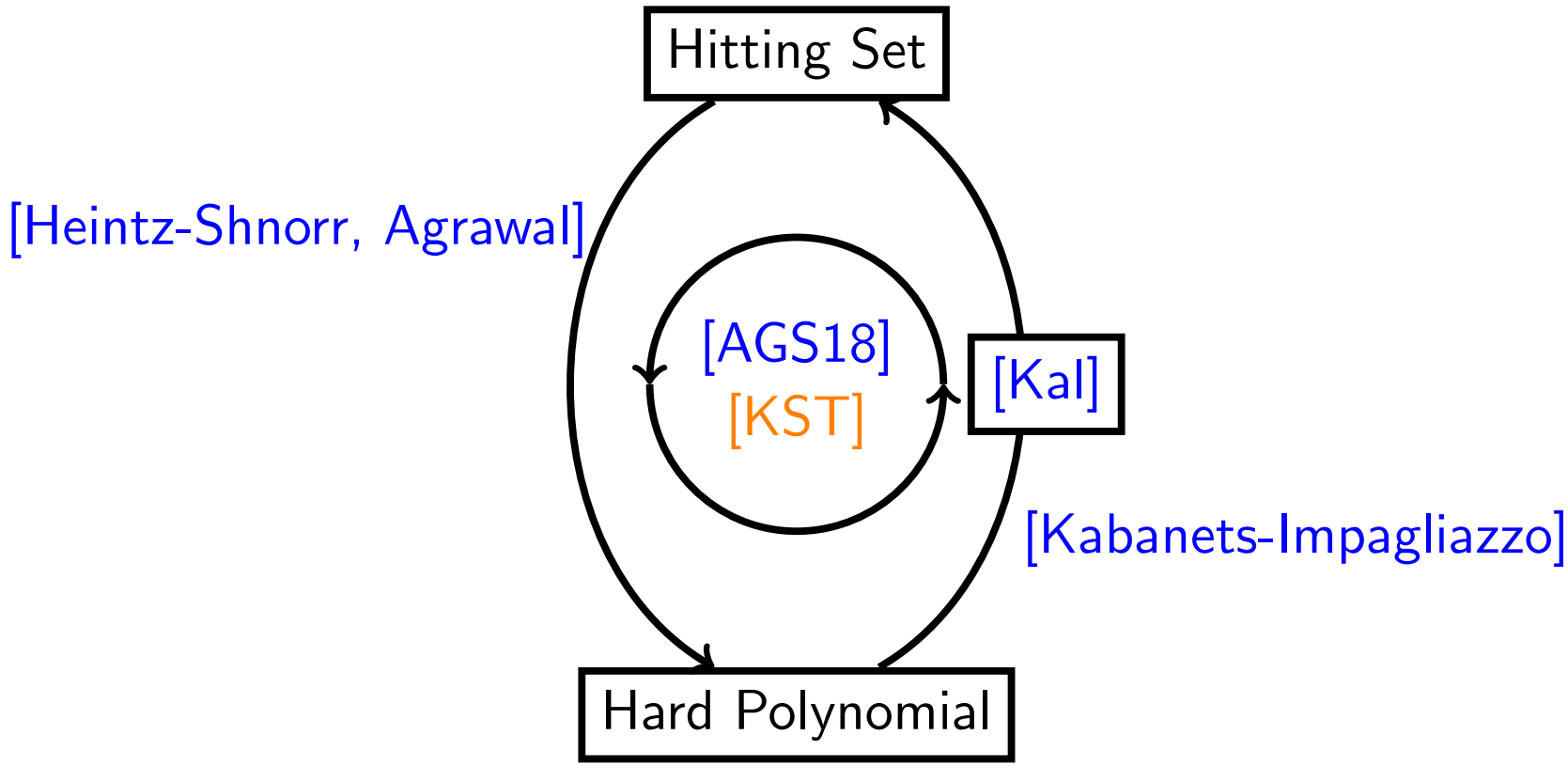

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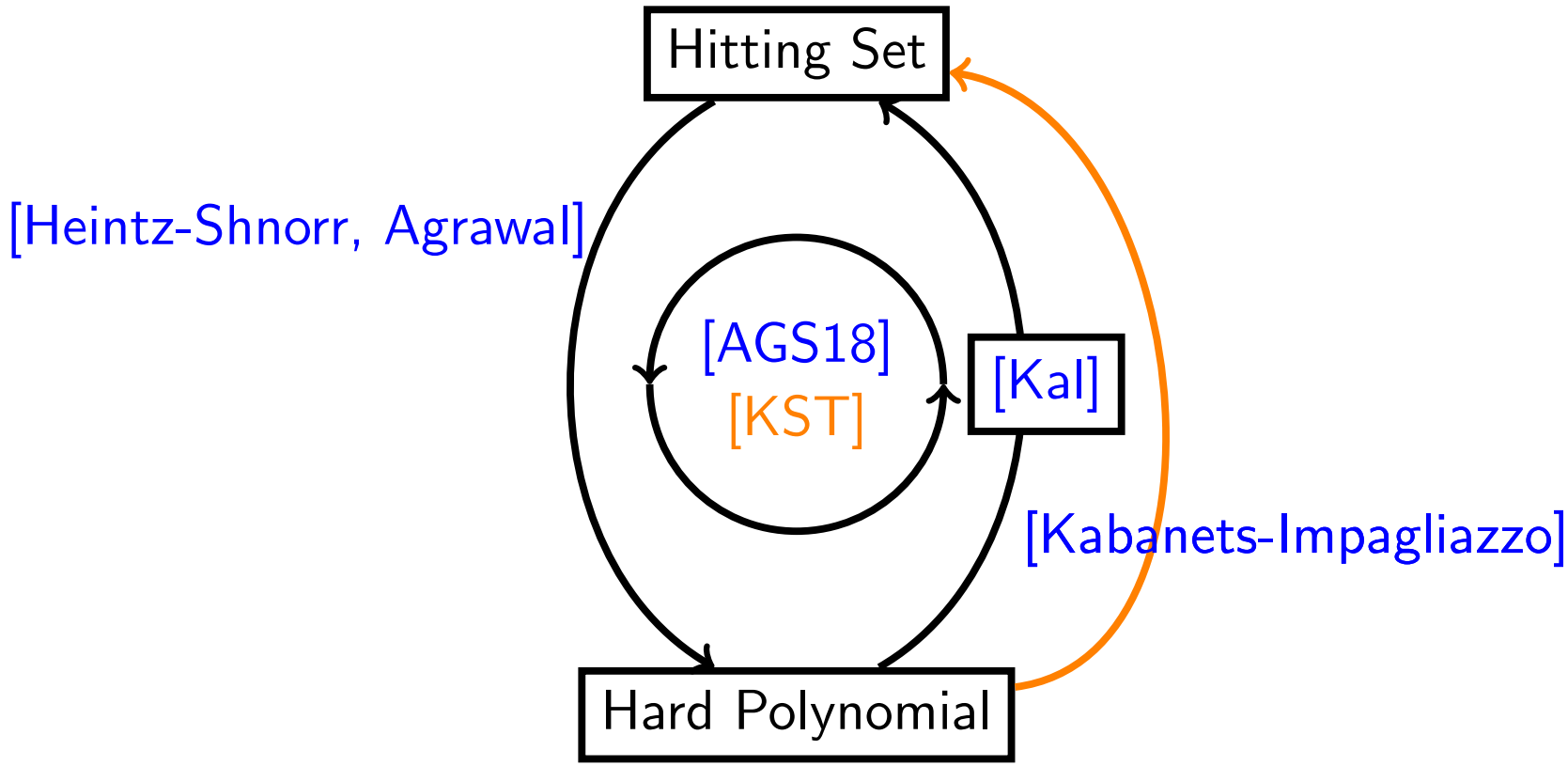

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Proof Idea: Use interpolation to get a Q_k that vanishes on H.

