High trace methods in random matrix theory

Charles Bordenave

CNRS & Aix-Marseille Université

The nonbacktracking matrix

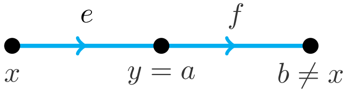
Nonbacktracking matrix

Let H be a matrix in $M_n(\mathbb{C})$.

Consider the matrix B in $M_{n^2}(\mathbb{C})$ with entries

$$B_{ef} = H_{ab}\mathbf{1}(y = a)\mathbf{1}(x \neq b),$$

where $e = (x, y)$ and $f = (a, b)$.



Beware that if H is Hermitian, B is not ! (not even normal).

Hashimoto (1989).

Nonbacktracking matrix on a graph

Variant: H is a matrix in $M_n(\mathbb{C})$ whose non-zeros entries (x,y) are edges of an undirected graph $G=(V,E)$ with vertices $V=\{1,\cdots,n\}$ and edges $E\subset\{\{x,y\}:x,y\in V\}$.

Then, the set of oriented edges of G is

$$\vec{E}=\{(x,y):\{x,y\}\in E\}$$

Define the matrix $\widetilde{B}$ which acts on $\vec{E}$ and with entries

$$\widetilde{B}_{ef}=H_{ab}\mathbf{1}(y=a)\mathbf{1}(x\neq b),$$

where $e=(x,y)$ and $f=(a,b)$ in $\vec{E}$.

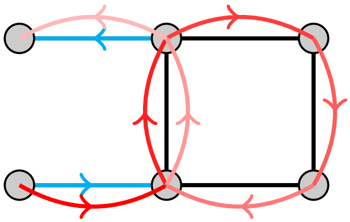
The two definitions of B coincides: $F=\mathrm{span}(\delta_{(x,y)}:\{x,y\}\notin E)$ is invariant by B and B^* and $B_{|F}=0$, $B_{|F^\perp}=\widetilde{B}$.

Nonbacktracking matrix and geodesics

For any $k \in \mathbb{N}$,

$$B^k_{ef} = \sum_{\gamma} \prod_{t=1}^{k} H_{\gamma_t \gamma_{t+1}}$$

where the sum is over nonbacktracking paths from e to f of length $k+1$, i.e. paths $(\gamma_0, \gamma_1, \ldots, \gamma_{k+1})$ such that $(\gamma_0, \gamma_1) = e$, $(\gamma_k, \gamma_{k+1}) = f$ and $\gamma_{t-1} \neq \gamma_{t+1}$. This is a discrete geodesic.



On a tree, nonbacktracking paths are shortest paths.

Nonbacktracking spectral identities

Despite its non-normality, due to its strong geometric flavour, nonbacktracking matrices are often easier to study.

There exists a familly of identities between eigenvalues and eigenvectors of a matrix and eigenvalues and eigenvectors of nonbacktracking matrices.

It allows to study the spectrum of matrix through its *nonbactracking spectrum*.

We will follow this strategy for computing largest eigenvalues.

Hashimoto-Ihara-Bass identity

Assume that $A \in M_n(\mathbb{C})$ is the adjacency matrix of a graph $G = (V, E)$.

Let Q be the diagonal matrix : $Q_{xx} = \deg(x) - 1$. We have

$$\det(zI_{\vec{E}} - B) \quad = \quad (z^2 - 1)^{|E|-|V|} \det(z^2 I_V - Az + Q).$$

If G is a d-regular graph, that is for all $x \in V$, $\deg(x) = d$, then $Q = (d-1)I_V$ and

$$\sigma(B) = \{\pm 1\} \cup \left\{\mu : \mu^2 - \lambda\mu + (d-1) = 0 \text{ avec } \lambda \in \sigma(A)\right\}.$$

Lemma

Let H be Hermitian with nonbacktracking matrix B and let $\mu \in \mathbb{C}$, $\mu > |H_{xy}|$ for all x, y. Define H_μ and D_μ diagonal

$$(H_\mu)_{xy} = \frac{H_{xy}}{1 - \mu^{-2}|H_{xy}|^2}\,, \quad (D_\mu)_{xx} = \mu + \frac{1}{\mu}\sum_y \frac{|H_{xy}|^2}{1 - \mu^{-2}|H_{xy}|^2}\,.$$

Then $\mu \in \sigma(B)$ if and only if $0 \in \sigma(H_\mu - D_\mu)$.

There is also a determinantal identity which extends the Hashimoto-Ihara-Bass identity.

From nonbacktracking to classical spectrum

Let $v \in \mathbb{C}^{n^2}$. Introduce the divergence vector $u \in \mathbb{C}^n$,

$$u_x = \sum_y H_{xy} v_{xy}.$$

Assume that $Bv = \mu v$ then

$$\mu v_{yx} = \sum_{y' \neq y} H_{xy'} v_{xy'} = u_x - H_{xy} v_{xy}.$$

Switching x and y,

$$\mu v_{xy} = u_y - \bar{H}_{xy} v_{yx}.$$

Hence $\mu^2 v_{xy} = \mu u_y - \bar{H}_{xy} u_x + |H_{xy}|^2 v_{xy}$ and (as $\mu \neq |H_{xy}|$)

$$v_{xy} = \frac{\mu u_y - \bar{H}_{xy} u_x}{\mu^2 - |H_{xy}|^2}.$$

$$v_{xy} = \frac{\mu u_y - \bar{H}_{xy} u_x}{\mu^2 - |H_{xy}|^2}.$$

We have $u \neq 0$ iff $v \neq 0$.

Writing the eigenvalue equation $Bv = \mu v$ in terms of u, we arrive at . . .

$$(H_\mu - D_\mu)u = 0.$$

with

$$(H_\mu)_{xy} = \frac{H_{xy}}{1 - \mu^{-2}|H_{xy}|^2} \qquad (D_\mu)_{xx} = \mu + \frac{1}{\mu} \sum_y \frac{|H_{xy}|^2}{1 - \mu^{-2}|H_{xy}|^2}.$$

As requested. □

From classical to nonbacktracking spectrum

Let H be Hermitian with nonbacktracking matrix B and let $\mu \in \mathbb{C}$, $\mu > |H_{xy}|$ for all x, y. Define H_μ and D_μ diagonal

$$(H_\mu)_{xy} = \frac{H_{xy}}{1 - \mu^{-2}|H_{xy}|^2}\ , \quad (D_\mu)_{xx} = \mu + \frac{1}{\mu}\sum_y \frac{|H_{xy}|^2}{1 - \mu^{-2}|H_{xy}|^2}\ .$$

Then $\mu \in \sigma(B)$ if and only if $0 \in \sigma(H_\mu - D_\mu)$.

It is possible to invert the statement and obtain a claim like:

Let $H \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{R} \backslash S$, there exists $\hat{H}_\lambda$ with associated nonbacktracking matrix $\hat{B}_\lambda$ such that $\mu \in \sigma(H)$ if and only if $1 \in \sigma(\hat{B}_\lambda)$.

We will see an explicit form of such statement later on.

