3.3 Stochastic duality

Definition. For two Markov processes X, Y and some function $D(x, y)$, X and Y are dual wrt the duality function D if

$$E^x[D(X(t), Y)] = E^y[D(X, Y(t))]$$

holds.

- The self-duality of ASEP, q-TASEP and so on can be formulated in this way.
- The calculation of n-point function is reduced to an n-particle problem. For example, for SEP, the average density (1pt function) $\langle \eta_x(t) \rangle$ satisfies the master equation for the one particle continous time random walker.

Systematic way to construct processes with duality

- Finding a duality is nontrivial. For ASEP, its self-duality is related to $U_q(sl_2)$ symmetry. (The SEP is related to the sl_2 symmetry.)
- A general scheme to construct Markov processes with self duality from a quantum group was proposed in [CGRS2016].
- As an application we found an asymmetric version of the KMP process with $U_q(su(1,1))$. This is an interesting example which has a quantum group symmetry but is not integrable. A question is if one can study the asymptotics (KPZ or not?).

3.4 Macdonald process

- TASEP is related to the Schur measure and process, which are written in terms of the Schur function.
- As a generalization, one can naturally consider the Macdonald measure

$$\frac{1}{Z} P_\lambda(a) Q_\lambda(b)$$

  and Macdonald process.
- The Macdonald polynomials have two parameters t, q. The $t = 0$ case is called the q-Whittaker function. Hence we can consider q-Whittaker measure and process.

q-Whittaker process

- Borodin and Corwin found that the marginal dynamics on the left diagonal of a Markov dynamics related to the q-Whittaker process is the q-TASEP. Hence one can study fluctuation properties of q-TASEP by considering q-Whittaker measure.
- A difficulty of studying the q-Whittaker measure compared to the Schur case is that for the q-Whittaker function, no single determinant formula has been known. So the q-Whittaker measure is not directly related to the determinantal point process.

- Still there are various nice properties for the q-Whittaker (and for Macdonald) polynomials. Borodin and Corwin found that by using the Macdonald operator (whose eigenfunctions are the Macdonald polynomials), one can find a multiple integral formula for the q moment, which is the same as the one derived by the duality in the fourth lecture.

- A difficulty. For the random initial condition with parameter α, the q-moment

$$\langle q^{nh(x,t)} \rangle = (-1)^n q^{\frac{n(n-1)}{2}} \int \prod_{j=1}^{n} dz_j \prod_{j<k} \frac{z_j - z_k}{z_j - q z_k} \prod_{j=1}^{n} \frac{e^{(q-1)z}}{(z_j - \alpha/q)(1-z)^x}.$$

diverges for a large enough n!

4. An approach without q moment

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(Based on a collaborations with T. Imamura, M. Mucciconi)

