Integrable stochastic interacting systems

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Abstract

Stochastic interacting particle systems show many intriguing phenomena due to the interaction among particles and have wide applications in various fields of science, but in general it is quite difficult to study their properties in detail. Over the last few decades, however, it has been gradually recognized that certain stochastic interacting particle systems can be “solved exactly”, meaning that they admit explicit calculations of various probabilities and expectation values, and behind this tractability lies the integrability of these systems. In particular there have been remarkable progress in the understanding of growth and transport models in the Kardar-Parisi-Zhang (KPZ) universality class, which have turned out to have deep connections with random matrix theory, representation theory, special functions and so on.

In these lectures, we will explain these developments, mainly focusing on the models in the KPZ class. In the first lecture, we introduce one of the most fundamental models on the subject, the asymmetric simple exclusion process (ASEP) and explain the connection between its totally asymmetric version (TASEP) and random matrix theory[1]. In the second lecture we introduce various models and discuss their integrability [2]. In the third lecture we introduce the notion of stochastic duality and explain its relation to the replica approach of using moments [3]. Then we elucidate how an exact formula for ASEP can be obtained by combining the duality and integrability of the model. In the fourth lecture we explain our approach introduced in [4], which does not rely on the moment calculation. We emphasize that the method can be applied to many models in parallel, including the stationary situation. In the last lecture we discuss our recent extension of the techniques to study a two species exclusion process [5]. The implications to the nonlinear fluctuating hydrodynamics, which was proposed recently by H. van Beijren and H. Spohn, is also explained. Lastly we discuss some outstanding problems.

Contents

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0 Introduction

In nature all matters consist of huge number of atoms and molecules, exhibiting various intriguing phenomena such as phase transitions, turbulence and so on. To have better understanding of such systems, we often construct models which should retain essential ingredients of the original systems. In many cases, one can model a system as a stochastic process of many particles which are interacting with each other, namely, a stochastic interacting system. Mathematical foundation of stochastic interacting systems had been set long time ago [14]. See Appendix A for a short explanation about the formulation.

There have been extensive amount of results on such systems [15], but it is often difficult to study their properties in detail. Recently, it has been realized that certain stochastic interacting systems allow quite exact and explicit analysis from which one can obtain very fine properties of the systems and there have been remarkable progress on the subject. In this development, a particularly important role was played by the asymmetric exclusion process and the Kardar-Parisi-Zhang (KPZ) equation.

The Asymmetric simple exclusion process (ASEP) is a stochastic process of many (infinite number of) particles on $\mathbb{Z}$ in which each particle tries to perform asymmetric random walk, with hopping rate to the right p and to the left q, under the volume exclusion interaction. The special case of the ASEP in which either $p = 0$ or $q = 0$ is called TASEP (Totally ASEP). When $p > q$, there is a net particle current to the right.

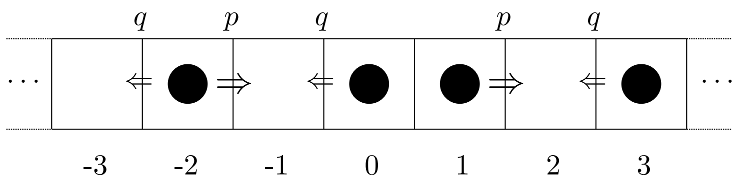

Figure 1: ASEP

By replacing empty sites by the upward slope / and occupied sites by the downward slope \, the ASEP is mapped to a surface growth model, called the single step model.

In the studies of surface growth, the standard model is the KPZ equation, which was introduced in 1986 [13]. In the case of one dimension, for the height of the surface $h(x, t)$ at position $x \in \mathbb{R}$ and at time $t \geq 0$, it reads,

$$\partial_t h(x,t) = \tfrac{1}{2}(\partial_x h(x,t))^2 + \tfrac{1}{2}\partial_x^2 h(x,t) + \eta(x,t)$$

where $\eta(x, t)$ is a Gaussian white noise with $\langle \eta(x,t)\eta(x',t')\rangle = \delta(x - x')\delta(t - t')$. Here $\langle\cdot\rangle$ means the average wrt η. In their original paper [13], Kardar, Parisi and Zhang applied a dynamical version of the renormalization group analysis to show that the height fluctuations scale like $O(t^{1/3})$ as $t \to \infty$. There have been a lot of (physical) analysis of this equation, but there had not been much information available for the height distribution until recently.

In 2010, the first explicit distribution has been found for special initial condition called the narrow wedge initial condition $e^{h(x,0} = \delta(x) = \lim_{\delta\to 0} c_\delta e^{-|x|/\delta}$ [2, 21–24]. The formula takes a simple form for a particular expectation:

$$\langle e^{-e^{h(x,t)+\frac{x^2}{2t}+\frac{t}{24}-\gamma_t s}}\rangle = \det(1 - K_{s,t})_{L^2(\mathbb{R}_+)} \tag{0.1}$$

where $\gamma_t = (t/2)^{1/3}$ and the kernel $K_{s,t}$ is

$$K_{s,t}(x,y) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{e^{\gamma_t(s-\lambda)}+1}. \tag{0.2}$$

In the long time limit, we get

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{h(x,t)+\frac{x^2}{2t}+\frac{t}{24}}{\gamma_t} \le s\right] = F_2(s) \tag{0.3}$$

where F_2 is the GUE Tracy-Widom distribution from random matrix theory, see Appendix B. The result (0.3) says that the height of an interface for large $t$and the largest eigenvalue of GUE of large matrix dimension share the same distribution in the limit. Long time and large scale behaviors of a wide variety of systems are expected to be the same, constituting the KPZ universality class.

In the last decade, many generalizations and related results have been found. Behind their solvability lies the integrability of the systems, which means that generators of stochastic process have certain high symmetries. The theory of integrable systems also have a long history and there have been remarkable accumulation of knowledge on them. Whereas a lot of methods and results which have been developed in integrable system community are useful for studying stochastic interacting systems, the integrability for stochastic systems have some novel feature and provide new direction for the studies of integrable systems as well.

In this series of lectures, we discuss the integrable stochastic interacting systems, by mainly focusing on how the formulas above can be found and generalized.