A first application

For $A \in M_n(\mathbb{C})$, the spectral radius is

$$\rho(A) = \max\{|\mu| : \mu \in \sigma(A)\}.$$

The operator norm is

$$\|A\| = \|A\|_{2\to 2} = \sup_{f\neq 0} \frac{\|Af\|_2}{\|f\|_2},$$

and

$$\|A\|_{2\to\infty} = \max_x \sqrt{\sum_y |A_{xy}|^2}\,, \qquad \|A\|_{1\to\infty} = \max_{x,y} |A_{xy}|\,.$$

Lemma

If H is Hermitian with non-backtracking matrix B, then, with $f(\mu) = \mu + 1/\mu$ for $\mu \geqslant 1$ and $f(\mu) = 2$ for $\mu \leqslant 1$,

$$\|H\| \leqslant .$$

A first application

For $A \in M_n(\mathbb{C})$, the spectral radius is

$$\rho(A) = \max\{|\mu| : \mu \in \sigma(A)\}.$$

The operator norm is

$$\|A\| = \|A\|_{2\to 2} = \sup_{f\neq 0} \frac{\|Af\|_2}{\|f\|_2},$$

and

$$\|A\|_{2\to\infty} = \max_x \sqrt{\sum_y |A_{xy}|^2}\,, \qquad \|A\|_{1\to\infty} = \max_{x,y} |A_{xy}|\,.$$

Lemma

If H is Hermitian with non-backtracking matrix B, then, with $f(\mu) = \mu + 1/\mu$ for $\mu \geqslant 1$ and $f(\mu) = 2$ for $\mu \leqslant 1$,

$$\|H\| \leqslant \|H\|_{2\to\infty} f\left(\frac{\rho(B)}{\|H\|_{2\to\infty}}\right) + 3\|H\|_{1\to\infty}.$$

A first application

For $A \in M_n(\mathbb{C})$, the spectral radius is

$$\rho(A) = \max\{|\mu| : \mu \in \sigma(A)\}.$$

The operator norm is

$$\|A\| = \|A\|_{2\to 2} = \sup_{f \neq 0} \frac{\|Af\|_2}{\|f\|_2},$$

and

$$\|A\|_{2\to\infty} = \max_x \sqrt{\sum_y |A_{xy}|^2}\,, \qquad \|A\|_{1\to\infty} = \max_{x,y} |A_{xy}|\,.$$

Lemma

If H is Hermitian with non-backtracking matrix B, then, with $f(\mu) = \mu + 1/\mu$ for $\mu \geqslant 1$ and $f(\mu) = 2$ for $\mu \leqslant 1$,

$$\|H\| \leqslant 2\|H\|_{2\to\infty} + \frac{(\rho(B) - \|H\|_{2\to\infty})_+^2}{\|H\|_{2\to\infty}} + 3\|H\|_{1\to\infty}.$$

A first application

Assume $\|H\|_{2\to\infty} = 1$. We set $\delta = \max |H_{xy}| = \|H\|_{1\to\infty}$ and

$$\mu_0 = \max(1+\delta, \rho(B)).$$

Recall

$$(H_\mu)_{xy} = \frac{H_{xy}}{1-\mu^{-2}|H_{xy}|^2}\ , \quad (D_\mu)_{xx} = \mu + \frac{1}{\mu}\sum_y \frac{|H_{xy}|^2}{1-\mu^{-2}|H_{xy}|^2}\,.$$

From the lemma: we have $\det(H_\mu - D_\mu) \neq 0$ for all $\mu \in (\mu_0, \infty)$.

Since $H_\mu - D_\mu = I + O(\mu^{-1})$ as $\mu \to \infty$,

$$H_{\mu_0} - D_{\mu_0} \succeq 0.$$

A first application

Recall, $\mu_0 = \max(1+\delta, \rho(B))$.

From the formulas of H_μ and D_μ, we find, for $\mu \geqslant \mu_0$,

$$|(H_\mu)_{xy} - H_{xy}| = \left| \frac{H_{xy}}{1 - \mu^{-2}|H_{xy}|^2} - H_{xy} \right| = \frac{|H_{xy}|^3}{\mu^2 - |H_{xy}|^2} \leqslant \delta |H_{xy}|^2.$$

$$(D_\mu)_{xx} \leqslant \left(\mu + \frac{1}{\mu} \right) + \delta.$$

Recall $H_{\mu_0} - D_{\mu_0} \succeq 0$ and $\sum_y |H_{xy}|^2 \leqslant 1$. From Gershgorin circle theorem, we deduce that

$$H \preceq \left(\mu_0 + \frac{1}{\mu_0} \right) + 2\delta.$$

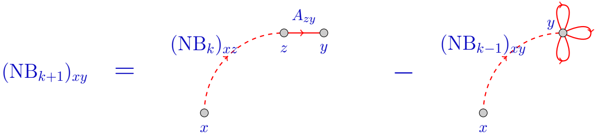
The conclusion $\lambda_1(H) \leqslant f(\rho(B)) + 3\delta$ follows easily. □

Geronimus Polynomials

For the adjacency matrix A of a d-regular graph, we may have at the same time Hermitian and non-backtracking paths!

Let $(\mathrm{NB}_k)_{x,y}$ be the number of non-backtracking paths of length k between x and y in G: we have the matrix identities $\mathrm{NB}_0 = I_V$, $\mathrm{NB}_1 = A$ and for $k \geqslant 2$,

$$\mathrm{NB}_{k+1} = \mathrm{NB}_k \cdot A - (d-1)\mathrm{NB}_{k-1}.$$



Geronimus Polynomials

It follows that for a monic polynomial of degree k of A:

$$\mathrm{NB}_k = G_k(A).$$

From the three-terms recurrence relation:

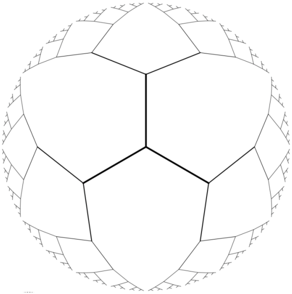
$$G_{k+1}(\lambda) = \lambda G_k(\lambda) - (d-1)G_{k-1}(\lambda),$$

we find

$$G_k(\lambda) = (d-1)^{\frac{k}{2}} U_k\left(\frac{\lambda}{2\sqrt{d-1}}\right) - (d-1)^{\frac{k}{2}-1} U_{k-2}\left(\frac{\lambda}{2\sqrt{d-1}}\right),$$

where $U_k(\cos\theta) = \sin((k+1)\theta)/\sin(\theta)$ is the Chebychev polynomial of the second kind.

Geronimus Polynomials



If A is the adjacency operator of the infinite d-regular tree, then

$$(G_k(A)G_\ell(A))_{xx} = \sum_y G_k(A)_{xy}G_\ell(A))_{xy} = d(d-1)^{k-1}\mathbf{1}(k=\ell).$$

since $G_k(A)_{xy} \in \{0,1\}$ is 1 is x and y are at distance k.

Geronimus Polynomial

The spectral measure of the adjacency operator A of the d-regular tree is defined by, for all $k \in \mathbb{N}$,

$$\int \lambda^k d\mu(\lambda) = (A^k)_{xx}.$$

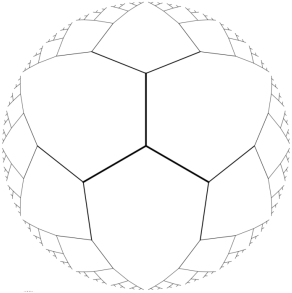
In particular,

$$(G_k(A)G_\ell(A))_{xx} = d(d-1)^{k-1}\mathbf{1}(k=\ell) = \int G_k(\lambda)G_\ell(\lambda)d\mu(\lambda).$$

The polynomials G_k are thus orthogonal with respect to μ.

Kesten-McKay distribution

$$\int \lambda^k d\mu = (A^k)_{xx}.$$



Kesten (1959): μ has support $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ and density

$$\frac{d}{2\pi}\frac{\sqrt{4(d-1)-\lambda^2}}{d^2-\lambda^2}.$$

Some references

M. Kotani and T. Sunada (2000). *Zeta functions of finite graphs*. Journal of Mathematical Sciences.

A. Terras, (2010). *Zeta Functions of Graphs: A Stroll through the Garden*. Cambridge University Press.

F. Benaych-Georges, C. Bordenave, A. Knowles (2017). *Spectral radii of sparse random matrices*, arXiv:1704.02945.

N. Anantharaman, M. Sabri (2017). *Quantum Ergodicity on Graphs: from spectral to spatial delocalization*, arXiv:1704.02953.