25 Jan 2018 @ ICTS Bangalore

Reference: arXiv: 1701.05991, published in PTRF
arXiv:1901.08381

4.0 Partitions and Gelfand-Tsetlin cone

Partition of length n

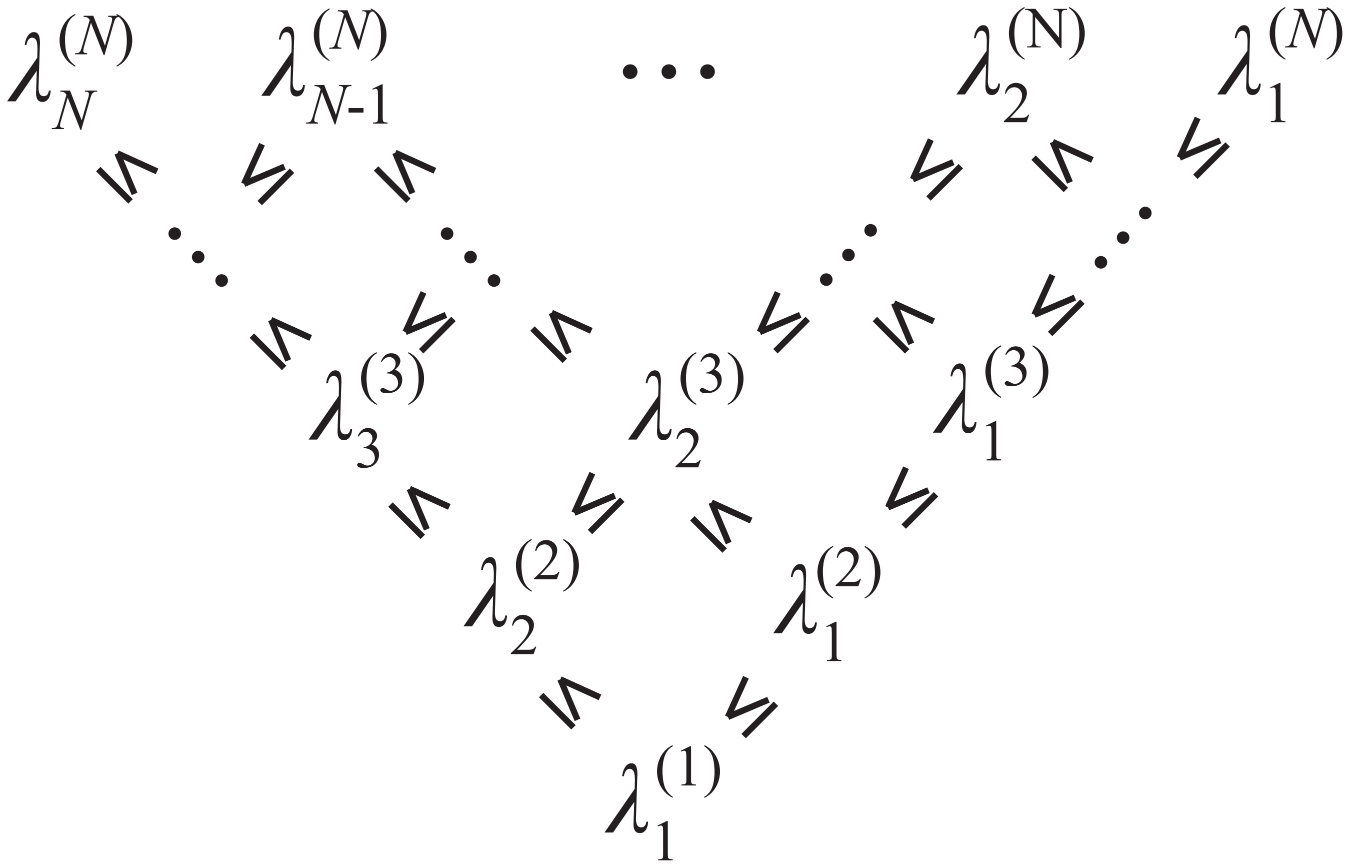
$$\mathcal{P}_n := \{\lambda = (\lambda_1, \cdots, \lambda_n) \in {\mathbb{Z}_+}^n | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$$

Gelfand-Tsetlin cone

$$\mathsf{GT}_N := \{(\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(N)}) \in \mathbb{Z}_+^{N(N+1)/2} | \lambda_{\ell+1}^{(m+1)} \leq \lambda_\ell^{(m)} \leq \lambda_\ell^{(m+1)}\}$$

An element is denoted by $\underline{\lambda}_N$.

Gelfand-Tsetlin cone



4.1 (Skew) q-Whittaker functions

The skew q-Whittaker function (with 1 variable)

$$P_{\lambda/\mu}(a) = \prod_{i=1}^{n} a^{\lambda_i} \cdot \prod_{i=1}^{n-1} \frac{a^{-\mu_i}(q;q)_{\lambda_i-\lambda_{i+1}}}{(q;q)_{\lambda_i-\mu_i}(q;q)_{\mu_i-\lambda_{i+1}}}$$

q-Whittaker function with N variables

$$P_\lambda(a) = \sum_{\substack{\lambda_i^{(k)}, 1\le i\le k\le N-1 \\ \lambda_{i+1}^{(k+1)} \le \lambda_i^{(k)} \le \lambda_i^{(k+1)}}} \prod_{j=1}^{N} P_{\lambda^{(j)}/\lambda^{(j-1)}}(a_j)$$

where the sum is over GT with $\lambda = \lambda^{(N)}$ and $a = (a_1, \ldots, a_N)$.