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when $\mathbf{y}_1, \ldots, \mathbf{y}_m$ *are nearly disjoint.*

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Requires closure under factoring!

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Proof: On the board.

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  $P' \in \mathcal{C}(k^2, s', s')$, hitting set of size $(s')^{k^2}$.

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  $\text{size}(P') = \text{size}(P) \cdot \text{size}(Q_k) = s'$.

  $\deg(P') \leq \deg(P) \cdot k \cdot d = d' < s'$.

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Q. What if we have a non-trivial hitting set for $\mathcal{C}(k^2, s, s)$?

Template for Bootstrapping

Hyp.: Q_k of ind. deg. $< d$ vanishes on $\mathcal{H}(k^2, s^{10}, s^{10})$.

Goal: Hitting set for $\mathcal{F}(m, s, s)$, for $m = 2^{\sqrt{k}}$, all s.

- Variable reduction:

  Obtain $\mathbf{y}_1, \ldots, \mathbf{y}_m \subseteq \{y_1, \ldots, y_{k^2}\}$. Nonzero $P \in \mathcal{F}(m, s, s)$,

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A. We get a $s^{30g(k)^2}$ hitting set for $\mathcal{C}(2^{\sqrt{k}}, s, s)$, for all s.

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Unlike the boolean case, nothing stops us.

High Level Overview

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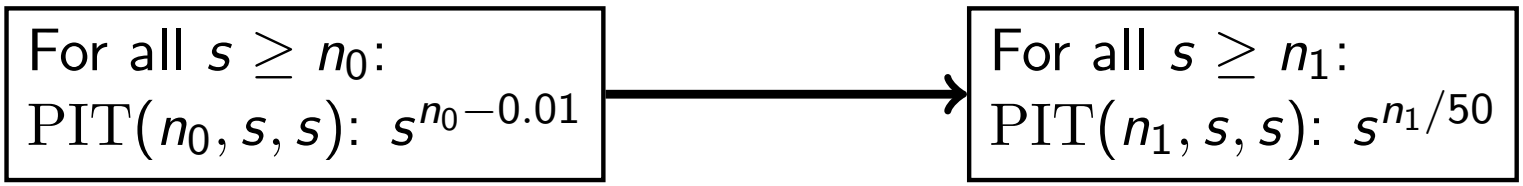
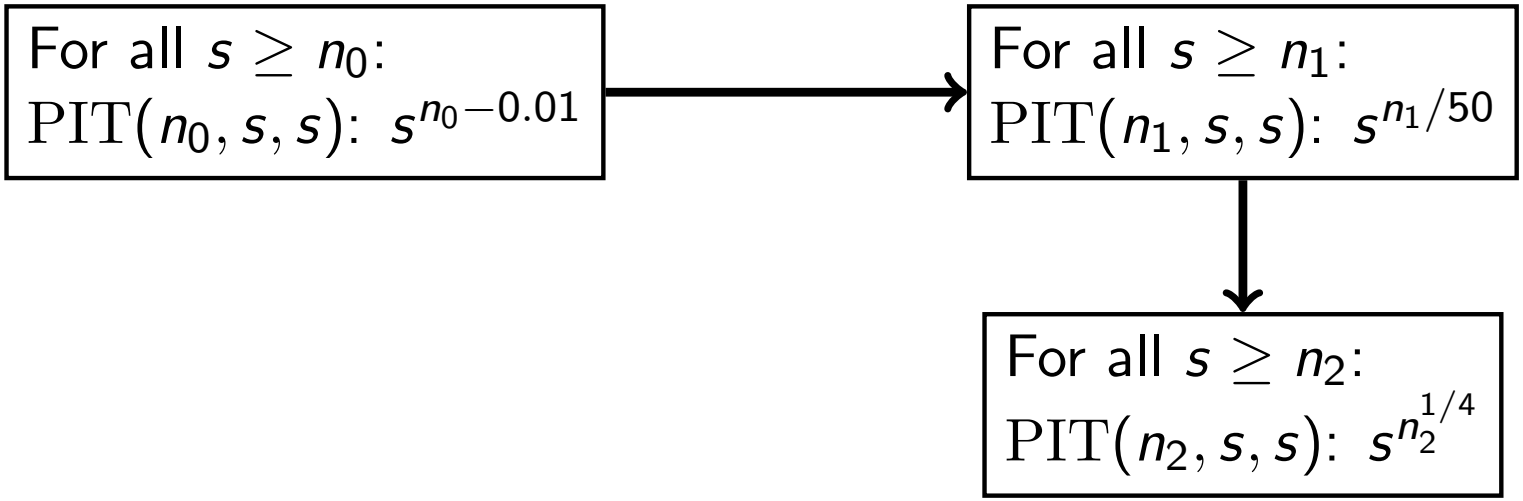
For all $s \geq n_0$:
$\mathrm{PIT}(n_0, s, s)$: $s^{n_0 - 0.01}$