C. Bordenave, B. Collins (2018). *Eigenvalues of random lifts and polynomial of random permutations matrices*, arXiv:1801.00876.

Fűredi-Komlós bound revisited

Let $H \in M_n(\mathbb{C})$ be an Hermitian random matrix with independent centered entries $(H_{xy})_{x \geqslant y}$ above the diagonal,

$$\text{for all } x, y, \quad \mathbb{E}|H_{xy}|^2 \leqslant \frac{1}{n} \quad \text{and} \quad a.s.-\max_{x,y} |H_{xy}| \leqslant \frac{1}{q}.$$

Let B be the nonbactracking matrix of H. Recall $\|H\|_{2\to\infty} = \max_x \sqrt{\sum_y |H_{xy}|^2}$

Theorem

Let $q' = \min(q, n^{1/10})$, *with high probability,*

$$\rho(B) \leqslant 1 + \frac{C}{q'}.$$

Spectral radius of random nonbacktracking matrices

For the Erdős-Renyi graph with average degree d and $H = (A - \mathbb{E}A)/\sqrt{d}$, we have that $\|H\|_{2\to\infty}^2 \sim \max_x \deg(x)/d$ concentrates around 1 iif $q^2 = d \gg \log n$. Then $\|H\| \leqslant 2 + o(1)$. For $q^2 = d = O(\log n)$, the bound on $\|H\|$ is off by a multiplicative factor.

In the regime $d \ll \log n$, for the non-backtracking matrix of A or H, we have $\rho(B) = O(1) \ll \|B\| \sim \max_x \sqrt{\deg(x)}$. This is an effect of the non-normality of B.

The bound on $\rho(B)$ is not optimal for $d = O(1)$.

Expected high trace method

We have for any $\ell \in \mathbb{N}$

$$\rho(B) \leqslant \|B^\ell\|^{\frac{1}{\ell}}.$$

Since $\|A\|^2 = \|AA^*\|$, for even k,

$$\rho(B)^k \leqslant \|B^{k/2}(B^{k/2})^*\| \leqslant \mathrm{Tr}\Big(B^{k/2}(B^{k/2})^*\Big).$$

We aim at, for some $k \gg \log n$,

$$\mathbb{E}\mathrm{Tr}\Big(B^{k/2}(B^{k/2})^*\Big) \leqslant Cn^2k^2.$$

Expected high trace method

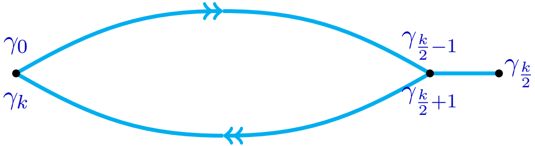
Expanding the trace

$$\begin{aligned}\mathbb{E}\mathrm{Tr}\left(B^{k/2}(B^{k/2})^*\right) &= \mathbb{E}\sum_{e,f}\left(B^{k/2}\right)_{ef}\left(B^{k/2}\right)^*_{fe} \\ &\leqslant n^2\sum_{\gamma\in\mathcal{N}_k} n^{-(e(\gamma)-v(\gamma)+1)}q^{-(k-2e(\gamma))},\end{aligned}$$

where $\mathcal{N}_k$ is the set of unlabeled paths $\gamma = (\gamma_0, \ldots, \gamma_k)$ which visits each edge at least twice,

$$\gamma_{t+1} \neq \gamma_{t-1} \quad \text{for all} \quad t \neq \frac{k}{2},$$

and the boundary conditions

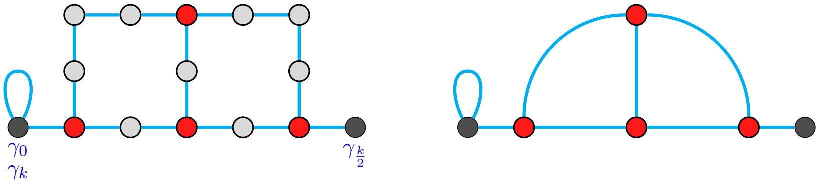


Expected high trace method

$$\mathbb{E}\mathrm{Tr}\Big(B^{k/2}(B^{k/2})^*\Big) \leqslant n^2 \sum_{\gamma\in\mathcal{N}_k} n^{-(e(\gamma)-v(\gamma)+1)} q^{-(k-2e(\gamma))},$$

For nonbacktracking paths, we can estimate $\mathcal{N}_k$ by genus $g = e - v + 1$ and visited edges $k - 2e$.

Let γ in $\mathcal{N}_k$ which visits $e \leqslant k/2$ edges and v vertices. Set $g = e - v + 1 \geqslant 0$. We build a reduced graph $\widehat{G}(\gamma)$ by removing inner vertices of degree 2.

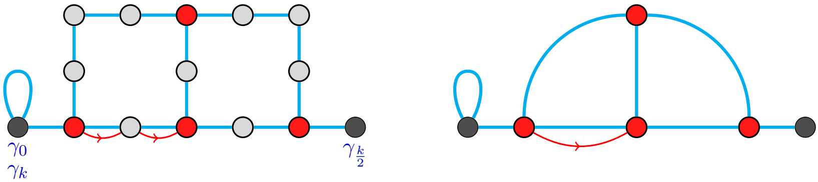


The path $\hat{\gamma} = (\hat{\gamma}_0, \dots, \hat{\gamma}_{\hat{k}})$ in the reduced graph $\widehat{G}(\gamma)$ determines the original path.

Fact: *$\widehat{G}(\gamma)$ has genus $\hat{g} = g$, $\hat{e} \leqslant 3g + 1$ edges, $\hat{v} \leqslant 2g + 2$ vertices.*

Expected high trace method

Let γ in $\mathcal{N}_k$ which visits $e \leqslant k/2$ edges and v vertices. Set $g = e - v + 1 \geqslant 0$. We build a reduced graph $\widehat{G}(\gamma)$ by removing inner vertices of degree 2.



The path $\hat{\gamma} = (\hat{\gamma}_0, \dots, \hat{\gamma}_{\hat{k}})$ in the reduced graph $\widehat{G}(\gamma)$ determines the original path.

Fact: $\widehat{G}(\gamma)$ *has genus* $\hat{g} = g$, $\hat{e} \leqslant 3g + 1$ *edges,* $\hat{v} \leqslant 2g + 2$ *vertices.*

The reduced graph $\widehat{G}(\gamma)$ *has* $\hat{e} \leqslant 3g+1$ *edges and* $\hat{v} \leqslant 2g+2$ *vertices*:

We have $2\hat{e} = \sum_x \deg(x)$. Since all but two vertices have degree at least 3:

$$2\hat{e} \geqslant 3(\hat{v}-2)+2 = 3\hat{v}-4.$$

$$2\hat{e} - 2\hat{v} + 2 = 2\hat{g} = 2g,$$

we get $\hat{v} \leqslant 2g+2$.

Consequently, $\hat{e} = \hat{g} + v - 1 \leqslant 3g+1$.

Expected high trace method

The number of reduced paths $\hat{\gamma} = (\hat{\gamma}_0, \ldots, \hat{\gamma}_{\hat{k}})$ of length $\hat{k}$ with genus g is at most

$$\hat{e}^{\hat{k}} \hat{v}^{\hat{e}},$$

(at each time $1 \leqslant s \leqslant \hat{k}$, we choose one of the $\hat{e} \leqslant 3g + 1$ edges and choose the end vertex of each new edge).

