5. Nonlinear Fluctuating Hydrodynamics and Two Species ASEP

T. Sasamoto

(Based on collaborations with Z. Chen, J. de Gier, I. Hiki)

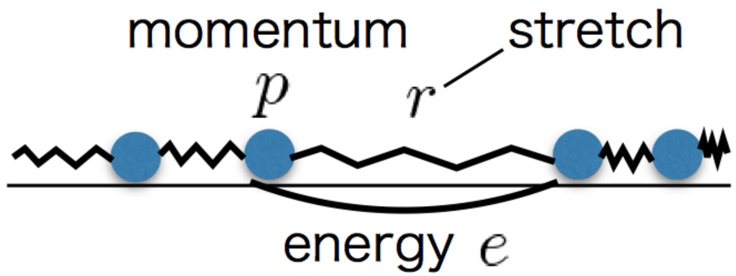
25 Jan 2019 @ ICTS Bangalore

Reference: arXiv:1803.06829 (PRL 120 (2018) 240601)

5.1. Nonlinear fluctuating hydrodynamics

Fermi-Pasta-Ulam chain

$$H = \sum_j \left(\frac{p_j^2}{2} + V(x_{j+1} - x_j) \right)$$



with $$V(x) = \frac{1}{2}x^2 + \frac{\alpha}{3}x^3 + \frac{\beta}{4}x^4$$

- FPU tried to see thermalization numerically, but recurrence seemed to occur. ($\rightarrow$ Chaos, Soliton, ...)
- Three conserved quantities (stretch, momentum, energy)
- Shows anomalous heat transport.
- Still difficult to study large time behaviors of the model, but some aspects may be understood by using a connection to KPZ through nonlinear fluctuating hydrodynamics.

Nonlinear fluctuating hydrodynamics

A conjectural theory for 1D multi-component systems which predicts that the distributions and correlations of "normal modes" are described by the ones of the single-component KPZ equation.

van Beijreren 2011, Spohn 2013-

For an anharmonic chain, there are three conserved fields.

$$r_j = x_{j+1} - x_j, \quad p_j, \quad e_j = \frac{p_j^2}{2} + V(r_j)$$

Equations of motion

$$\begin{aligned} \dot{r}_j &= p_{j+1} - p_j \\ \dot{p}_j &= V'(r_j) - V'(r_{j-1}) \\ \dot{e}_j &= p_{j+1} V'(r_j) - p_j V'(r_{j-1}) \end{aligned}$$

This can be summarized as the continuity equation

$$\frac{d}{dt}\vec{G}(j,t) + \vec{J}(j+1,t) - \vec{J}(j,t) = 0$$

where

$$\vec{G} = (r_j, p_j, e_j) \quad \vec{J} = (-p_j, -V'(r_{j-1}), -p_j V'(r_{j-1}))$$

Hydrodynamics: Euler equation

$$\frac{\partial}{\partial t}\vec{g} + \frac{\partial}{\partial x}\vec{j} = 0$$

For taking into account the fluctuations effectively, we add noise (η) and a diffusion term to get

$$\frac{\partial}{\partial t}\vec{g} + \frac{\partial}{\partial x}(\vec{j} + \partial_x D\vec{g} + B\vec{\eta}) = 0$$

Expanding around equilibrium up to the 2nd order, we find

$$\frac{\partial}{\partial t}\vec{u} + \frac{\partial}{\partial x}\left(A\vec{u} + \frac{1}{2}\langle \vec{u}, H\vec{u}\rangle + \partial_x D\vec{u} + B\vec{\eta}\right) = 0$$

Diagonalizing A as $RAR^{-1} = \mathrm{diag}(c_1, c_0, c_{-1})$ and setting $\vec{\phi} = R\vec{u}$ (normal modes), we have

$$\frac{\partial}{\partial t}\phi_\alpha + \frac{\partial}{\partial x}\left(c_\alpha \phi_\alpha + \frac{1}{2}\langle \vec{\phi}, G^\alpha \vec{\phi}\rangle + \partial_x (D\vec{\phi})_\alpha + (B\vec{\eta})_\alpha\right) = 0$$

If we assume that the main contributions from the nonlinear term come from the diagonal terms, then the equation for a component is noisy-Burgers equation.