High Level Overview



High Level Overview



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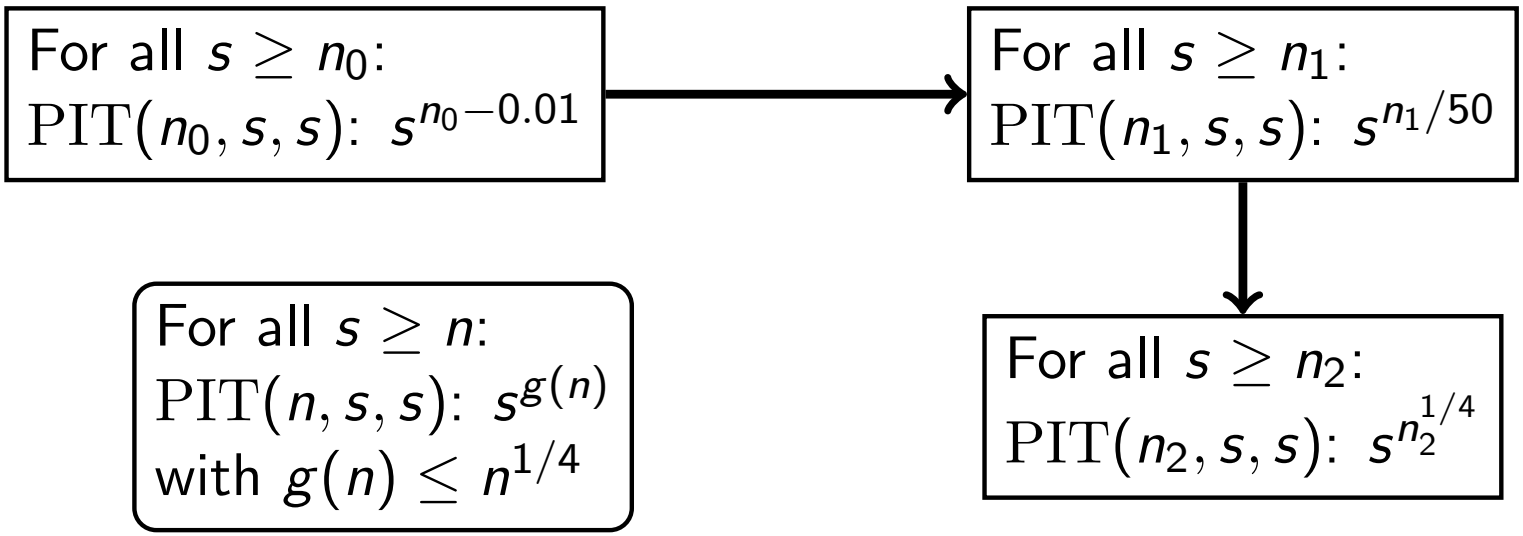