Moreover, since $k - 2e = \sum_e (m_e - 2)$,

$$k - 2e \geqslant \hat{k} - 2\hat{e} \geqslant k - 6g.$$

Expected high trace method

The number of reduced paths $\hat{\gamma} = (\hat{\gamma}_0, \dots, \hat{\gamma}_{\hat{k}})$ of length $\hat{k}$ with genus g is at most

$$\hat{e}^{\hat{k}} \hat{v}^{\hat{e}} \leqslant (3g+1)^{\hat{k}} (2g+2)^{3g+1},$$

(at each time $1 \leqslant s \leqslant \hat{k}$, we choose one of the $\hat{e} \leqslant 3g+1$ edges and choose the end vertex of each new edge).

Moreover, since $k - 2e = \sum_e (m_e - 2)$,

$$k - 2e \geqslant \hat{k} - 2\hat{e} \geqslant k - 6g.$$

Expected high trace method

The number of reduced paths $\hat{\gamma} = (\hat{\gamma}_0, \ldots, \hat{\gamma}_{\hat{k}})$ of length $\hat{k}$ with genus g is at most

$$\hat{e}^{\hat{k}} \hat{v}^{\hat{e}} \leqslant (3g+1)^{\hat{k}} (2g+2)^{3g+1},$$

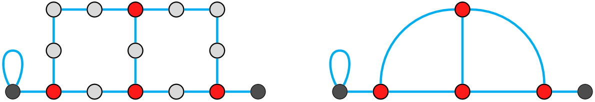
(at each time $1 \leqslant s \leqslant \hat{k}$, we choose one of the $\hat{e} \leqslant 3g+1$ edges and choose the end vertex of each new edge).

Moreover, since $k - 2e = \sum_e (m_e - 2)$,

$$k - 2e \geqslant \hat{k} - 2\hat{e} \geqslant k - 6g.$$

Expected high trace method

We estimate the number of paths $\gamma \in \mathcal{N}_k$ associated to a reduced path $\hat{\gamma}$.



If n_i is the number of edges in $G(\gamma)$ associated to the i-th edge of $\widehat{G}(\gamma)$ and $m_i \geqslant 2$ its multiplicity, we have

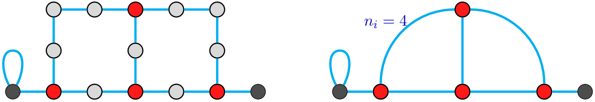
$$\sum_{i=1}^{\hat{e}} n_i m_i = k.$$

Hence, our number is at most the number of positive integer vectors (p_i) such that $\sum_i p_i \geqslant k$:

$$\binom{k-1}{\hat{e}-1} \leqslant \left(\frac{3(k-1)}{\hat{e}-1}\right)^{\hat{e}-1} \leqslant \left(\frac{k}{g}\right)^{3g}.$$

Expected high trace method

We estimate the number of paths $\gamma \in \mathcal{N}_k$ associated to a reduced path $\hat{\gamma}$.



If n_i is the number of edges in $G(\gamma)$ associated to the i-th edge of $\widehat{G}(\gamma)$ and $m_i \geqslant 2$ its multiplicity, we have

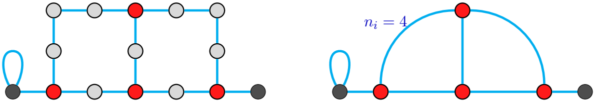
$$\sum_{i=1}^{\hat{e}} n_i m_i = k.$$

Hence, our number is at most the number of positive integer vectors (p_i) such that $\sum_i p_i \geqslant k$:

$$\binom{k-1}{\hat{e}-1} \leqslant \left(\frac{3(k-1)}{\hat{e}-1}\right)^{\hat{e}-1} \leqslant \left(\frac{k}{g}\right)^{3g}.$$

Expected high trace method

We estimate the number of paths $\gamma \in \mathcal{N}_k$ associated to a reduced path $\hat{\gamma}$.



If n_i is the number of edges in $G(\gamma)$ associated to the i-th edge of $\widehat{G}(\gamma)$ and $m_i \geqslant 2$ its multiplicity, we have

$$\sum_{i=1}^{\hat{e}} n_i m_i = k.$$

Hence, our number is at most the number of positive integer vectors (p_i) such that $\sum_i p_i \geqslant k$:

$$\binom{k-1}{\hat{e}-1} \leqslant \left(\frac{3(k-1)}{\hat{e}-1}\right)^{\hat{e}-1} \leqslant \left(\frac{k}{g}\right)^{3g}.$$

Finally,

$$\mathbb{E}\mathrm{Tr}\Big(B^{k/2}(B^{k/2})^*\Big) \leqslant n^2 \sum_{\gamma\in\mathcal{N}_k} n^{-(e(\gamma)-v(\gamma)+1)} q^{-(k-2e(\gamma))}$$
$$\leqslant n^2 \sum_{g=0}^{\infty} n^{-g} \sum_{\hat{k}=g}^{k} q^{-(\hat{k}-6g)} \left(\frac{k}{g}\right)^{3g} (3g+1)^{\hat{k}} (2g+2)^{3g+1}.$$

The computation is then straightforward: we find, if $k \leqslant c \min(q \log n, n^{0.33} q^{-2})$,

$$\mathbb{E}\mathrm{Tr}\Big(B^{k/2}(B^{k/2})^*\Big) \leqslant Cn^2k^2.$$

□

Remarks

The same argument works for inhomogeneous Wigner matrices with bounded row variances:

$$\text{for all } x, \quad \mathbb{E}\sum_{y} |H_{xy}|^2 \leqslant 1 \quad \text{and} \quad a.s.{-}\max_{x,y} |H_{xy}| \leqslant \frac{1}{q}.$$

Provided that $\max \mathbb{E}|H_{xy}|^2$ is not too large.

Some references

J. Friedman (1991).*On the second eigenvalue and random walks in random d-regular graphs*, Combinatorica.

S. Sodin (2007) Random matrices, nonbacktracking walks, and orthogonal polynomials. J. Math. Phys., 48(12):123503, 21, 2007.

F. Benaych-Georges, C. Bordenave, A. Knowles (2017). *Spectral radii of sparse random matrices*, arXiv:1704.02945.

Diluted random matrices

Diluted random matrices

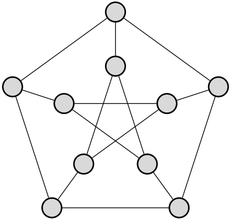
We will now study random matrices with $O(1)$ non-zero entries on each row. For example, adjacency matrix of a random 4-regular graph on n vertices.

For the random matrices of interest, classical expected high trace method will not work properly, even when applied to nonbacktracking matrices.

Two extra technical problems: usually, we cannot recenter easily the entries of the matrices, and for many models of interest, the entries are not independent.

Uniform regular graphs

Regular graph



For $2 \leqslant d \leqslant n-1$ and nd even, the set $\mathcal{G}(n,d)$ of d-regular graphs on the vertex set $\{1,\dots,n\}$ is not empty.

A uniform d-regular graph on n is a random graph sampled according to the uniform distribution on $\mathcal{G}(n,d)$.

Eigenvalues

Consider the adjacency matrix A of a d-regular graph on n vertices with eigenvalues

$$d = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n,$$

(we have $A\mathbf{1} = d\mathbf{1}$).

Recall that

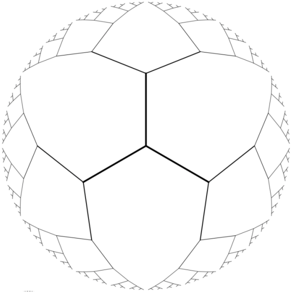
$$\mu_A = \frac{1}{n} \sum_k \delta_{\lambda_k}$$

is the empirical distribution of eigenvalues.