Another function.

$$Q_\lambda(t) = \prod_{i=1}^{N-1} (q^{\lambda_i - \lambda_{i+1}+1}; q)_\infty \int_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dz_i}{z_i} \cdot P_\lambda\left(\frac{1}{z}\right) \Pi(z;t)\, m_N^q(z)$$

where

$$\Pi(a;t) = \prod_{j=1}^{N} e^{a_j t}$$

$$m_N^q(z) = \frac{1}{(2\pi i)^N N!} \prod_{1\le i<j\le N} (z_i/z_j;q)_\infty (z_j/z_i;q)_\infty$$

Remark: May look a bit strange but Q_λ is related to the initial condition. Recall $\tilde{s}_\lambda$ for TASEP.

4.2 q-Whittaker process

Definition. For a set of N parameters a and $t \geq 0$, set

$$P_t(\underline{\lambda}_N) := \frac{\prod_{j=1}^{N} P_{\lambda^{(j)}/\lambda^{(j-1)}}(a_j) \cdot Q_{\lambda^{(N)}}(t)}{\Pi(a;t)}$$

Proposition.

$P_t(\underline{\lambda}_N)$ satisfies the Kolmogorov forward equation (master equation) for the Markov dynamics introduced before on GT cone.

One can also check the step initial condition on the q-TASEP marginal $\lambda_i^{(i)}$.

To summarize, if we can study the q-Whittaker process, we can study the N-particle q-TASEP with step i.c.

q-Whittaker measure

$x_N(t)(=\lambda_N^{(N)}-N)$ can be studied by focusing on $\lambda^{(N)}(t)$.

Marginal for $\lambda^{(N)}(t)$ is given by q-Whittaker measure:

$$\mathbb{P}[\lambda^{(N)}(t)=\lambda]=\frac{P_\lambda(a)Q_\lambda(\alpha,t)}{\Pi(a;t)}$$

Let us recall the Cauchy identity

$$\sum_{\lambda\in\mathcal{P}_N}P_\lambda(x)Q_\lambda(y)=\prod_{ij=1}^{N}\frac{1}{(x_iy_j;q)_\infty}$$

where $Q_\lambda(y)$ is the ordinary q-Whittaker function.

4.3. Nth particle position

By writing $P_\lambda(x) = X^{\lambda_N} R_\ell(x)$, $\ell_j = \lambda_j - \lambda_{j+1}$ the Cauchy identity can be rewritten as

$$\sum_{\ell_1,\cdots,\ell_{N-1}=0}^{\infty} R_\ell(x) R_\ell(y) \prod_{j=1}^{N-1} \frac{1}{(q;q)_{\ell_j}} = \frac{(XY;q)_\infty}{\prod_{ij=1}^{N}(x_i y_j;q)_\infty}$$

with $X = x_1 \cdots x_N, Y = y_1 \cdots y_N$. Using this we find a multiple integral formula for the particle position,

$$\mathbb{P}[\lambda_N^{(N)}(t) = l]$$

$$= (q;q)_\infty^{N-1} \int_{\mathbb{T}^N} \prod_{j=1}^{N} \frac{dz_j}{z_j} \cdot \left(\frac{A}{Z}\right)^l m_N^q(z) \frac{\Pi(z;t)}{\Pi(a;t)} \cdot \frac{(A/Z;q)_\infty}{\prod_{ij=1}^{N}(a_i/z_j;q)_\infty}$$

where $A = \prod_{i=1}^N a_i$ and $Z = \prod_{i=1}^N z_i$.