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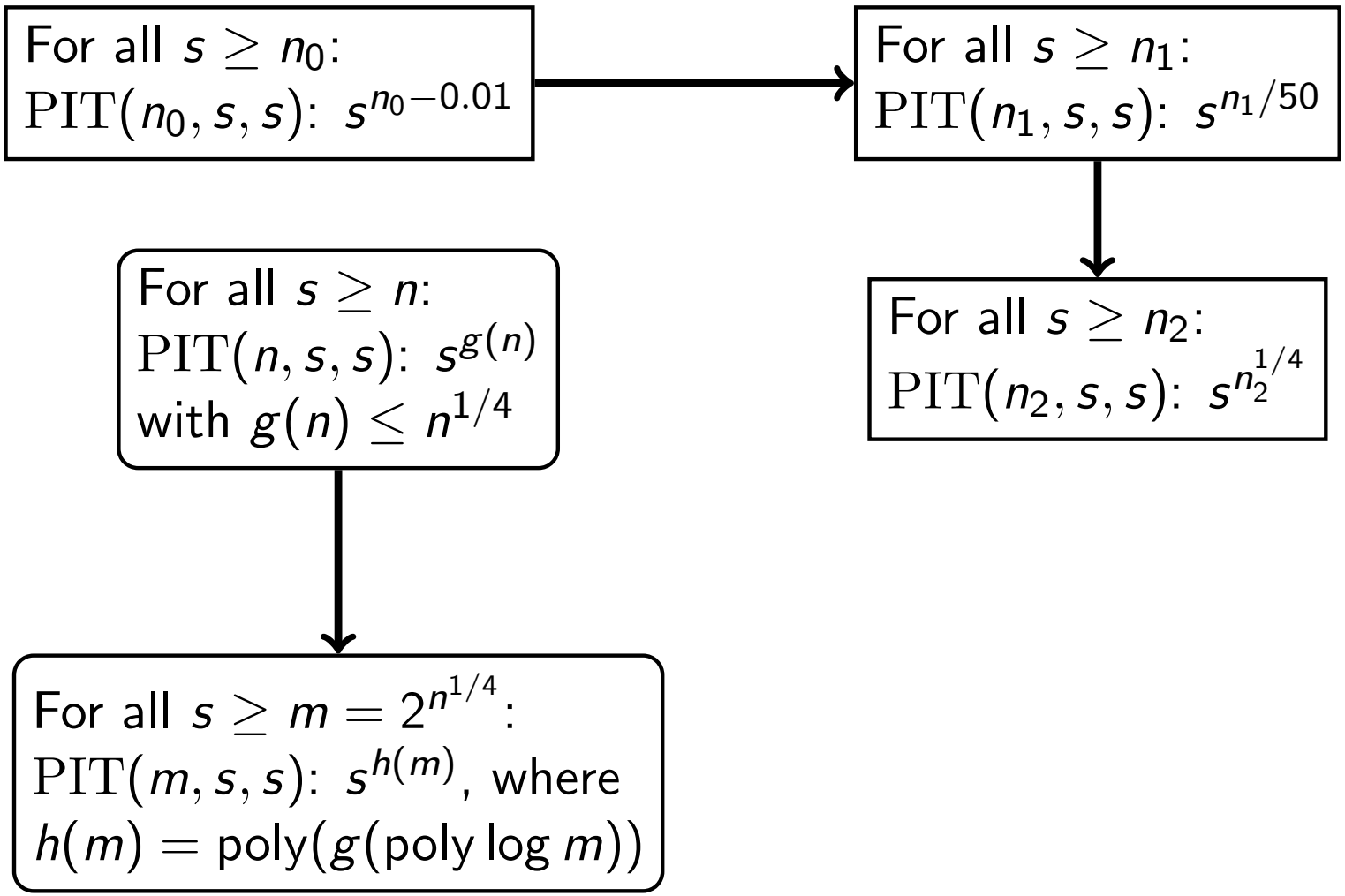