Kesten-McKay distribution

The spectral measure μ_d of the infinite d-regular tree $\mathcal{T}_d$ is

$$\int \lambda^k d\mu_d = (A^k_{\mathcal{T}_d})_{xx}.$$



Kesten (1959): μ_d has support $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ and density

$$\frac{d}{2\pi}\frac{\sqrt{4(d-1)-\lambda^2}}{d^2-\lambda^2}.$$

Theorem (McKay (1981))

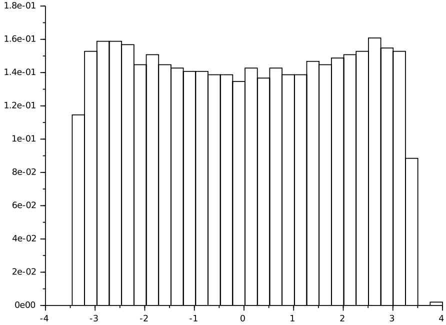
Let $d \geqslant 2$ and $G = G_n$ a sequence of d-regular graphs on n vertices. Assume that for any integer ℓ, the number of cycles of length ℓ in G is $o(n)$. Then, if A is the adjacency matrix of G, weakly,

$$\lim_{n \to \infty} \mu_A = \mu_d.$$

We may apply this result to a uniform d-regular graph on n vertices.

McKay Theorem

Take $d = 4$, $n = 2000$ and G a uniformly sampled d-regular graph.



McKay Theorem

Let G be a d-regular graph on n vertices and A its adjacency matrix. For any fixed ℓ, the nb of cycles of length $\leqslant \ell$ is $C_\ell = o(n)$.

If a vertex x is at distance at least k to any cycle of length at most $2k$, then the k-neighborhood of x is a d-regular tree of depth k. In particular,

$$(A^k)_{xx} = (A^k_{\mathcal{T}_d})_{oo} = \int \lambda^k d\mu_d.$$

The number of such vertices is at least $n - C_k k(d-1)^k$.

$$\left| \frac{1}{n} \mathrm{Tr} A^k - \int \lambda^k d\mu_d \right| = \left| \frac{1}{n} \sum_x (A^k)_{xx} - \int \lambda^k d\mu_d \right| \leqslant \frac{C_k k(d-1)^k d^k}{n} = o(1).$$

□

Alon-Boppana lower bound

Consider the adjacency matrix A of a d-regular graph on n vertices with eigenvalues

$$d = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n.$$

Theorem (Alon - Boppana (1986), Mohar (2010))

For any d-regular on n vertices,

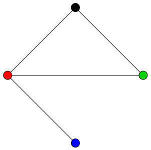
$$\lambda_2 \geqslant 2\sqrt{d-1} - \frac{c_d}{(\log n)^2}.$$

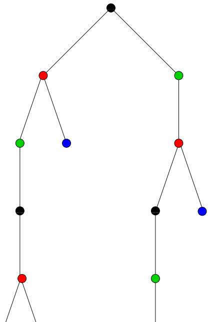
The spectral radius of $A_{\mathcal{T}_d}$ is a lower bound on λ_2.

Alon-Boppana lower bound

Every graph has a uninversal covering tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$

A construction of $\mathcal{T}$*:* take $o \in G$, $\mathcal{V}$ is the set of all nonbacktracking paths $(x_0, \cdots, x_k)$ starting from $x_0 = o$ $(x_{i-1} \neq x_{i+1})$. Two paths share an edge if one is the largest prefix of the other.





Alon-Boppana lower bound

Weaker result on $\lambda_\star = \max_{i \geqslant 2} |\lambda_i| = \lambda_2 \vee (-\lambda_n)$.

The universal covering tree of G is $\mathcal{T}_d$.

The nb of closed walks starting from x in G of length k is at least the nb of closed walks starting from the root in $\mathcal{T}_d$ of length k:

$$\frac{1}{n}\mathrm{Tr}(A^k) = \frac{1}{n}\sum_x (A^k)_{xx} \geqslant (A^k)_{oo} = \int \lambda^k d\mu_d.$$

For k even,

$$\int \lambda^k d\mu_d \geqslant \frac{c}{k^{3/2}}\left(2\sqrt{d-1}\right)^k.$$

For even k,

$$\mathrm{Tr}(A^k) = \sum_j \lambda_j^k \leqslant d^k + n\lambda_\star^k.$$

So finally,

$$\frac{c}{k^{3/2}}\left(2\sqrt{d-1}\right)^k \leqslant \frac{d^k}{n} + \lambda_\star^k.$$

Take $k = \log_d n$. □

Replacing $\lambda_\star$ *by* λ_2 *requires a refinement of this strategy (without trace).*

Ramanujan graphs

Let G be a d-regular graph on n vertices. Consider its adjacency matrix A

$$d = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n.$$

$\lambda_n = -d$ is equivalent to G bipartite.

The largest non-trivial eigenvalue is

$$\lambda_\star = \max_i \{|\lambda_i| : |\lambda_i| \neq d\}.$$

G is Ramanujan if

$$\lambda_\star \leqslant 2\sqrt{d-1}.$$

Ramanujan = non trivial eigenvalues bounded by the spectral radius of the adjacency operator of the universal covering tree.

Alon's conjecture (1986)

Theorem (Friedman (2008))

Fix an integer $d \geqslant 3$*. Let* G_n *is a sequence of uniformly distributed* d*-regular graphs on* n *vertices, then with high probability,*

$$\lambda_2 \vee |\lambda_n| \leqslant 2\sqrt{d-1} + o(1).$$

Most regular graphs are nearly Ramanujan!

We can take $o(1) = c(\log\log n)/(\log n)^2$.

Expected high trace method

If A is the adjacency matrix of G_n we would like to prove that for even $k \gg \log n$,

$$d^k + \lambda_2^k + \lambda_n^k \leqslant \mathrm{Tr}(A^k) \overset{?}{\leqslant} d^k + n\Big(2\sqrt{d-1} + o(1)\Big)^k.$$

Friedman's Theorem would follow.

Since $A\mathbf{1} = d\mathbf{1}$, it is wiser to project orthogonally on $\mathbf{1}^\perp$:

$$\mathrm{Tr}(A^k) - d^k = \mathrm{Tr}\left(A - \frac{d}{n}\mathbf{1}\mathbf{1}^*\right)^k \overset{?}{\leqslant} n\Big(2\sqrt{d-1} + o(1)\Big)^k.$$

Expected high trace method

For a first moment estimate, we would aim at

$$\mathbb{E}\mathrm{Tr}(A^k) - d^k = \mathbb{E}\mathrm{Tr}\left(A - \frac{d}{n}\mathbf{1}\mathbf{1}^*\right)^k \overset{?}{\leqslant} n\left(2\sqrt{d-1} + o(1)\right)^k$$

for $k \gg \log n$.

This is wrong !

The probability that the graph contains K_{d+1} as subgraph is at least n^{-c}. On this event $\lambda_2 = d$. Hence, for even $k \gg \log n$,

$$\mathbb{E}\mathrm{Tr}\left(A - \frac{d}{n}\mathbf{1}\mathbf{1}^*\right)^k \geqslant n^{-c} d^k \gg n\left(2\sqrt{d-1} + o(1)\right)^k.$$

Subgraphs which have polynomially small probability compromise the expected high trace method. Called Tangles.

Strategy

1. Use the nonbacktracking matrix B instead of A.
2. Remove the tangles.
3. Project on $\mathbf{1}^{\perp}$.
4. Use the expected high trace method to evaluate the remainder terms.