Some notations. $\mathbb{N} = \{0,1,2,\cdots\}, \mathbb{Z}_+ = \{1,2,\cdots\}, \mathbb{T} = \{e^{i\theta}, 0 \le \theta < 2\pi\}$ Z is used as a normalization in general. For q-notation, see Appendix E.

1 TASEP

The main goal of this section is to get the result about the limiting distribution like (0.3) for the case of the TASEP with step initial condition in which all sites on $x \le 0$ are occupied and all the rest are empty at $t =$. For the integrated current $N(t)$ at the origin, which is the same as the number of particles at t on $x \ge 1$ for our setting, the result reads

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{N(t)-t/4}{2^{-4/3}t^{1/3}} \ge -s\right] = F_2(s). \tag{1.1}$$

What we actually try to show in the following is

$$\mathbb{P}[N(t) \ge N] = \frac{1}{Z} \sum_{\substack{x_i=N \\ 1\le i\le N}}^{\infty} \prod_{1\le j<k\le N} (x_j - x_k)^2 \prod_{j=1}^{N} \frac{t^{x_j} e^{-t}}{x_j!} \tag{1.2}$$

This formula looks very similar to (B.3) with $\beta = 2$ for the largest eigenvalue of GUE. One can follow the same steps (by replacing Hermite polynomials with Chariler polynomials) to show (1.1). The result (1.1) was first established by Johansson [12], by considering a discrete time version of TASEP and using combinatorial arguments. Here we show (1.2) by using the transition probability (or Green's function). A reference for this section is [20].

1.1 Time evolution equation for the transition probability

The transition probability for ASEP with N particles on $\mathbb{Z}$ is the probability that N particles starting from $y_1, \dots, y_N$ at time 0 are on sites $x_1, \dots, x_N$ at time t. (We assume $x_i < x_{i+1}, y_i < y_{i+1}, 1 \le i \le N-1$.) Let $X_i(t), 1 \le i \le N$ be the position of the i th particle at time t. Then the transition probability is

$$G(x_1, \dots, x_N; t | y_1, \dots, y_N; 0) = \mathbb{P}[X_i(t) = x_i, 1 \le i \le N | X_i(0) = y_i, 1 \le i \le N]. \tag{1.3}$$

We often omit the dependence on y. In this section we take the time scale s.t. $p + q = 1$.

First let us write down the time evolution (Kolmogorov forward equation, master equation) satisfied by the transition probability. The equation for one particle ($N = 1$) case reads

$$\frac{d}{dt} G(x; t) = pG(x-1; t) + qG(x+1; t) - G(x; t). \tag{1.4}$$

Next for $N = 2$, we have to consider two cases separately. When $x_2 - x_1 \ge 2$, the forward equation reads

$$\begin{aligned} \frac{d}{dt} G(x_1, x_2; t) &= pG(x_1 - 1, x_2; t) + qG(x_1 + 1, x_2; t) + pG(x_1, x_2 - 1; t) \\ &\quad + qG(x_1, x_2 + 1; t) - 2G(x_1, x_2; t). \end{aligned} \tag{1.5}$$

When $x_2 = x_1 + 1$, due to the exclusion rule, the equation is

$$\frac{d}{dt} G(x_1, x_1 + 1; t) = pG(x_1 - 1, x_1 + 1; t) + qG(x_1, x_1 + 2; t) - G(x_1, x_2; t). \tag{1.6}$$

The initial condition for the transition probability is

$$G(x_1, x_2; t | y_1, y_2; 0) = \delta_{x_1 y_1} \delta_{x_2 y_2}. \tag{1.7}$$

The transition probability is determined as the solution to (1.5),(1.6),(1.7). It is a little cumbersome that one has to deal with the two equations (1.5),(1.6) separately. But the second one can be replaced by a boundary condition for $G(x_1, x_2; t)$. Setting $x_2 = x_1 + 1$ in (1.5) one gets

$$\begin{aligned} \frac{d}{dt} G(x_1, x_1 + 1; t) &= pG(x_1 - 1, x_1 + 1; t) + qG(x_1 + 1, x_1 + 1; t) + pG(x_1, x_1; t) \\ &\quad + qG(x_1, x_1 + 2; t) - 2G(x_1, x_1 + 1; t). \end{aligned} \tag{1.8}$$

Comparing (1.8) with (1.6), we have

$$pG(x_1, x_1, t) + qG(x_1 + 1, x_1 + 1; t) = G(x_1, x_1 + 1; t). \tag{1.9}$$

This means that instead of considering (1.5),(1.6) for $x_1 < x_2$, one can consider (1.5) with the boundary condition (1.9) for $x_1 \le x_2$ and focus on the case $x_1 < x_2$.

For general N the situation is similar to the $N = 2$ case. The main forward equation reads

$$\begin{aligned} \frac{d}{dt} G(x_1, \dots, x_N; t) = \sum_{i=1}^{N} \Big(& pG(\dots, x_i - 1, \dots; t) + qG(\dots, x_i + 1, \dots, t) \\ & - G(\dots, x_i, \dots; t) \Big). \end{aligned} \tag{1.10}$$

One has to solve this with the boundary condition

$$pG(\dots, x_i, x_i, \dots; t) + qG(\dots, x_i + 1, x_i + 1, \dots; t) = G(\dots, x_i, x_{i+1}, \dots; t) \tag{1.11}$$

and the initial condition,

$$G(x_1, \dots, x_N; t = 0) = \prod_{i=1}^{N} \delta_{x_i y_i}. \tag{1.12}$$

1.2 Bethe ansatz for ASEP

Formally the time evolution equation can be written in a form

$$\frac{d}{dt} G = L^* G \tag{1.13}$$

with L the time evolution generator of the process. For solving the eigenvalue problem for L^*,

$$L^* \Psi = \Lambda \Psi, \tag{1.14}$$

for ASEP, we apply the Bethe ansatz [7], in which assume the form of the eigenfunction to be in the form

$$\Psi_z(x) = \sum_{\sigma \in S_N} A_\sigma \prod_{j=1}^{N} z_{\sigma_j}^{x_j}. \tag{1.15}$$