4.4 Fredholm determinant for the q-Laplace transform

Theorem. For $\zeta \neq q^n, n \in \mathbb{Z}$

$$\left\langle \frac{1}{(\zeta q^{x_N(t)+N};q)_\infty} \right\rangle = \det(1 - fK)_{L^2(\mathbb{Z})}$$

where $\langle \cdots \rangle$ is the expectation and

$$f(n) = \frac{1}{1 - q^n/\zeta}, \quad K(n,m) = \sum_{l=0}^{N-1} \phi_l(m)\psi_l(n)$$

$$\phi_l(n) = \sqrt{a_{l+1}} \int_D dv \frac{e^{-vt}}{v^{n+N}} \frac{1}{v - a_{l+1}} \prod_{j=1}^{l} \frac{v}{v - a_j} \prod_k \frac{1}{(qv/a_k;q)_\infty}$$

$$\psi_l(n) = \sqrt{a_{l+1}} \int_{C_r} dz \frac{e^{zt} z^{n+N}}{z} \prod_{j=1}^{l} \frac{z - a_j}{z} \prod_k (qz/a_k;q)_\infty$$

Here C_r, D is around $\{0\}, \{a_i q^j\}$ respectively.

4.5 Ramanujan's summation formula and theta function

Theorem. For $|q| < 1, |b/a| < |z| < 1$,

$$\sum_{n\in\mathbb{Z}} \frac{(bq^n;q)_\infty}{(aq^n;q)_\infty} z^n = \frac{(az;q)_\infty(\frac{q}{az};q)_\infty(q;q)_\infty(\frac{b}{a};q)_\infty}{(a;q)_\infty(\frac{q}{a};q)_\infty(z;q)_\infty(\frac{b}{az};q)_\infty}$$

We introduce a modified Jacobi theta function

$$\theta(z) = (z,q)_\infty(q/z;q)_\infty.$$

Also

$$\tilde{\theta}(1/z) = \frac{1}{\sqrt{z}}\tilde{\theta}(z)$$

which satisfies $\tilde{\theta}(1/z) = \tilde{\theta}(z)$.

Frobenius determinant (Cauchy det for theta function)

Let $[x]$ satisfy $[-x] = -[x]$ and the Riemann relation

$$\begin{aligned} &[x+y][x-y][u+v][u-v] \\ &= [x+u][x-u][y+v][y-v] - [x+v][x-v][y+u][y-u] \end{aligned}$$

$[x]$ satisfying the above two relations is necessarily in the form $e^{ax^2+b} f(cx)$ where $f(x)$ is either $f(x) = x$, $\sin \pi x$ or $\sigma(x)$, the Weierstrass sigma function. $\tilde{\theta}(q^x)$ is an example of $[x]$.

Theorem. (1882 Frobenius) For $[x]$ above, the Cauchy determinant type formula holds. With $B = \sum_i b_i, C = \sum_i c_i$,

$$\frac{[\lambda + B - C] \prod_{i<j} [b_i - b_j][c_j - c_i]}{[\lambda] \prod_{i,j} [b_i - c_j]} = \det \left(\frac{[\lambda + b_i - c_j]}{[\lambda][b_i - c_j]} \right)$$

Mutliple integral formula for q-Laplace transform

We consider the quantity.

$$\langle \frac{1}{(\zeta q^{\lambda_N};q)_\infty} \rangle = \sum_{l\in\mathbb{Z}} \int_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dz_i}{z_i} \left(\frac{A}{Z}\right)^l m_N(z) \frac{\Pi(z;t)}{\Pi(a;t)} \frac{(q;q)_\infty^{N-1}(A/Z;q)_\infty}{\prod_{i,j}(a_i/z_j;q)_\infty}$$

Here we use the Ramanujan's formula with $a=\zeta, b=0, z=A/Z$.

$$\sum_{l\in\mathbb{Z}} \frac{1}{(\zeta q^l;q)_\infty} \left(\frac{A}{Z}\right)^l = \frac{(\frac{\zeta A}{Z};q)_\infty(\frac{qZ}{\zeta A};q)_\infty(q;q)_\infty}{(\zeta,q)_\infty(\frac{q}{\zeta};q)_\infty(\frac{A}{Z};q)_\infty} = \frac{\theta(\frac{\zeta A}{Z})(q;q)_\infty}{\theta(\zeta)(\frac{A}{Z};q)_\infty},$$