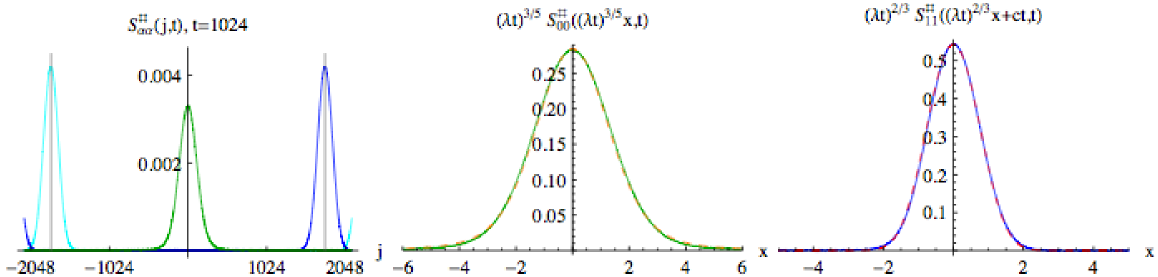
$\Rightarrow$ Correlations and distributions for the normal modes are expected to be described by the ones of the KPZ equation.

Simulations

For an anharmonic chain, $c_{\pm} = \pm 1, c_0 = 0$, corresponding to two sound modes and one heat mode. The correlation of each sound mode seems to be given by the stationary KPZ 2pt function.

MD simulations for shoulder potential (Mendl Spohn)

$V(x) = \infty \ (0 < x < \frac{1}{2}), 1(\frac{1}{2} < x < 1), \ 0(x > 1)$

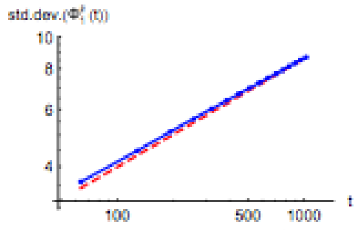


(a) overview (b) heat, $\lambda = 1.624$ (c) sound, $\lambda = 1.442$

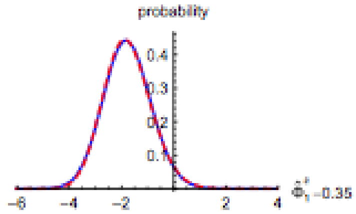
Universal KPZ distributions?

Mendl Spohn

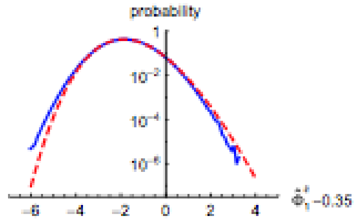
- By considering the integrated current for normal modes, one can also observe $\boldsymbol{F_0}$.
- For step type initial condition in which the macro paramters (temperature, pressure) change at the origin, one observes GUE TW.



(a) std. dev. of $\Phi_1^\sharp(t)$



(d) $(\Gamma_1 t)^{-1/3}\,\Phi_1^\sharp(t)$

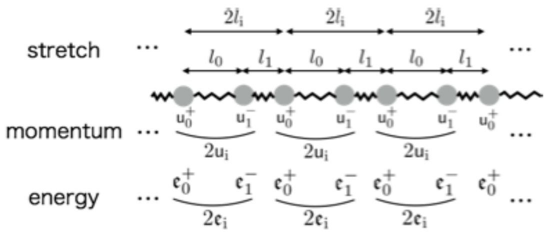


(g) $(\Gamma_1 t)^{-1/3}\,\Phi_1^\sharp(t)$

Flat type i.c.

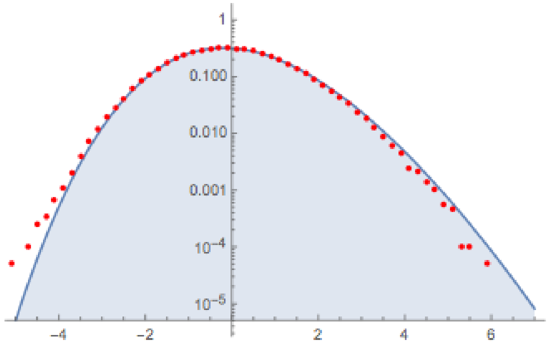
Hiki-TS

For a pure periodic initial condition, there is no randomness and hence no fluctuation.