When $N = 1$, the equation is

$$p\Psi(x - 1) + q\Psi(x + 1) - \Psi(x) = \Lambda \Psi(x), \ \ x \in \mathbb{Z}. \tag{1.16}$$

The solution is, with $z \in \mathbb{C}$,

$$\Psi(x) = \Psi_z(x) \tag{1.17}$$

with the eigenvalue

$$\Lambda = p/z + qz - 1 =: \epsilon(z). \tag{1.18}$$

When $N = 2$, we wand to solve

$$p\Psi(x_1 - 1, x_2) + q\Psi(x_1, x_2 - 1) - 2\Psi(x_1, x_2) = \Lambda \Psi(x_1, x_2), \tag{1.19}$$

$$p\Psi(x, x) + q\Psi(x + 1, x + 1) = \Psi(x, x + 1). \tag{1.20}$$

According to the Bethe ansatz (1.15), we make an ansatz that the eigenfunction is given in the form,

$$\Psi(x_1, x_2) = \Psi_{z_1, z_2}(x_1, x_2) = A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_2^{x_1} z_1^{x_2}. \tag{1.21}$$

From (1.19) one sees $\Lambda = \epsilon(z_1) + \epsilon(z_2)$. From (1.20), one can determine

$$\frac{A_{21}}{A_{12}} = -\frac{p + qz_1 z_2 - z_2}{p + qz_1 z_2 - z_1}. \tag{1.22}$$

For general N, A_σ can also be written as

$$A_\sigma = \mathrm{sgn}\sigma \frac{\prod_{i<j}(p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - (p + q)\xi_{\sigma(i)})}{\prod_{i<j}(p + q\xi_i \xi_j - (p + q)\xi_i)}. \tag{1.23}$$

Remark: when we consider ASEP on a period lattice of size L, we also have to put the periodic boundary condition on the eigenfunction, from which we get some conditions which z_i's should satisfy. This is called the Bethe ansatz equation (BAE). This is a set of coupled algebraic equations which are usually unsolvable. In these lectures, we mostly consider ASEP on $\mathbb{Z}$.

The transition probability for ASEP can be constructed by taking a linear combination of eigenfunctions appropriately. It is given by [31]

$$G(x_1,\ldots,x_N;t|y_1,\ldots,y_N;0) = \sum_{\sigma\in S_N}\int_{C_r}\cdots\int_{C_r} d\xi_1\ldots d\xi_N A_\sigma \prod_{i=1}^N \xi_{\sigma(i)}^{x_i-y_{\sigma(i)}-1} e^{\sum_{i=1}^N \epsilon(\xi_i)t} \tag{1.24}$$

where C_r is a contour enclosing the origin anticlockwise with a radius small enough that all the poles in A_σ are not included in C_r. $\epsilon(\xi)$ is defined in (1.18). It is highly nontrivial to utilize this formula to study asymptotic behavior of ASEP [30–32].

1.3 Determinantal formula for TASEP

For TASEP, the Bethe eigenfunction simplifies a lot and the transition probability can be written as a single determinant [26].

Proposition 1.1.

$$G(x_1,\ldots,x_N;t|y_1,\ldots,y_N;0) = \det\left(F_{k-j}(x_k-y_j;t)\right)_{1\le j,k\le N} \tag{1.25}$$

Here the function $F_n(x,t)$ appearing as a matrix element of the determinant is

$$F_n(x,t) = \frac{1}{2\pi i}\int_{0,1} dz \frac{1}{z^{x+1}}(1-1/z)^{-n} e^{-(1-z)t}$$

where the contour enclosing the poles at $z=0,1$ of the integrand anticlockwise

The formula is found by using Bethe ansatz, one can prove it directly. To prove the formula it is useful to list a few properties of $F_n(x,t)$ [excercise].

Lemma 1.2. *(i)*

$$F_{n+1}(x,t) = \sum_{y=x}^\infty F_n(y,t) \tag{1.26}$$

(ii)

$$\frac{d}{dt}F_n(x,t) = F_n(x-1,t) - F_n(x,t) \tag{1.27}$$

Proof of Prop. 1.1 It is enough to check the forward equation and the initial conditions. Here we only consider the $N=2$ case in which (1.25) reads

$$G(x_1,x_2;t) = \begin{vmatrix} F_0(x_1-y_1;t) & F_1(x_2-y_2;t) \\ F_{-1}(x_1-y_2;t) & F_0(x_2-y_2;t)\end{vmatrix}. \tag{1.28}$$

For our special case where $p = 1, q = 0$, (1.5),(1.9), (1.7) read

$$\frac{d}{dt}G(x_1, x_2; t) = G(x_1 - 1, x_2; t) + G(x_1, x_2 - 1; t) - 2G(x_1, x_2; t), \tag{1.29}$$

$$G(x_1, x_1, t) = G(x_1, x_1 + 1; t), \tag{1.30}$$

$$G(x_1, x_2; t|y_1, y_2; 0) = \delta_{x_1 y_1}\delta_{x_2 y_2}. \tag{1.31}$$

We check (1.28) satisfies them. First about (1.29), we see

$$\begin{aligned}
\frac{d}{dt}G(x_1, x_2; t) &= \begin{vmatrix} \frac{d}{dt}F_0(x_1 - y_1; t) & F_1(x_2 - y_1; t) \\ \frac{d}{dt}F_{-1}(x_1 - y_2; t) & F_0(x_2 - y_2; t) \end{vmatrix} + \begin{vmatrix} F_0(x_1 - y_1; t) & \frac{d}{dt}F_1(x_2 - y_1; t) \\ F_{-1}(x_1 - y_2; t) & \frac{d}{dt}F_0(x_2 - y_2; t) \end{vmatrix} \\
&= \begin{vmatrix} F_0(x_1 - y_1 - 1; t) - F_0(x_1 - y_1; t) & F_1(x_2 - y_1; t) \\ F_{-1}(x_1 - y_2; t) - F_{-1}(x_1 - y_2 - 1; t) & F_0(x_2 - y_2; t) \end{vmatrix} \\
&+ \begin{vmatrix} F_0(x_1 - y_1; t) & F_1(x_2 - y_1; t) - F_1(x_2 - y_1 - 1; t) \\ F_{-1}(x_1 - y_2; t) & F_0(x_2 - y_2; t) - F_0(x_2 - y_2 - 1; t) \end{vmatrix} \\
&= G(x_1 - 1, x_2; t) + G(x_1, x_2 - 1; t) - 2G(x_1, x_2; t).
\end{aligned} \tag{1.32}$$