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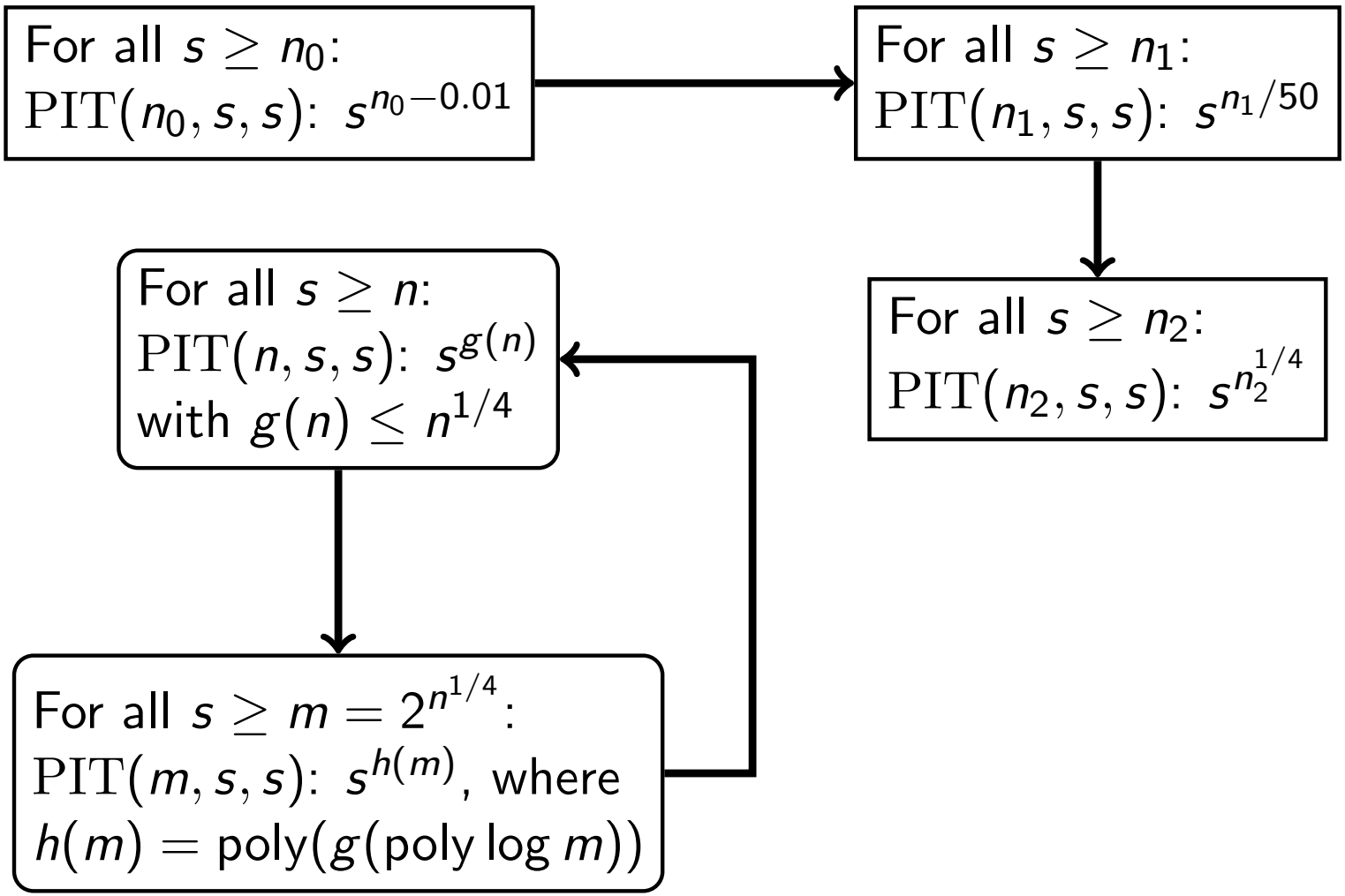