Proposition. The following multiple integral formula holds.

$$\langle \frac{1}{(\zeta q^{\lambda_N};q)_\infty} \rangle = \frac{(q;q)_\infty^N}{N!} \int_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dz_i}{z_i} \frac{\theta(\frac{\zeta A}{Z})}{\theta(\zeta)} \frac{\prod_{i\neq j}(z_i/z_j;q)_\infty}{\prod_{i,j}(a_i/z_j;q)_\infty} \frac{\Pi(z;t)}{\Pi(a;t)}.$$

After some calculations, we find

$$\left\langle \frac{1}{(\zeta q^{\lambda_N};q)_\infty} \right\rangle = \frac{(q;q)_\infty^N}{N!} \int_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dz_i}{z_i} \frac{\prod_{i<j}(a_i - a_j) \prod_{i<j}(z_i - z_j)}{\prod_{i,j}(a_i - z_j)}$$
$$\times \frac{\prod_{i<j} \tilde{\theta}(a_i/a_j) \prod_{i<j} \tilde{\theta}(z_i/z_j)}{\prod_{i,j} \tilde{\theta}(a_i/z_j)}$$
$$\times \frac{\tilde{\theta}(\frac{\zeta A}{Z})}{\tilde{\theta}(\zeta)} \prod_i \frac{a_i \prod_k (z_i/a_k;q)_\infty g(z_i;t)}{\prod_{k\neq i}(a_i/a_k;q)_\infty g(a_i;t)}$$

where

$$g(z;t) = e^{zt}.$$

By the Frobenius determinant formula,

$$\langle \frac{1}{(\zeta q^{\lambda_N};q)_\infty} \rangle = \frac{1}{N!} \int_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dz_i}{z_i} \det(\frac{a_i}{a_i - z_j}) \det(\frac{\tilde{\theta}(\zeta a_i/z_j)}{\tilde{\theta}(\zeta)\tilde{\theta}(a_i/z_j)})$$

$$\times \prod_i \frac{\prod_k (z_i/a_k;q)_\infty g(z_i;t)(q;q)_\infty}{\prod_{k\neq i}(a_i/a_k;q)_\infty g(a_i;t)}$$

At this point we find a product of two determinants. Now we can apply the standard machinery of random matrix theory!

By using the Cauchy-Binet identity,

$$= \det \left(\int_{\mathbb{T}} \frac{dz}{z} \frac{a_i}{a_i - z} \frac{\theta(\zeta a_i/z)}{\theta(\zeta)\theta(a_i/z)} \frac{(q;q)_\infty \prod_k (z/a_k;q)_\infty g(z;t)}{\prod_{k\neq i}(a_i/a_k;q)_\infty g(a_i;t)} \right)$$

By making the contour smaller and taking the pole at $z = a_i$

$$= \det \left(\delta_{ij} - \int_{C_r} \frac{dz}{z} \frac{a_i}{a_i - z} \frac{\theta(\zeta a_i/z)}{\theta(\zeta)\theta(a_i/z)} \frac{(q;q)_\infty \prod_k (z/a_k;q)_\infty g(z;t)}{\prod_{k\neq i}(a_i/a_k;q)_\infty g(a_i;t)} \right)$$

Here using the Ramanujan's formula again with $a = 1/\zeta, b = q/\zeta, z \to z/a_j$,

$$\sum_{n\in\mathbb{Z}} \frac{1}{1-q^n/\zeta} \left(\frac{z}{a_j}\right)^n = \frac{(\frac{z}{\zeta a_j})_\infty (\frac{q\zeta a_j}{z};q)_\infty (q;q)_\infty^2}{(1/\zeta;q)_\infty (q\zeta;q)_\infty (z/a_j;q)_\infty (qa_j/z;q)_\infty}$$

$$= \frac{\theta(\frac{z}{\zeta a_j})}{\theta(1/\zeta)\theta(z/a_j)} (q;q)_\infty^2,$$