For (1.30), we see

$$\begin{aligned}
G(x_1, x_1; t) &= \begin{vmatrix} F_0(x_1 - y_1; t) & F_1(x_1 - y_1; t) \\ F_{-1}(x_1 - y_2; t) & F_0(x_1 - y_2; t) \end{vmatrix} \\
&= \begin{vmatrix} F_0(x_1 - y_1; t) & F_1(x_1 - y_1; t) - F_0(x_1 - y_1; t) \\ F_{-1}(x_1 - y_2; t) & F_0(x_1 - y_2; t) - F_{-1}(x_1 - y_2; t) \end{vmatrix} \\
&= G(x_1, x_1 + 1; t)
\end{aligned} \tag{1.33}$$

where in the second last equality we used (1.26).

About the initial condition, one sees

$$G(x_1, x_2; t = 0) = \begin{vmatrix} F_0(x_1 - y_1; 0) & F_1(x_2 - y_2; 0) \\ F_{-1}(x_1 - y_2; 0) & F_0(x_2 - y_2; 0) \end{vmatrix} = \begin{vmatrix} \delta_{x_1 y_1} & \sum_{z=x_2}^{\infty} \delta_{x_2 y_1} \\ \delta_{x_1 y_2} - \delta_{x_1, y_2+1} & \delta_{x_2 y_2} \end{vmatrix}. \tag{1.34}$$

Since $x_2 \geq y_2 > y_1$, the second term is zero.
Using these one can show that the determinant in (1.25) satisfies (1.5),(1.9), (1.7), i.e. it gives the transition probability for TASEP.

□

1.4 Charlier ensemble representation

In this section we get (1.2). First note that the distribution of the current can be obtained by taking an appropriate sum of the transition probability.

$$\mathbb{P}[N(t) \geq N] = \mathbb{P}[x_1(t) \geq 1] = \sum_{1 \leq x_1 < \cdots < x_N} G(x, t) \tag{1.35}$$

For $N = 2$, we see

$$\begin{aligned}
\mathbb{P}[x_1(t) \geq 1] &= \sum_{1 \leq x_1 < x_2} \begin{vmatrix} F_0(x_1 + 1; t) & F_1(x_2 + 1; t) \\ F_{-1}(x_1; t) & F_0(x_2; t) \end{vmatrix} \\
&= \sum_{1 \leq x_1 < x_2} \sum_{y_2 = x_2}^{\infty} \begin{vmatrix} F_0(x_1 + 1; t) & F_0(y_2 + 1; t) \\ F_{-1}(x_1; t) & F_{-1}(y_2; t) \end{vmatrix}
\end{aligned} \tag{1.36}$$

Writing the determinant in the last expression as $f(x_1, y_2)$, this is calculated as

$$\sum_{x_1=1}^{\infty} \sum_{x_2=x_1+1}^{\infty} \sum_{y_2=x_2}^{\infty} f(x_1, y_2) = \sum_{x_1=1}^{\infty} \sum_{y_2=x_1+1}^{\infty} \sum_{x_2=x_1+1}^{y_2} f(x_1, y_2) \tag{1.37}$$

$$= \sum_{x_1=1}^{\infty} \sum_{y_2=x_1+1}^{\infty} (y_2 - x_1) f(x_1, y_2) = \sum_{x_1=1}^{\infty} \sum_{x_2=x_1+1}^{\infty} (x_2 - x_1) f(x_1, x_2). \tag{1.38}$$

Here

$$f(x_1, x_2) = \begin{vmatrix} F_0(x_1+1; t) & F_0(x_2+1; t) \\ F_{-1}(x_1; t) & F_{-1}(x_2; t) \end{vmatrix} = \begin{vmatrix} F_0(x_1+1; t) & F_0(x_2+1; t) \\ F_0(x_1; t) - F_0(x_1+1; t) & F_0(x_2; t) - F_0(x_2+1; t) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{t^{x_1+1}}{(x_1+1)!} e^{-t} & \frac{t^{x_2+1}}{(x_2+1)!} e^{-t} \\ \frac{t^{x_1}}{(x_1)!} e^{-t} & \frac{t^{x_2}}{(x_2)!} e^{-t} \end{vmatrix} = \frac{t^{x_1+x_2+1}}{(x_1+1)!(x_2+1)!} e^{-2t}. \tag{1.39}$$

Substituting this into (1.38) one finds

$$\mathbb{P}[N(t) \geq 2] = \sum_{x_1=1}^{\infty} \sum_{x_2=x_1+1}^{\infty} (x_1 - x_2)^2 \frac{t^{x_1+x_2+1}}{(x_1+1)!(x_2+1)!} e^{-2t} = \frac{1}{2t} \sum_{x_1=2}^{\infty} \sum_{x_2=2}^{\infty} (x_1 - x_2)^2 \frac{t^{x_1+x_2}}{x_1! x_2!} e^{-2t}, \tag{1.40}$$

which is nothing but the $N = 2$ case of (1.2). Generalization to general N is left as an exsersize.

1.5 Schur measure and process

The Schur measure is a measure on the set of partitions, in the form

$$\frac{1}{Z} s_\lambda(a_1, \cdots, a_N) s_\lambda(b_1, \cdots, b_M). \tag{1.41}$$

Here s_λ is the Schur function, see Appendix C. By the Cauchy identity (C.23), the normalization here is

$$Z = \prod_{j,k} \frac{1}{1 - a_j b_k}. \tag{1.42}$$

The Schur process is a measure on GT_N, in the form

$$\frac{1}{Z} s_{\lambda^{(1)}}(a_1) s_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \cdots s_{\lambda^{(N)}}(a_N) s_{\lambda^{(N)}}(b_1, \cdots, b_M) \tag{1.43}$$

By (C.21), the marginal measure about $\lambda^{(N)}$ is nothing but the Schur measure (1.41).