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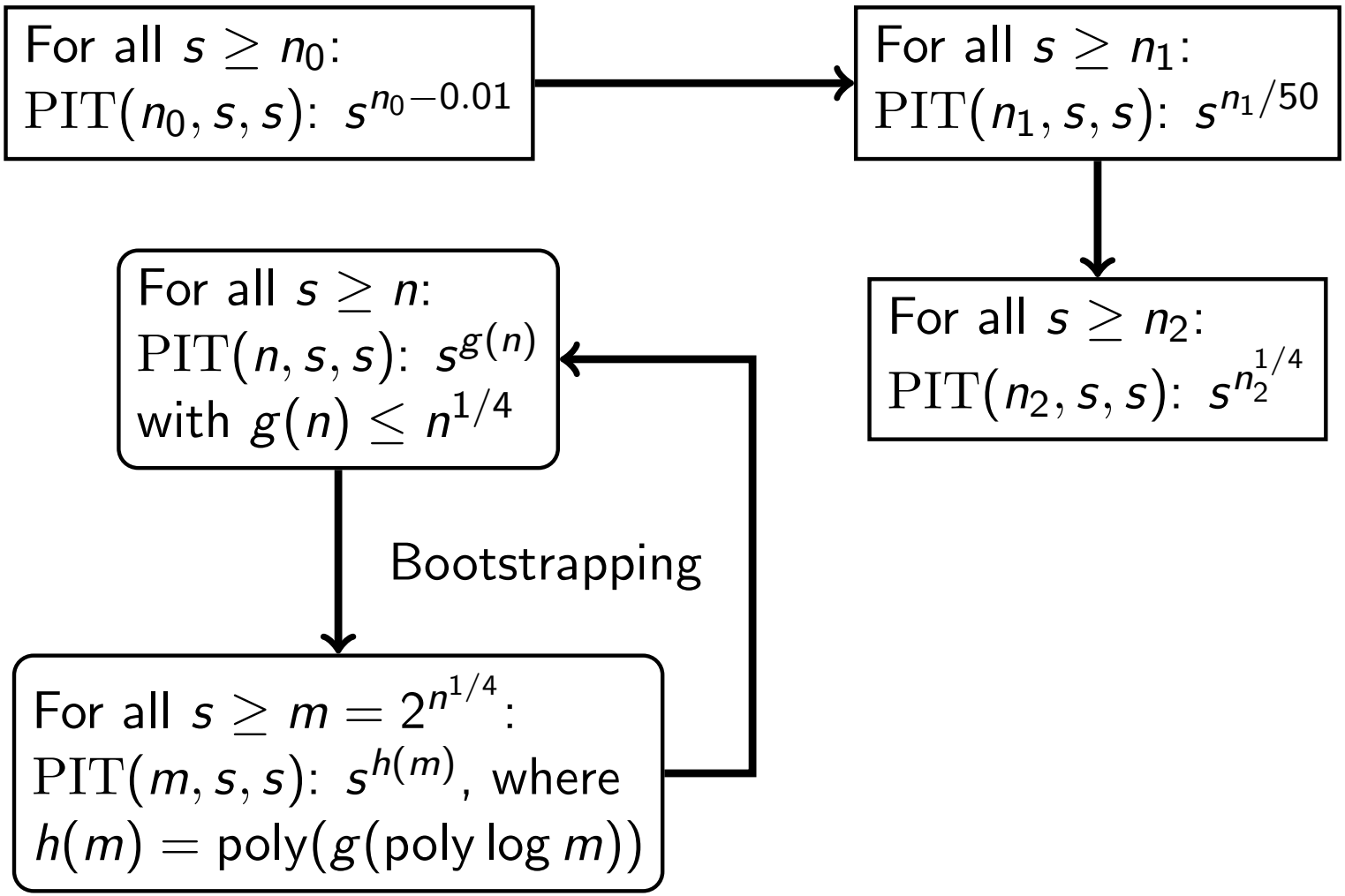