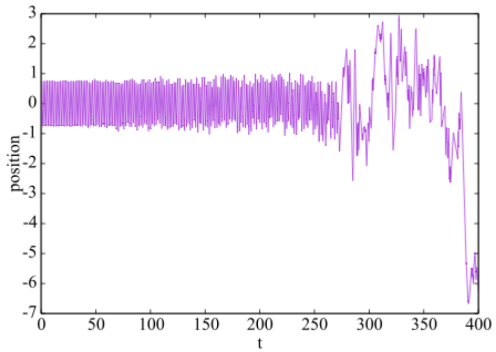


With tiny random perturbation, the GOE (flat KPZ) fluctuation is observed, due to effective noise generated by molecular chaos.

GOE TW



A particle trajectory



Problem 1: Can we prove them (or Show them analytically) ?

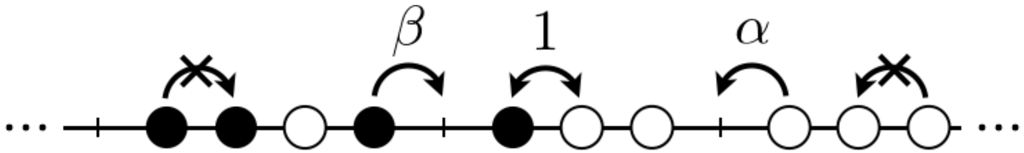
5.2 NLFHD for stochastic models

NLFHD theory applies to more general multi-component systems with more than one conserved quantities.

Stochastic systems should be easier to treat.

Remark. Kubo formula, fluctuation theorem etc have been proved for stochastic systems.

Two species ASEP (1998 Arndt-Heinzel-Rittenberg (AHR))

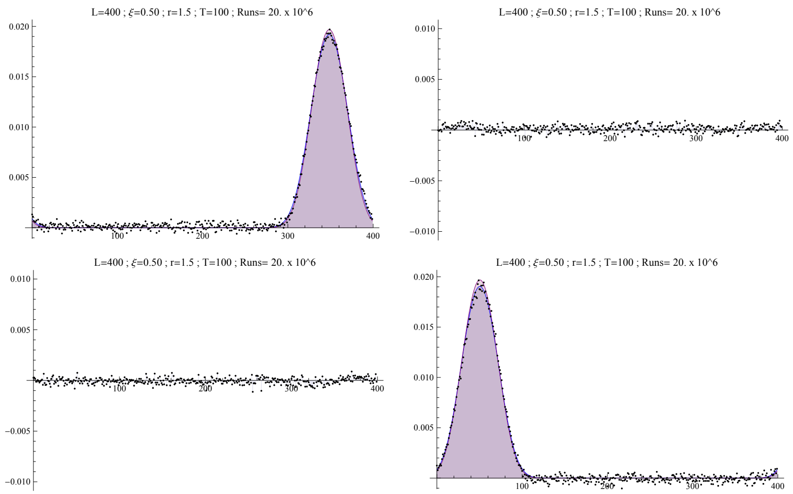


$$\boldsymbol{\alpha + \beta = 1}$$

- Two conserved quantities (numbers of $+$ and $-$ particles).
- Exactly solvable

Monte Carlo simulation for AHR model

For AHR model, stationary KPZ 2pt function had been observed in MC simulations (Ferrari TS Spohn 2014).



For step i.c. GUE TW was observed (Mendl-Spohn)

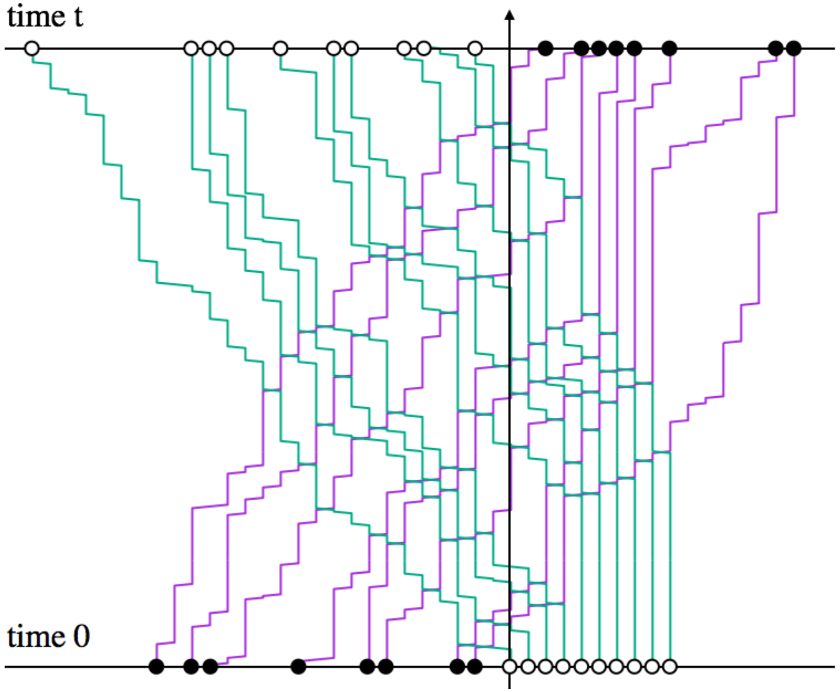
For periodic i.c. GOE TW was observed (Hiki-TS)