we see

$$\langle \frac{1}{(\zeta q^{\lambda_N}; q)_\infty} \rangle = \det \left(\delta_{ij} - \sum_{n \in \mathbb{Z}} \frac{1}{1 - q^n/\zeta} \int_{C_r} \frac{dz}{z} \frac{a_i}{a_i - z} \right.$$

$$\left. \times \; \frac{z^n \prod_k (z/a_k; q)_\infty g(z;t)}{a_j^n (q;q)_\infty \prod_{k \neq i} (a_i/a_k; q)_\infty g(a_i;t)} \right)$$

$$= \det(\delta_{ij} - \sum_{n \in \mathbb{Z}} A(i,n) B(n,j))$$

with

$$A(i,n) = \frac{1}{1 - q^n/\zeta} \int_{C_r} \frac{dz}{z} \frac{a_i}{a_i - z} z^n \prod_k (z/a_k; q)_\infty g(z;t)$$

$$B(n,j) = \frac{1}{(q;q)_\infty (a_i/a_k; q)_\infty g(a_i;t)}$$

Here use $\det(1-AB)=\det(1-BA)$. We see

$$(BA)(m,n)=\sum_{i=1}^{N}B(m,i)A(i,n)$$

$$=\sum_{i=1}^{N}\frac{1}{a_i^m(q;q)_\infty(a_i/a_k;q)_\infty g(a_i;t)}\frac{1}{1-q^n/\zeta}$$

$$\times\int_{C_r}\frac{dz}{z}\frac{a_i}{a_i-z}z^n\prod_k(z/a_k;q)_\infty g(z;t)$$

$$=\frac{-1}{1-q^n/\zeta}\int_D dv\int_{C_r}\frac{dz}{z}\frac{1}{v-z}\frac{z^n\prod_k(z/a_k;q)_\infty g(z;t)}{v^n\prod_k(v/a_k;q)_\infty g(v;t)}$$

where the contour D is around $\{a_i\}$. Here

$$\frac{\prod_k (z/a_k;q)_\infty g(z;t)}{\prod_k (v/a_k;q)_\infty g(v;t)} = \frac{\prod_k (qz/a_k;q)_\infty}{\prod_k (qv/a_k;q)_\infty} \frac{(z-a_k)e^{zt}}{(v-a_k)e^{vt}}$$

Hence

$$\langle \frac{1}{(\zeta q^{\lambda_N};q)_\infty} \rangle = \frac{1}{1-q^n/\zeta} \int_D dv \int_{C_r} \frac{dz}{z} \frac{z^{n+N} e^{zt} \prod_k (qz/a_k;q)_\infty}{v^{n+N} e^{vt} \prod_k (qv/a_k;q)_\infty} \times \left(\frac{1}{z-v} \prod_k \frac{1-a_k/z}{1-a_k/v} - 1 \right)$$

By using

$$\frac{1}{z-v}\prod_k \frac{1-a_k/z}{1-a_k/v} - 1$$

$$= \sum_{l=0}^{N-1} \frac{a_{l+1}}{z(v-a_{l+1})} \prod_{j=1}^{l} \frac{(z-a_j)v}{(v-a_j)z}$$

we arrive at the desired Fredholm determinant expression.

Comments

- Stationary case can be equally studied by simply replacing $g(z,t) = e^{zt}$ by $g(z,t) = \frac{e^{zt}}{(\alpha/z;q)_\infty}$.
  There is no difficulty of diverging moments!

- Many models can be studied in a unified way. For example setting $q = 0$ gives results for TASEP.

- Stationary HS6VM can also by studied by using this approach. Matteo's talk in the following session.

Multiple integral formula for TASEP case

By taking $q \to 0$ limit, we find

$$\mathbb{P}[N(t) \geq N] = \frac{1}{N!} \int_0 \prod_{j=1}^{N} dz_j \frac{e^{\epsilon(z_j)t}}{(1-z_j)^N} \prod_{j<k} (z_k - z_j)^2.$$

This formula can be found from the Schutz determinantal transition probability and using the integral representation of F_n function. [cf. Talk by Lee]