Nonbacktracking matrix

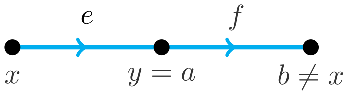
Oriented edge set :

$$\vec{E} = \{(x,y) : \{x,y\} \in E\},$$

Consider the matrix B acting on $\mathbb{R}^{\vec{E}}$ with entries

$$B_{ef} = \mathbf{1}(y = a)\mathbf{1}(x \neq b),$$

where $e = (x,y)$ and $f = (a,b)$.



Nonbacktracking version of Alon's conjecture

Complex eigenvalues, $|\vec{E}| = nd$,

$$d - 1 = \mu_1 \geqslant |\mu_2| \geqslant \cdots \geqslant |\mu_{nd}|.$$

Using the Hashimoto-Ihara-Bass identities:

Theorem (Friedman (2008))

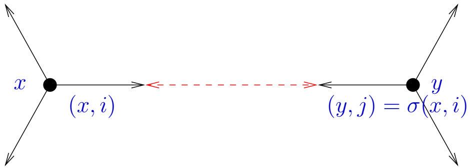
Fix an integer $d \geqslant 3$. Let G_n is a sequence of uniformly distributed d-regular graphs on n vertices, then with high probability,

$$|\mu_2| \leqslant \sqrt{d-1} + o(1).$$

Configuration model

The oriented edge set $\vec{E}$, $|\vec{E}| = nd$ is written as, with $V = \{1, \ldots, n\}$,

$$\vec{E} = V \times \{1, \ldots, d\}.$$



A matching σ on $\vec{E}$ defines a multigraph $G = G(\sigma)$ where a matching is a permutation such that $\sigma^2(x) = x$ and $\sigma(x) \neq x$.

Configuration model

We take σ a uniform random matching on $\vec{E}$.

Conditioned on the multigraph $G = G(\sigma)$ to be simple, $G(\sigma)$ is uniformly distributed on $\mathcal{G}(n, d)$, d-regular graphs on $V = \{1, \ldots, n\}$.

The probability for $G = G(\sigma)$ to be simple is lower bounded uniformly in n.

Since $\mathbb{P}(E^c|F) \leqslant \mathbb{P}(E^c)/\mathbb{P}(F)$, it is enough to prove Friedman's Theorem for the configuration model.

Configuration model

The nonbacktracking matrix with $f = (y, i)$,

$$B_{ef} = \mathbf{1}(\sigma(e) = (y, j) \text{ for some } j \neq i).$$

can be written as

$$B = MN$$

where

$$N_{ef} = \mathbf{1}(e_1 = f_1, e \neq f) = N_{fe}.$$

and M is the permutation matrix associated to σ,

$$M_{ef} = \mathbf{1}(\sigma(e) = f) = M_{fe}.$$

Restricted spectral radius

Since $B\mathbf{1} = B^*\mathbf{1} = (d-1)\mathbf{1}$, $|\mu_2|$ is the spectral radius of $B_{\mathbf{1}^\perp}$.

For any integer ℓ, the second largest eigenvalue of B is thus bounded by

$$|\mu_2|^\ell \leqslant \max_{v:\langle \mathbf{1}, v\rangle = 0} \frac{\|B^\ell v\|_2}{\|v\|_2}.$$

We prove if σ is a uniform random matching that with high probability

$$\max_{v:\langle \mathbf{1}, v\rangle = 0} \frac{\|B^\ell v\|_2}{\|v\|_2} \leqslant (\log n)^c (d-1)^{\ell/2}.$$

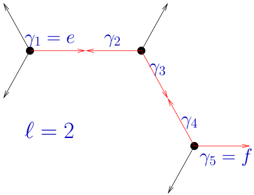
with $\ell \simeq \log n$.

Path decomposition

Recall $M_{ef} = \mathbf{1}(\sigma(e) = f)$, $N_{ef} = \mathbf{1}(e_1 = f_1, e \neq f)$

$$B^{\ell}_{ef} = \Big((MN)^{\ell}\Big)_{ef} = \sum_{\gamma \in \Gamma^{\ell}_{ef}} \prod_{s=1}^{\ell} M_{\gamma_{2s-1}\gamma_{2s}},$$

where Γ^{ℓ}_{ef} is the set of paths $\gamma = (\gamma_1, \ldots, \gamma_{2\ell+1}) \in (\vec{E})^{2\ell+1}$ such that $\gamma_1 = e$, $\gamma_{2k+1} = f$ and $N_{\gamma_{2s}\gamma_{2s+1}} = 1$.

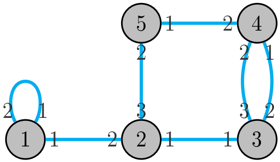


Path decomposition

$$B^{\ell}_{ef} = \sum_{\gamma \in \Gamma^{\ell}_{ef}} \prod_{s=1}^{\ell} M_{\gamma_{2s-1}\gamma_{2s}},$$

The set of paths Γ^{ℓ}_{ef} is independent of σ: combinatorial part.

The summand is the probabilistic part.



$\gamma = (1,1)(1,2)(1,1)(2,2)(2,1)(3,1)(3,2)(4,1)(4,2)(3,3)(3,2)(4,1)(4,2)(5,1)(5,2)(2,3)(2,1)(3,1)$

Path decomposition

$$B_{ef}^{\ell} = \left((MN)^{\ell}\right)_{ef} = \sum_{\gamma \in \Gamma_{ef}^{\ell}} \prod_{s=1}^{\ell} M_{\gamma_{2s-1}\gamma_{2s}},$$

The projection of M on $\mathbf{1}^{\perp}$ is,

$$\underline{M} = M - \frac{\mathbf{1}\mathbf{1}^*}{nd}.$$

Hence, if $\langle v, \mathbf{1} \rangle = 0$, we get

$$B^{\ell} v = \underline{B}^{\ell} v,$$

where $\underline{B} = \underline{M}N$ and

$$\underline{B}_{ef}^{\ell} = \left((\underline{M}N)^{\ell}\right)_{ef} = \sum_{\gamma \in \Gamma_{ef}^{\ell}} \prod_{s=1}^{\ell} \underline{M}_{\gamma_{2s-1}\gamma_{2s}}.$$

Tangles

A multi-graph (or a path) is tangle-free if it contains at most one cycle.

A multi-graph (or a path) is ℓ-tangle-free if all vertices have at most at most one cycle in their ℓ-neighborhood.

We denote by F_{ef}^{ℓ} the subset of tangle-free paths Γ_{ef}^{ℓ}.

Path decomposition

Assume that $G = G(\sigma)$ is ℓ-tangle-free. Then, for $0 \leqslant k \leqslant \ell$,

$$B^k = B^{(k)},$$

where

$$(B^{(k)})_{ef} = \sum_{\gamma \in F_{ef}^k} \prod_{s=1}^{k} M_{\gamma_{2s-1}\gamma_{2s}}.$$

Recall $\underline{M} = M - \mathbf{1}\mathbf{1}^*/(nd)$. For $0 \leqslant k \leqslant \ell$, we define the "projected" matrix

$$(\underline{B}^{(k)})_{ef} = \sum_{\gamma \in F_{ef}^k} \prod_{s=1}^{k} \underline{M}_{\gamma_{2s-1}\gamma_{2s}}.$$

Path decomposition

Beware that $\underline{B}^k \neq \underline{B}^{(k)}$, this is only approximately true!