Problem 2: Can we prove these?

5.3 $\rho - 1$ step i.c.

Chen, de Gier, Hiki, TS

Infinite + particles (●) with density ρ on the left and infinite − particles (○) packed on the right.



Hydrodynamics

Macroscopic densities of $\pm$ particles, $\boldsymbol{u(t,x) = (\rho_+, \rho_-)}$, satisfy

$$\frac{\partial u(t,x)}{\partial t} + \frac{\partial \mathbf{j}(u(t,x))}{\partial x} = 0$$

where $\mathbf{j}_\pm(\boldsymbol{u})$ represent macroscopic current of $\pm$ particle,

$$\mathbf{j}_+(u) = \rho_+(1 - \rho_+ - \rho_-) + 2\rho_+\rho_-$$
$$\mathbf{j}_-(u) = -(1 - \rho_+ - \rho_-)\rho_- - 2\rho_+\rho_-$$

This set of coupled equations can be solved by switching to the normal modes which diagonalize the Jacobian $\boldsymbol{\partial \mathbf{j}/\partial \mathbf{u}}$.

The average currents of $\pm$ particles are
$j_+ = \frac{\rho(3-\rho)^2}{16}$, $j_- = \frac{(1+\rho)^2(2-\rho)}{16}$.

Normal modes for $\rho - 1$ step i.c.

The NLFHD predicts

$$\lim_{t\to\infty} P_{\infty,\infty}[s_-(n,m,\rho,t) \le s_-] = F_2(s_-)$$

$$\lim_{t\to\infty} P_{\infty,\infty}[s_+(n,m,\rho,t) \le s_+] = F_G(s_+)$$

where n, m are the numbers of $\pm$ particles which passed the origin up to time t and scaled variables $s_\pm(n,m,\rho,t)$ are given by

$$s_-(n,m,\rho,t) = \frac{(1+\rho)\cdot n - (3-\rho)\cdot m + (1-\rho)(1-\frac{(1-\rho)^2}{4})t}{(3/16)^{1/3}(1-\rho)(3-\rho)^{2/3}(1+\rho)^{2/3}t^{1/3}}$$

$$s_+(n,m,\rho,t) = \frac{-2(2-\rho)\cdot n + 2\rho\cdot m + 2(2-\rho)(1-\rho)\rho t}{3(1-\rho)^{3/2}\sqrt{\rho(2-\rho)}t^{1/2}}$$

5.4 Bethe ansatz for AHR model

We set $\alpha + \beta = 1$.

Let x_j: position of $+$ particle and y_j: position of $-$ particle.

For $n_+ = 1$, $n_- = 1$.

For $x \neq y \pm 1$ the eigenvalue equation reads

$$\Lambda\psi(x;y) = \alpha\psi(x;y+1) + \beta\psi(x-1;y) - (\alpha+\beta)\psi(x;y),$$

while if the two particles are nearest neighbours, the equation is

$$\Lambda\psi(x;x+1) = \alpha\psi(x;x+2) + \beta\psi(x-1;x+1) - \psi(x;x+1),$$
$$\Lambda\psi(x+1;x) = \psi(x;x+1) - (\alpha+\beta)\psi(x+1;x).$$

When $\alpha + \beta = 1$ these can be solved using the ansatz

$$\psi(x; y) = \begin{cases} A_{+-} z^x w^{-y} & x < y \\ A_{-+} z^x w^{-y} & x > y \end{cases}$$