There are various variants of the Schur measure and process, in which $s_\lambda(b_1, \cdots, b_M)$ is replaced by a Schur function related to a more general specialization ρ. Such measures appear for Markov dynamics on GT_N and TASEP can be realized as a marginal dynamics on the left diagonal $\lambda_i^i, 1 \leq i \leq N$.

Because the Schur function is written as a determinant by the Jacobi-Trudi formula (C.19), the Schur measure can be studied by the same methods as for GUE.

A Formulation of lattice gases

In this section we explain the formulation of the lattice gas on $\mathbb{Z}$ for the case of exclusion process. We do not give the details but refer the readers to existing literature [14–16, 27].

Let us introduce a notation,

$$\eta(x) = \begin{cases} 0, & \text{site } x(\in \mathbb{Z}) \text{ is empty,} \\ 1, & \text{site } x(\in \mathbb{Z}) \text{ is occupied.} \end{cases} \tag{A.1}$$

Our state space is $X = \{0,1\}^{\mathbb{Z}}$, which is compact in the product topology. $\eta(\in X)$ is called a configuration (of particles). We set

$$\eta^{xy}(u) = \begin{cases} \eta(y), & \text{if } \ u = x, \\ \eta(x), & \text{if } \ u = y, \\ \eta(u), & \text{if } \ u \neq x, y. \end{cases} \tag{A.2}$$

Let us set $C(X)$ to be the space of continuous functions on X. Let us consider a Feller process η_t: $[0,\infty) \to X$ for which $E^\eta f(\eta_t) \in C(X)$ for any $f \in C(X)$. Let P^μ be the distribution of the process with initial measure μ and E^μ be the corresponding expectation. We sometimes consider the situation in which the process starts from a single configuration η for which case we abuse the notation like P^η and E^η. For η_t, one can define the semigroup $S(t)$ on $C(X)$ by

$$S(t)f(\eta) = E^\eta f(\eta_t), \quad f \in C(X). \tag{A.3}$$

It is known that there is a one-to-one correspondence between the semigroup $S(t)$ and the generator L defined by

$$Lf = \lim_{t\to 0} \frac{S(t)f - f}{t}. \tag{A.4}$$

The lattice gas can be constructed by giving its generator. Let $c(x,y,\eta)$ be the rate of exchange of the occupancies at x and y. The generator of the process is given by introducing an operator

$$Lf(\eta) = \frac{1}{2} \sum_{x,y\in\mathbb{Z}} c(x,y,\eta)[f(\eta^{xy}) - f(\eta)] \tag{A.5}$$

for f a cylinder function (which depends only on finitely many coordinates) and then taking its closure. For this construction to work the rate $c(x,y,\eta)$ should satisfy certain conditions but here we simply assume that they are satisfied.

In the $t \to \infty$ limit, the system approaches the stationary measure μ, which is defined by

$$E^\mu[Lf] = 0 \tag{A.6}$$

For the ASEP the operator L on the cylinder set is given by

$$Lf(\eta) = \frac{1}{2} \sum_{x\in\mathbb{Z}} \left(p\eta(x)(1-\eta(x+1)) + q(1-\eta(x))\eta(x+1)\right)[f(\eta^{xy}) - f(\eta)]. \tag{A.7}$$

For ASEP on $\mathbb{Z}$, there are two series of the stationary measures. One is the product measure with $E[\eta(x)] = 1/(1+(q/p)^x)$ and its translations. The other is the Bernoulli measure with density $\rho, 0 \leq \rho \leq 1$. It is known that these exhaust the all extremal stationary measures.

B GUE and Tracy-Widom distribution [3, 10, 17]

In random matrix theory, Gaussian ensembles play a prominent role.

Definition B.1. *(Gaussian ensembles)*

In Gaussian ensembles, the measure for $N \times N$ matrix H is given in the form,

$$P(H)dH = \frac{1}{Z} e^{-\frac{\beta}{2}\mathrm{Tr}H^2} dH, \quad \beta = 1, 2, 4. \tag{B.1}$$

For GOE($\beta = 1$, Gaussian orthogonal ensemble(GOE)*), H is taken to be a real symmetric matrix and the measure dH is $dH = \prod_{j=1}^{N} dH_{jj} \prod_{j<l} dH_{jl}$. For GUE($\beta = 2$,* Gaussian unitary ensemble(GUE)*), H is taken to be a hermitian matrix and the measure dH is $dH = \prod_{j=1}^{N} dH_{jj} \prod_{j<l} dH_{jl}^R \prod_{j<l} dH_{jl}^I$ where H_{jl}^R and H_{jl}^I denotes the real and imaginary part of H_{jl} respectively. GSE($\beta = 4$) means the* Gaussian symplectic ensemble(GSE)*). For a precise definition see a reference.*

For Guassian ensembles, the joint eigenvalue density can be written down explicitly.

Proposition B.2. *The probability density of eigenvalues, $x_i, 1 \le i \le N$, ($x_i \le x_{i+1}, 1 \le i \le N-1$) is*

$$P(x_1, \ldots, x_N) = \frac{1}{Z} \prod_{j<l} |x_l - x_j|^\beta \prod_{j=1}^{N} e^{-\frac{\beta}{2}x_j^2}, \quad \beta = 1, 2, 4. \tag{B.2}$$

For a proof see for instance [3].

By using this joint distribution one can in principle study all properties of the eigenvalues of Gaussian ensembles. For example, the Wigner's semin-circle law can be derived. In our discussions, the probabilistic properties of the largest eigenvalue are important. From the above joint distribution, one finds that the distribution function of the largest eigenvalue of GUE can be written in the following N fold integral.

Corollary B.3. *The distribution function of the largest eigenvalue $x_{\max}$*

$$\mathbb{P}_{N2}[x_{\max} \le u] = \frac{1}{Z} \int_{(-\infty,u]^N} \prod_{1 \le j < l \le N} |x_l - x_j|^\beta \prod_{j=1}^{N} e^{-\frac{\beta}{2}x_j^2} dx_1 \cdots dx_N. \tag{B.3}$$

We are interested in the large N asymptotics of this quantity. This N fold integral expression is not very suited for doing this. We rewrite it into the Fredholm determinant. For the moment let us focus on the $\beta = 2$ case.