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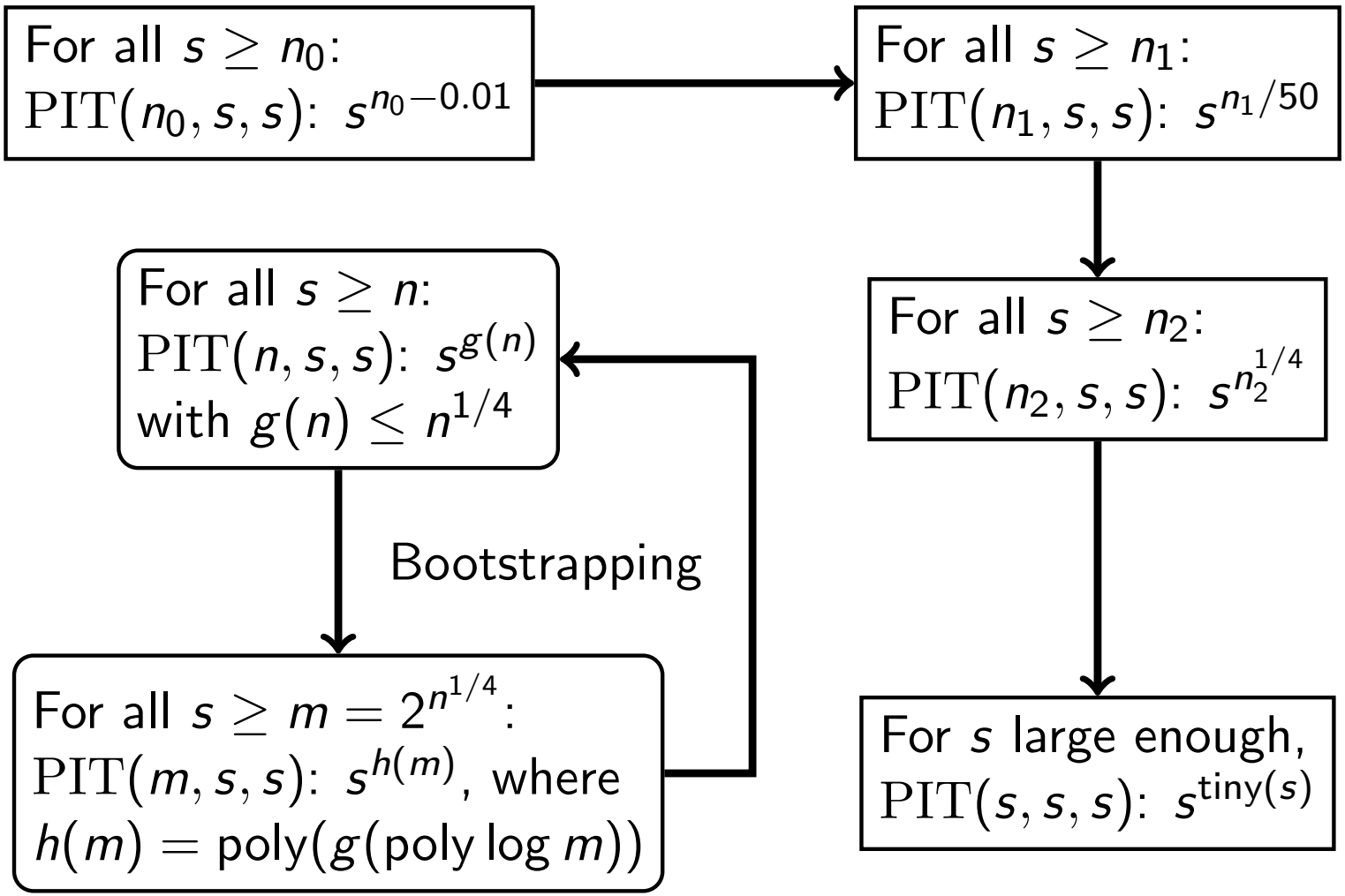

Summary and Open Questions

Theorem [Kumar, Saptharishi, T]

Suppose for a constant $n \geq 2$, *some* $\epsilon > 0$ *and all* $s \geq n$, *there is an explicit hitting set of size*

$$s^{n-\epsilon} \quad \text{for} \quad \mathcal{F}(n, s, s).$$

Then for all large s, *there is an explicit hitting set of size*

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Thank You!