It then follows that the eigenvalue is given by

$$\Lambda = \alpha w^{-1} + \beta z^{-1} - \alpha - \beta.$$

We have

$$(\alpha + \beta - 1) A_{+-} z^x w^{-x-1} = 0 \quad \text{for} \quad \alpha + \beta = 1,$$

and

$$A_{-+}(\alpha z + \beta w) = A_{+-}.$$

General case

$$\psi(x_1,\ldots,x_n;y_1,\ldots,y_m;\{z\},\{w\})$$

$$= \sum_{\pi\in S_n} \mathrm{sign}(\pi) \prod_{j=1}^{n} \left(\frac{1}{z_{\pi_j}-1}\right)^{j-1} z_{\pi_j}^{x_j}$$

$$\times \sum_{\rho\in S_m} \mathrm{sign}(\rho) \prod_{k=1}^{m} \left(\frac{1}{w_{\rho_k}-1}\right)^{m-k} w_{\rho_k}^{-y_k}$$

$$\times \prod_{k=1}^{m} \prod_{j=1}^{p_k} \frac{1}{\alpha z_{\pi_{n-j+1}} + (1-\alpha) w_{\rho_k}^{-1}},$$

with eigenvalue

$$\boldsymbol{\Lambda = \alpha \sum_{k=1}^{m} (w_k^{-1} - 1) + (1 - \alpha) \sum_{j=1}^{m} (z_j^{-1} - 1).}$$

Here we define $\boldsymbol{p_k}$ as the number of $+$ particles to the right of the $\boldsymbol{k}$th $-$ particle, i.e.

$$\boldsymbol{p_k := \# x_i > y_k,}$$

5.5 Green's function

Let $x_i^{(0)}, y_j^{(0)}$ be initial positions of $+$ and $-$ particles and x_i, y_j final positions at time t. Using the Bethe ansatz eigenfunctions, for certain special cases, one can write down the Green's function (probability of x_i, y_j at t given $x_i^{(0)}, y_j^{(0)}$ at $t = 0$) as a multiple integral.

Assume $x_1^{(0)} < \ldots < x_N^{(0)} < y_1^{(0)} < \ldots < y_M^{(0)}$.

The Green's function for the case (the order has not changed)

$$x_1 < \ldots < x_N < y_1 < \ldots < y_M,$$

is given by, with $\boldsymbol{\Lambda = \beta \sum_{i=1}^{N}(z_i^{-1} - 1) + \alpha \sum_{i=1}^{M}(w_i^{-1} - 1)}$,

$$\begin{aligned}
&\boldsymbol{G(\{x_j - x_j^{(0)}\}, \{y_k - y_k^{(0)}\}, t)} \\
&\boldsymbol{= \oint \prod_{j=1}^{N} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \prod_{k=1}^{M} \frac{\mathrm{d}w_k}{2\pi \mathrm{i}}\ \mathrm{e}^{\Lambda t}} \\
&\boldsymbol{\times \sum_{\pi \in S_N} \mathrm{sign}(\pi) \prod_{j=1}^{N} \left(\frac{z_j - 1}{z_{\pi_j} - 1}\right)^{j-1} z_{\pi_j}^{x_j} z_j^{-x_j^{(0)}-1}} \\
&\boldsymbol{\times \sum_{\rho \in S_M} \mathrm{sign}(\rho) \prod_{k=1}^{M} \left(\frac{w_k - 1}{w_{\rho_k} - 1}\right)^{M-k} w_{\rho_k}^{-y_k} w_k^{y_k^{(0)}-1}},
\end{aligned}$$

where all contour integrals are around the origin.

$+- \;\to\; -+$ boundary conditions

For the ordering (complete exchange of $+$ and $-$ particles)

$$y_1 < \ldots < y_M < x_1 < \ldots < x_N,$$

$$G(\{x_j - x_j^{(0)}\}, \{y_k - y_k^{(0)}\}, t)$$

$$= \oint \prod_{j=1}^{N} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \prod_{k=1}^{M} \frac{\mathrm{d}w_k}{2\pi\mathrm{i}}\, \mathrm{e}^{\Lambda t} \prod_{k=1}^{M} \prod_{j=1}^{N} \frac{1}{\beta z_j + \alpha w_k}$$

$$\times \sum_{\pi \in S_N} \mathrm{sign}(\pi) \prod_{j=1}^{N} \left(\frac{z_j - 1}{z_{\pi_j} - 1}\right)^{j-1} z_{\pi_j}^{x_j} z_j^{-x_j^{(0)}-1}$$

$$\times \sum_{\rho \in S_M} \mathrm{sign}(\rho) \prod_{k=1}^{M} \left(\frac{w_k - 1}{w_{\rho_k} - 1}\right)^{M-k} w_{\rho_k}^{-y_k} w_k^{y_k^{(0)}-1}$$