By rewriting the determinant (using Heine identity below and $\det(1+AB) = \det(1+BA)$), one can show

Proposition B.4.

$$\mathbb{P}_{N2}[x_{\max} \le u] = \det(1 - \chi_u K_{N2} \chi_u) \tag{B.4}$$

where

$$K_{N2}(x, y) = e^{-\frac{x^2+y^2}{2}} \sum_{n=0}^{N-1} \frac{H_n(x) H_n(y)}{\sqrt{\pi} 2^n n!}. \tag{B.5}$$

Here $H_n(x)$ is the n th Hermite *polynomial*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{B.6}$$

Note that the kernel (B.5) can be written in the form $K(x,y) = \sum_{n=0}^{N-1} \phi_n(x)\psi_n(y)$ if we set $\phi_n(x) = \psi_n(x) = \frac{e^{-x^2/2}}{\pi^{1/4}\sqrt{2^n n!}} H_n(x)$. In addition, by using the integral representations of the Hermite polynomial,

$$H_n(x) = \frac{e^{x^2} 2^n}{\sqrt{\pi} i} \int_{i\mathbb{R}} dz e^{z^2 - 2xz} z^n = \frac{n!}{2\pi i} \int_0 \frac{dz}{z^{n+1}} e^{2xz - z^2}, \tag{B.7}$$

one can find a double contour integral formula of the kernel,

$$K_{N2}(x,y) = 2e^{-y^2} \int_{i\mathbb{R}} dz \int_0 dw \frac{z^N}{w^N} \frac{e^{z^2 - 2zx - w^2 + 2wy}}{z - w} \tag{B.8}$$

Now it has become easier to consider the large N asymptotics. The basic asymptotics we need is that of the Hermite polynomials [28]. The Airy function $\mathrm{Ai}(x)$ is defined by

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty + i\epsilon}^{+\infty + i\epsilon} e^{ixz + \frac{iz^3}{3}} dz, \ \epsilon > 0. \tag{B.9}$$

Let us write

$$\phi_n(x) = \frac{H_n(x) e^{-x^2/2}}{\pi^{1/4}\sqrt{2^n n!}}. \tag{B.10}$$

Then for the scaling $n = N - N^{1/3}\lambda, x = \sqrt{2N} + \frac{\xi}{\sqrt{2}N^{1/6}}$, we have

$$\phi_N(x) \sim 2^{1/4} N^{-1/12} \mathrm{Ai}(\xi + \lambda). \tag{B.11}$$

This asymptotics can be shown by applying saddle point method to the integral representations (B.7).

To describe the results we introduce a kernel and the corresponding distribution.

Definition B.5. *The* Airy *kernel $K_2(x,y)$ is defined by*

$$K_2(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x - y} = \int_0^\infty \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda) d\lambda. \tag{B.12}$$

The GUE Tracy-Widom *distribution $F_2(s)$ [29] is defined by*

$$F_2(s) = \det(1 - \chi_s K_2 \chi_s)_{L^2(\mathbb{R})}. \tag{B.13}$$

This is the limiting distribution for the appropriately scaled largest eigenvalue in GUE.

Theorem B.6.

$$\lim_{N\to\infty} \mathbb{P}_{N2} \left[(x_{\max} - \sqrt{2N})\sqrt{2}N^{1/6} \le s \right] = F_2(s). \tag{B.14}$$

This follows from

Proposition B.7.

$$\lim_{N\to\infty} \frac{1}{\sqrt{2}N^{1/6}} K_{N2}\left(\sqrt{2N} + \frac{\xi_1}{\sqrt{2}N^{1/6}}, \sqrt{2N} + \frac{\xi_2}{\sqrt{2}N^{1/6}} \right) = K_2(\xi_1, \xi_2), \tag{B.15}$$

which in turn is a consequence of (B.11).

So far our discussions are only for the GUE. One can generalize the discussions to other two cases, GOE,GSE. The limiting distribution is denoted by F_1, F_4

Theorem B.8.

$$\lim_{N\to\infty} \mathbb{P}_{N\beta}\left[(x_{\max} - \sqrt{2N})\sqrt{2}N^{1/6} \le s\right] = F_\beta(s), \quad \beta = 1, 4. \tag{B.16}$$

Before ending the section we state

Proposition B.9. *(Heine identity)*

$$\begin{aligned}
&\int \det\left(\Phi_j(x_{l+1})\right)_{0\le j,l\le N-1} \det\left(\Psi_j(x_{l+1})\right)_{0\le j,l\le N-1} dx_1\cdots dx_N \\
&= N! \det\left(\int \Phi_j(x)\Psi_l(x)dx\right)_{0\le j,l\le N-1}.
\end{aligned} \tag{B.17}$$

This is seen as follows

$$\begin{aligned}
&\det\left(\int \Phi_j(x)\Psi_l(x)dx\right)_{0\le j,l\le N-1} = \int dx_1\cdots dx_N \det\left(\Phi_i(x_j)\Psi_j(x_i)\right) \\
&= \int dx_1\cdots dx_N \prod_i \Phi_i(x_i)\det\left(\Psi_j(x_i)\right) = \int dx_1\cdots dx_N \prod_i \Phi_i(x_{\sigma(i)})\det\left(\Psi_j(x_{\sigma(i)})\right) \\
&= \int dx_1\cdots dx_N \mathrm{sgn}\sigma \prod_{\mathrm{i}} \Phi_{\mathrm{i}}(\mathrm{x}_{\sigma(\mathrm{i})})\det\left(\Psi_{\mathrm{j}}(\mathrm{x}_{\mathrm{i}})\right) \\
&= \frac{1}{N!}\int dx_1\cdots dx_N \sum_\sigma \mathrm{sgn}\sigma \prod_{\mathrm{i}} \Phi_{\mathrm{i}}(\mathrm{x}_{\sigma(\mathrm{i})})\det\left(\Psi_{\mathrm{j}}(\mathrm{x}_{\mathrm{i}})\right) \\
&= \int dx_1\cdots dx_N \det\left(\Phi_i(x_j)\right)\det\left(\Psi_i(x_j)\right).
\end{aligned} \tag{B.18}$$