Since $M_{ef} = \underline{M}_{ef} + 1/(nd)$,

$$(B^{(\ell)})_{ef} = (\underline{B}^{(\ell)})_{ef} + \sum_{\gamma \in F_{ef}^{\ell}} \sum_{k=1}^{\ell} \prod_{s=1}^{k-1} \underline{M}_{\gamma_{2s-1}\gamma_{2s}} \left(\frac{1}{nd}\right) \prod_{k+1}^{\ell} M_{\gamma_{2s-1}\gamma_{2s}},$$

which follows from the identity,

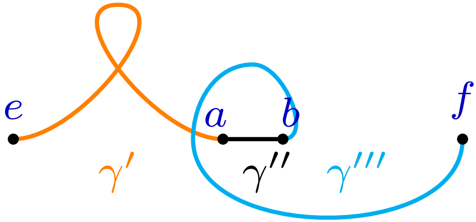
$$\prod_{s=1}^{\ell} x_s = \prod_{s=1}^{\ell} y_s + \sum_{k=1}^{\ell} \prod_{s=1}^{k-1} y_s (x_k - y_k) \prod_{k+1}^{\ell} x_s.$$

Path decomposition

A path $\gamma \in F_{ef}^{\ell}$ can be decomposed as the union of

$$\gamma' \in F_{ea}^{k-1}, \quad \gamma'' \in F_{ab}^{1} \quad \text{and} \quad \gamma''' \in F_{bf}^{\ell-k}.$$

with $a = \gamma_{2k-1}$, $b = \gamma_{2k+1}$.

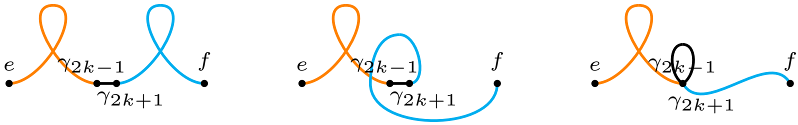


Path decomposition

For any e, f, we have $|\Gamma^1_{ef}| = (d-1)$. We find

$$\sum_{\gamma \in F^{\ell}_{ef}} \prod_{s=1}^{k-1} \underline{M}_{\gamma_{2s-1}\gamma_{2s}} \prod_{k+1}^{\ell} M_{\gamma_{2s-1}\gamma_{2s}} = (d-1)\Big(\underline{B}^{(k-1)}\mathbf{1}\mathbf{1}^* B^{(\ell-k)}\Big)_{ef} - \Big(R_k^{(\ell)}\Big)_{ef}$$

where $\left(R_k^{(\ell)}\right)_{ef}$ sums tangle-free paths whose union is tangled:



So finally,

$$B^{(\ell)} \quad = \quad \underline{B}^{(\ell)} + \frac{d-1}{nd} \sum_{k=1}^{\ell} \underline{B}^{(k-1)} \mathbf{1}\mathbf{1}^* B^{(\ell-k)} - \frac{1}{nd} \sum_{k=1}^{\ell} R_k^{(\ell)}.$$

Hence, if $\mathbf{1}^* v = \langle v, \mathbf{1} \rangle = 0$ and $G = G(\sigma)$ is ℓ-tangle-free, since $\mathbf{1}^* B^{(\ell-k)} = \mathbf{1}^* B^{\ell-k} = (d-1)^{\ell-k} \mathbf{1}^*$,

$$B^{(\ell)} v \quad = \quad \underline{B}^{(\ell)} v - \frac{1}{nd} \sum_{k=1}^{\ell} R_k^{(\ell)} v.$$

Path decomposition

We arrive at

$$|\mu_2|^\ell \leqslant \max_{v:\langle \mathbf{1}, v\rangle = 0} \frac{\|B^\ell v\|_2}{\|v\|_2} \leqslant \|\underline{B}^{(\ell)}\| + \frac{1}{nd} \sum_{k=1}^{\ell} \|R_k^{(\ell)}\|.$$

This inequality holds if $G(\sigma)$ *is* ℓ *tangle-free.*

Fact: For uniform random σ, $G(\sigma)$ *is* ℓ *tangle-free with high probability for* $\ell = 0.1 \log n / \log(d-1)$. (Lubetzky-Sly (2010))

Expected high trace method

$$|\mu_2|^\ell \leqslant \|\underline{B}^{(\ell)}\| + \frac{1}{nd}\sum_{k=1}^{\ell}\|R_k^{(\ell)}\|.$$

Our aim is then to prove that with high probability

$$\|\underline{B}^{(\ell)}\| \leqslant (\log n)^c (d-1)^{\ell/2} \quad \text{and} \quad \|R_k^{(\ell)}\| \leqslant (\log n)^c (d-1)^{\ell - k/2}$$

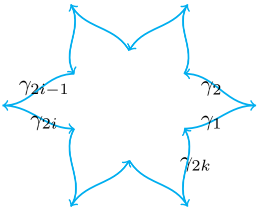
By estimating, for $S = \underline{B}^{(\ell)}$ or $S = R_k^{(\ell)}$.

$$\mathbb{E}\|S\|^{2k} \leqslant \mathbb{E}\mathrm{Tr}(SS^*)^k.$$

with $k \simeq \log n / (\log\log n)$: on the overall paths of length $2\ell k \gg \log n$.

For $S = \underline{B}^{(\ell)}$,

$$\mathbb{E}\|S\|^{2k} \leqslant \mathbb{E}\mathrm{Tr}(SS^*)^k \leqslant \sum_{\gamma} \mathbb{E} \prod_{i=1}^{2k} \prod_{t=1}^{\ell} \underline{M}_{\gamma_{i,2t-1}\gamma_{i,2t}}$$



The path $\gamma = (\gamma_{i,t})$ is made of $2k$ tangle-free paths of length ℓ. To control the nb of such paths with a given genus and given number of vertices, we use crucially the fact that each γ_i visits at most one cycle in the reduced graph of $G(\gamma)$.

Expected high trace method

For $S = \underline{B}^{(\ell)}$,

$$\mathbb{E}\|S\|^{2k} \leqslant \mathbb{E}\mathrm{Tr}(SS^*)^k \leqslant \sum_{\gamma} \mathbb{E} \prod_{i=1}^{2k} \prod_{t=1}^{\ell} \underline{M}_{\gamma_{i,2t-1}\gamma_{i,2t}}.$$

Recall $\underline{M}_{ef} = M_{ef} - 1/(dn)$. The probabilistic part relies on the claim: for $T \leqslant \sqrt{dn}$ and any $(e_t, f_t)_t \in \vec{E}^{2T}$,

$$\left| \mathbb{E} \prod_{t=1}^{T} \left(M_{e_t f_t} - \frac{1}{dn} \right) \right| \leqslant c \left(\frac{1}{dn} \right)^{a} \left(\frac{3T}{\sqrt{dn}} \right)^{a_1},$$

where a is the nb of distinct unordered pairs $\{e_t, f_t\}$ and a_1 is the nb of pairs appearing exactly once.

Some references

J. Friedman (2008). *A proof of Alon's second eigenvalue conjecture and related problems*. Mem. Amer. Math. Soc.

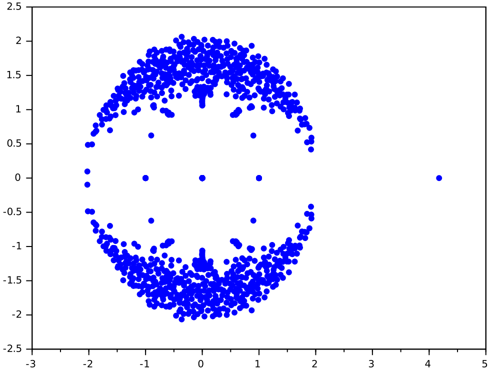
C. Bordenave (2015). *A new proof of Friedman's second eigenvalue Theorem and its extension to random lifts.* ArXiv:1502.04482.