5.6 Result: Multiple-integral formula for current distribution

Theorem. A step i.c. in which there are N + particles on the left with density ρ and M − particles are packed on the right.

When $\alpha + \beta = 1$, for the currents $N_{\pm}(t)$ at the origin,

$$P_{N,M}(N_+(t) = N, N_-(t) = M) = \frac{1}{N!M!} \oint \prod_{j=1}^{N} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \prod_{k=1}^{M} \frac{\mathrm{d}w_k}{2\pi \mathrm{i}}\, \mathrm{e}^{\Lambda t}$$

$$\frac{\rho^N \prod_{1\le i<j\le N} (z_i - z_j)^2 \prod_{1\le k<l\le M} (w_l - w_k)^2}{\prod_{j=1}^{n} (z_j - 1)^N (1-(1-\rho)z_j) \prod_{k=1}^{M} (w_k - 1)^M \prod_{j=1}^{n} \prod_{k=1}^{M} \left(\beta z_j + \alpha w_k\right)}$$

with $\Lambda = \sum_{j=1}^{N} \beta(1/z_j - 1) + \sum_{k=1}^{M} \alpha(1/w_k - 1)$.

5.7 NLFHD prediction for finite particles case

- The original NLFHD is formulated for infinite systems.

$$P_{\infty,\infty}[N_+(t) = n, N_-(t) = m] \simeq F'_G(s_+)F'_2(s_-)$$

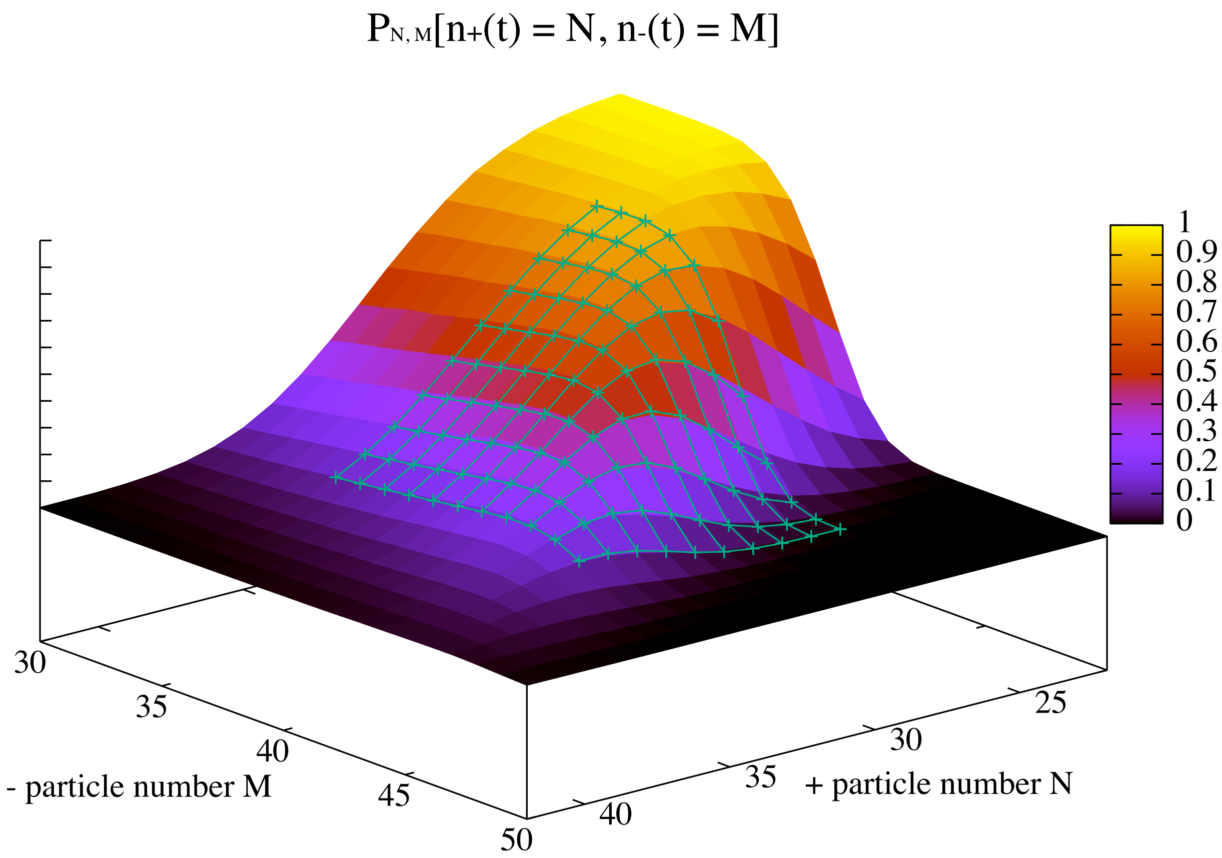
- Our formula is for finite number of particles. We can formulate a generalization of NLFHD prediction for finite case.