]

C Partition, Schur function, etc

First a partition is an n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $n \in \mathbb{N}, \lambda_j \in \mathbb{Z}_+, 1 \le j \le n$ s.t. $\lambda_1 \ge \ldots \ge \lambda_n$. n is called the length of the partition and is denoted as $\ell(\lambda)$. The set of all partitions of length n is denoted by $\mathcal{P}_n$. A partition λ can also be represented (and identified) as a Young diagram with n rows of length $\lambda_1, \ldots, \lambda_n$. The transpose λ' is the partition of length λ_1 defined as $\lambda'_i = \#\{j \in \mathbb{Z}_+ | \lambda_j \ge i\}, 1 \le i \le \lambda_1$. For two partitions $\lambda \in \mathcal{P}_n, \mu \in \mathcal{P}_m$ s.t. $m \le n$ and $\lambda_i - \mu_i \ge 0, 1 \le i \le n$ (with the understanding $\mu_i \equiv 0, m < i \le n$), a pair (λ, μ) is called a skew diagram and is denoted by λ/μ, with $|\lambda/\mu| = |\lambda| - |\mu|$. We say that λ/μ is a horizontal strip iff in each column λ/μ has at most one box, i.e., $\mu_i \le \lambda_i \le \mu_{i-1}$. A column-strict (skew) Young tableaux is a sequence of partitions

$$(\mu =)\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(N)} (= \lambda)$$

such that $\lambda^{(i)}/\lambda^{(i-1)}, 1 \le i \le N$ is a horizontal strip. If $\mu = \phi$, this is equivalent to the semi-standard Young tableaux with entries from $\{1, \cdots, N\}$. Put i to the horizontal strip of $\lambda^{(i)}/\lambda^{(i-1)}$.

(Skew) Schur function with N variables $x = (x_1, \ldots, x_N)$ for a skew Young diagram λ/μ is defined as

$$s_{\lambda/\mu}(x_1, \ldots, x_N) = \sum_T x^T, \quad x^T = \prod_{i=1}^N x_i^{|\lambda^{(i)} - \lambda^{(i-1)}|}$$

where the sum over T is wrt column strict tableaux with $\lambda^{(0)} = \mu, \lambda^{(N)} = \lambda$.

When $N = 1$

$$s_{\lambda/\mu}(x_1) = \begin{cases} x_1^{|\lambda-\mu|} & \text{If } \mu_i \leq \lambda_i \leq \mu_{i-1} \\ 0 & \text{otherwise} \end{cases}$$

$$s_{\lambda'/\mu'}(x_1) = \begin{cases} x_1^{|\lambda'-\mu'|} & \text{If } (\lambda_j = \mu_j \text{ or } \lambda_j = \mu_j + 1) \ \& \ \lambda_i \geq \mu_i \\ 0 & \text{otherwise} \end{cases}$$

The (skew) Schur function can be written as a single determinant by the Jacobi-Trudi formula,

$$s_{\lambda/\mu}(x) = \det(h_{\lambda_i - \mu_j - i + j}(x))_{1 \leq i,j \leq \ell(\lambda)} \tag{C.19}$$

where $h_k(x) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq N} x_{i_1} \cdots x_{i_k}$ is the complete homogeneous symmetric function. For the case of one variable $h_k(x_1) = x_1^k 1_{k \geq 0}$.

Properties of Schur function.

$$\sum_\mu s_{\mu/\lambda}(x) s_{\mu/\nu}(y) = \sum_\tau s_{\nu/\tau}(x) s_{\lambda/\tau}(y) \Pi(x, y) \tag{C.20}$$

$$\sum_\lambda s_\lambda(x) s_{\mu/\lambda}(y) = s_\mu(x, y) \tag{C.21}$$

where

$$\Pi(x, y) = \prod_{i=1}^N \prod_{j=1}^M \frac{1}{1 - x_i y_j} \tag{C.22}$$

Setting $\lambda = \nu = \phi$ in the first equality, we get the Cauchy identity,

$$\sum_\mu s_\mu(x) s_\mu(y) = \Pi(x, y). \tag{C.23}$$

The Gelfand-Tsetlin cone is defined to be

$$\mathrm{GT}_N := \{\lambda_i^{(j)}, 1 \leq i \leq j \leq N | \lambda_i^{(j)} \in \mathbb{N}, \lambda_i^{(j-1)} \leq \lambda_i^{(j)} \leq \lambda_{i-1}^{(j-1)}\}$$

Each $\lambda^{(j)} = \{\lambda_i^{(j)}, 1 \leq i \leq j\}, 1 \leq j \leq N$ is a Young diagram and an element of GT_N denoted by $\underline{\lambda}$ is considered as consisting of these N Young diagrams $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ and is represented as an array of a triangular shape.

D LUE formula for TASEP with step initial condition

In [12], the following LUE formula was shown.

$$
\begin{array}{ccccccccc}
\lambda_N^{(N)} & \lambda_{N-1}^{(N)} & & & \cdots & & & \lambda_2^{(N)} & \lambda_1^{(N)} \\
 & \ddots & & & \vdots & & & \iddots & \\
 & & \lambda_3^{(3)} & & \lambda_2^{(3)} & & \lambda_1^{(3)} & & \\
 & & & \lambda_2^{(2)} & & \lambda_1^{(2)} & & & \\
 & & & & \lambda_1^{(1)} & & & &
\end{array}
$$

Figure 2: Gelfand-Tsetlin cone

Proposition D.1.

$$
\mathbb{P}[N(t) \geq N] = \frac{1}{Z'_{N2}} \int_{[0,t]^N} \prod_{1 \leq j < k \leq N} (x_j - x_k)^2 \prod_{j=1}^{N} e^{-x_j} dx_1 \dots dx_N. \tag{D.24}
$$

Z'_{N2} is a normalization.