L. Massoulié (2014). *Community detection thresholds and the weak Ramanujan property*. STOC '14.

Non-backtracking spectrum of Erdős-Renyi graphs

Non-backtracking spectrum of Erdős-Renyi graphs

Eigenvalues of B for an Erdős-Rényi graph with average degree $d = 4$ and $n = 500$ vertices.



Erdős-Rényi Graph

Let B be the nonbacktracking matrix of the adjacency matrix A, with eigenvalues

$$\mu_1 \geqslant |\mu_2| \geqslant \cdots$$

Theorem

Let $d > 1$ and G_n be an Erdős-Rényi graph with average degree d. With high probability,

$$\begin{aligned} \mu_1 &= d + o(1) \\ |\mu_2| &\leqslant \sqrt{d} + o(1). \end{aligned}$$

Bordenave, Massoulié & Lelarge (2018)

Erdős-Rényi Graph

The bound $|\mu_2| \leqslant \sqrt{d} + o(1)$ is a Ramanujan property: the spectral radius of the nonbacktracking operator of the universal covering tree of G_n is $\sqrt{d} + o(1)$.

There is an analog result for the stochastic block model (inhomogeneous Erdős-Rényi random graphs with finite number of classes).

The proof follows the same strategy. The path decomposition is much more involved, the eigenvector associated to μ_1 is genuinely random.

Strong asymptotic freeness of uniform permutations

Algebra of permutation matrices

Let $\sigma_1, \dots, \sigma_q$ permutations on $\{1, \dots, n\}$.

Let $S_1, \dots, S_q$ their permutation matrices:

$$(S_i)_{xy} = \mathbf{1}(\sigma_i(x) = y).$$

For a given non-commutative polynomial P, we consider the matrix in $M_n(\mathbb{C})$

$$P = P(S_1, \dots, S_q, S_1^*, \dots, S_q^*).$$

Examples : $P = S_1 S_2^2 S_1^* - S_3 S_1^* S_3$ or $P = S_1 + S_2 + S_1^* + S_2^*$ (adjacency matrix of 4-regular graph).

The constant vector $\mathbf{1}$ is an eigenvector of P and P^*.

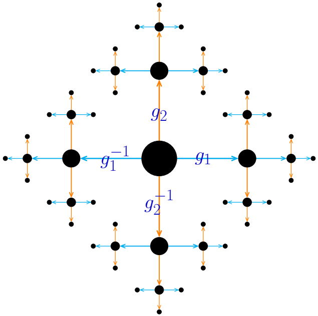
The operator norm of P on $\mathbf{1}^\perp$ is

$$\left\| P_{|\mathbf{1}^\perp} \right\| = \sup_{f \in \mathbf{1}^\perp} \frac{\|Pf\|_2}{\|f\|_2}.$$

What is the value of $\left\| P_{|\mathbf{1}^\perp} \right\|$ *when* n *is large and* $\sigma_1, \ldots, \sigma_q$ *uniform random permutations?*

Algebra of the free group

Let X be the free group with q generators $g_1, \ldots, g_q$ and their inverses.



Consider the operator on $\ell^2(X)$,

$$P_\star = P\big(\lambda(g_1), \ldots, \lambda(g_q), \lambda(g_1^{-1}), \ldots, \lambda(g_q^{-1})\big),$$

where $\lambda(\cdot)$ is the left-regular representation (left multiplication).

$$P = P(S_1, \dots, S_q, S_1^*, \dots, S_q^*).$$

$$\|T\| = \sup_{f \neq 0} \frac{\|Tf\|_2}{\|f\|_2}.$$

Theorem

Let $S_1, \cdots, S_q$ *be independent uniform permutation matrices in* $\mathcal{S}_n$. *Then with high probability, as* $n \to \infty$,

$$\left\| P_{|\mathbf{1}^\perp} \right\| = \|P_\star\| + o(1).$$

Bordenave & Collins (2018)

Strategy

Set $i^* = i + q$, $i^{**} = i$ and $S_{i^*} = S_i^*$.

Linearization trick: it is enough to consider *symmetric linear polynomials with matrix coefficients* :

$$A = a_0 + \sum_{i=1}^{2q} a_i \otimes S_i$$

where $a_i \in M_k(\mathbb{C})$ et $a_{i^*} = a_i^*$.

Claim: the convergence of the spectra of such matrices A implies the convergence of the operator norm of all non-commutative polynomial P.

Strategy

Nonbacktracking: we introduce the *nonbactracking matrix with matrix coefficients*:

$$B = \sum_{(i,j):i \neq j^*} a_i \otimes S_i \otimes E_{ij}.$$

Claim: the convergence of the spectral radii of all nonbactracking matrices implies the convergence of the spectrum of A *(Extensions of Hashimoto-Ihara-Bass identities).*

To deal with *nonbactracking matrix with matrix coefficients*, we adapt the strategy used in the proof for the uniform regular graphs: removing tangles / projection / expected high trace method. This is more involved, due to the matrices a_i.

Remarks

Extend to tensor products: polynomial in $S_i \otimes S_i$ and other random unitary matrices.

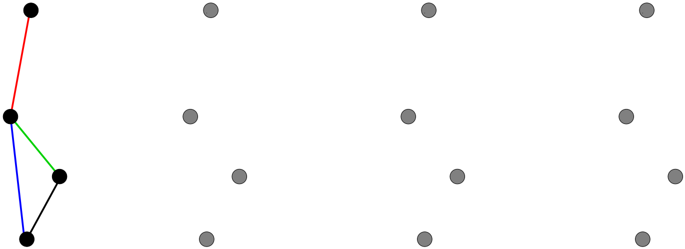
The matrix $A = \sum_{i=1}^{2q} a_i \otimes S_i$ is a random n-lift if $a_i = E_{x_i,y_i} \in M_k(\mathbb{C})$: $A_1 = \sum_i (a_i + a_i^*)$. is the adjacency matrix of a graph with k vertices and q edges.

The convergence of the non-trivial eigenvalues of A is a generalization of Alon's conjecture to random n-lifts.

Remarks

Extend to tensor products: polynomial in $S_i \otimes S_i$ and other random unitary matrices.

The matrix $A = \sum_{i=1}^{2q} a_i \otimes S_i$ is a random n-lift if $a_i = E_{x_i,y_i} \in M_k(\mathbb{C})$: $A_1 = \sum_i (a_i + a_i^*)$. is the adjacency matrix of a graph with k vertices and q edges.

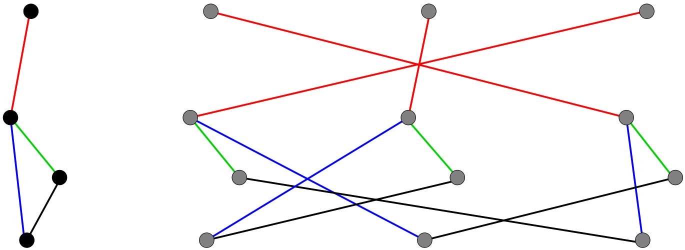


The convergence of the non-trivial eigenvalues of A is a generalization of Alon's conjecture to random n-lifts.

Remarks

Extend to tensor products: polynomial in $S_i \otimes S_i$ and other random unitary matrices.

The matrix $A = \sum_{i=1}^{2q} a_i \otimes S_i$ is a random n-lift if $a_i = E_{x_i,y_i} \in M_k(\mathbb{C})$: $A_1 = \sum_i (a_i + a_i^*)$. is the adjacency matrix of a graph with k vertices and q edges.



The convergence of the non-trivial eigenvalues of A is a generalization of Alon's conjecture to random n-lifts.

Concluding words

Thank you for your attention !