  Conjecture

$$\lim_{t\to\infty} P_{N,M}[N_+(t) = N, N_-(t) = M] = F_G(s_+)F_2(s_-)$$

  This may look very similar to the original conjecture but is in fact a very nontrivial generalization. (Note the difference of $P_*[N_+(t) = N, N_-(t) = M]$ as $t \to \infty$.)

Confirmation of the conjecture by simulation



5.8 Analytic confirmation by the multiple integral formula

In the multiple integral formula (with $\alpha = \beta = \frac{1}{2}$), we take the simple pole at $z_j = 1/(1-\rho)$ and find

$$P_{N,M}[N_+(t) = N, N_-(t) = M] = I_1 + J \times I_2$$

where

$$I_1 = \frac{1}{M!} \int \prod_{k=1}^{M} \frac{dw_k}{2\pi i} \frac{e^{\Lambda_{0,M} t} \prod_{1 \le k < l \le M} (w_l - w_k)^2}{\prod_{k=1}^{M} (w_k - 1)^M \prod_{k=1}^{M} (\frac{1}{2}(1 + w_k))^N}$$

$$J = \frac{\rho^{N-1}}{(1-\rho)^N} \left(\frac{2(1-\rho)}{2-\rho}\right)^M \frac{e^{-\rho t/2}}{(N-1)!} \int_1 \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i}$$

$$\times\, e^{\Lambda_{N-1,0}t} \frac{\prod_{1\le i<j\le N-1}(z_i - z_j)^2 \prod_{j=1}^{N-1}(1-(1-\rho)/z_j)}{\prod_{j=1}^{N-1}(z_j - 1)^N \prod_{j=1}^{N-1}(\frac{1}{2}(1+z_j))^M},$$

$$I_2 = \frac{1}{M!} \int \prod_{k=1}^{M} \frac{dw_k}{2\pi} e^{\Lambda_{0,M}t} \frac{\prod_{1\le k<l\le M}(w_l - w_k)^2}{\prod_{k=1}^{M}(w_k - 1)^M}$$

$$\times\, \frac{\prod_{j=1}^{N-1}(1+z_j)^M (\frac{1}{1-\rho}+1)^M}{\prod_{k=1}^{M}(\prod_{j=1}^{N-1}(z_j + w_k)(\frac{1}{1-\rho}+w_k))}.$$

We can study asymptotics of the integrals to show

$I_1, I_2 \simeq F_2(s_-), J \simeq F_G(s_+) - 1$.

The first analytic confirmation of the prediction of NLFHD!

6. Outstanding problems

- Other initial conditions. In particular the flat case. For determinantal case (e.g TASEP) ok: GOE TW appears. For the q-case, only partial results. Replica analysis for KPZ equation and moment formula for ASEP. No rigorous analysis.
- Other geometries. Semi-infinite systems. Slow bond problem.
- Multi-point correlations and joint distributions. For determinantal case ok (Airy processes) but not for the q-case.
- A deeper understanding of a reason why determinants appear.
- Universality for non-solvable models. Lattice KPZ equation.
- Multi-species models. In particular asymptotics.