The similarity of this expression to (B.3) is obvious.

The formula can be derived using the transition probability [1, 18] One can show [18]

Proposition D.2. *Under the condition $M > y_N - y_1$,*

$$
\begin{aligned}
&\mathbb{P}[X_1(t) \geq y_1 + M] \\
&= \sum_{y_1 + M \leq x_1 < x_2 < \cdots < x_N} G(x_1, x_2, \cdots, x_N; t | y_1, y_2, \cdots, y_N; 0) \\
&= \frac{1}{\prod_{j=1}^{N} j!} \int_{[0,t]^N} dt_1 \cdots dt_N \prod_{1 \leq j < k \leq N} (t_k - t_j) \det\left(F_{-j+1}(y_1 - y_j + M - 1; t_{N-k+1})\right).
\end{aligned} \tag{D.25}
$$

The step initial condition corresponds to setting $y_j = -j + 1, j = 1, \cdots, N$. In this case (D.25) reduces to (D.24). Here we only see how the computation proceeds for $N = 2$ with step i.c.

$$
\begin{aligned}
\mathbb{P}[N(t)\geq 2] &= \sum_{1\leq x_1<x_2} G(x_1,x_2;t|y_1=-1,y_2=0;0)\\
&= \begin{vmatrix} F_1(2;t) & F_2(3;t)\\ F_0(1;t) & F_1(2;t)\end{vmatrix} = \int_0^t dt_2 \int_0^t ds \begin{vmatrix} F_0(1;t_2) & F_1(2;s)\\ F_{-1}(0;t_2) & F_0(1;s)\end{vmatrix}\\
&= \int_0^t dt_2 \int_0^t ds \int_0^s dt_1 \begin{vmatrix} F_0(1;t_2) & F_0(1;t_1)\\ F_{-1}(0;t_2) & F_{-1}(0;t_1)\end{vmatrix}\\
&= \int_0^t dt_2 \int_0^t ds(t-t_1) \begin{vmatrix} F_0(1;t_2) & F_0(1;t_1)\\ F_{-1}(0;t_2) & F_{-1}(0;t_1)\end{vmatrix}\\
&= \frac{1}{2}\int_0^t dt_2 \int_0^t ds(t_2-t_1) \begin{vmatrix} F_0(1;t_2) & F_0(1;t_1)\\ F_{-1}(0;t_2) & F_{-1}(0;t_1)\end{vmatrix}\\
&= \frac{1}{2}\int_0^t dt_2 \int_0^t ds(t_2-t_1) \begin{vmatrix} F_0(1;t_2) & F_0(1;t_1)\\ F_0(0;t_2) & F_0(0;t_1)\end{vmatrix}\\
&= \frac{1}{2}\int_0^t dt_2 \int_0^t ds(t_2-t_1) \begin{vmatrix} t_2e^{-t_2} & t_1e^{-t_2}\\ e^{-t_2} & e^{-t_1}\end{vmatrix}\\
&= \frac{1}{2}\int_0^t dt_2 \int_0^t ds(t_2-t_1)^2 e^{-t_1-t_2}.
\end{aligned} \tag{D.26}
$$

The computation can be generalized to arbitrary N and in this way one can arrive at (D.24).

E Some q-functions and q-formulas

In this appendix, we summarize a few q-notations, q-functions and q-formulas. The first is the q-Pochhammer symbol, or the q shifted factorial. For $|q|<1$ and $n\in\mathbb{N}$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{E.27}$$

The q-binomial theorem will be useful in various places in the discussions,

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \ |z|<1. \tag{E.28}$$

In particular the $a=0$ case appears in many applications. Another q-binomial formula reads (see e.g. Cor.10.2.2.(b) in [4])

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n}z^n = (z;q)_\infty. \tag{E.29}$$

There is yet another version of the q-binomial theorem (see e.g. Cor.10.2.2.(c) in [4]),

$$\sum_{k=0}^{\ell}\frac{(-1)^k q^{k(k-1)/2}(q;q)_\ell}{(q;q)_k(q;q)_{\ell-k}}x^k = (1-x)(1-xq)\cdots(1-xq^{\ell-1}). \tag{E.30}$$

The q-exponential function, denoted as $e_q(z)$ is defined to be

$$e_q(z) := \frac{1}{((1-q)z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q;q)_n} z^n. \tag{E.31}$$

The second equality is by the above q-binomial theorem (E.28). From the series expansion expression, it is easy to see that this tends to the usual exponential function in the $q \to 1$ limit.

The q-Gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = (1-q)^{1-x} \frac{(q;q)_\infty}{(q^x;q)_\infty}. \tag{E.32}$$

The q-digamma function is defined by $\Phi_q(z) = \partial_z \log \Gamma_q(z)$. In the $q \to 1$ limit, they tends to the usual Γ function and the digamma function respectively.

Ramanujan's summation formula (cf [4] p502, [11] p138) is a two-sided generalization of the above q-binomial theorem (E.28). For $|q| < 1, |b/a| < |z| < 1$,

$$\sum_{n \in \mathbb{Z}} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(az;q)_\infty (\frac{q}{az};q)_\infty (q;q)_\infty (\frac{b}{a};q)_\infty}{(z;q)_\infty (\frac{q}{a};q)_\infty (b;q)_\infty (\frac{b}{az};q)_\infty}. \tag{E.33}$$

F Pauli matrices, Tensor product

Pauli matrices are

$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma^y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{F.34}$$

Tensor product notation. Suppose there are m dimensional vector spaces $V_i, 1 \le i \le N$. For an $m \times m$ matrix A, A_i acts nontrivially only on V_i. Similarly, for $m^2 \times m^2$ matrix B, $B_{i,j}$ acts nontrivially only on V_i, V_j.

For example, when there are three two-dimensional vector spaces $V_1 = V_2 = V_3 = \mathbb{C}^2$ and 4×4 matrix A, $\sigma_1^x = \sigma^x \otimes 1_2 \otimes 1_2, \sigma_2^z = 1_2 \otimes \sigma^z \otimes 1_2$, $A_{12} = A \otimes 1_2$, where 1_2 means the 2×2 identity matrix.

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