Simplicity of Spectral Edges and Applications to Homogenization

Vivek Tewary
Joint work with Prof. Sivaji Ganesh Sista
Department of Mathematics
Indian Institute of Technology Bombay

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Outline

- Recap of Bloch wave Method
- Spectrum of Periodic Elliptic Operators
- Applications to Homogenization

Recap of Bloch wave method

Bloch wave method

- We shall consider operators $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined by

$$\mathcal{A}u := -\frac{\partial}{\partial y_k}\left(a_{kl}(y)\frac{\partial u}{\partial y_l}\right)$$

where $A = (a_{kl})$ is a real symmetric matrix with entries in $L^\infty_\sharp(Y)$, $Y = [0, 2\pi)^d$.

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- The matrix A satisfies $A\xi \cdot \xi \geq \alpha|\xi|^2$ for some $\alpha > 0$ and all $\xi \in \mathbb{R}^d$.

- Also define $Y' := \left[-\frac{1}{2}, \frac{1}{2}\right)^d$.

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- It further induces a decomposition of the operator $\mathcal{A}$ in $L^2(\mathbb{R}^d)$ into a 'field' of operators indexed by $\eta \in Y^{'}$ that distribute over the 'field' of Hilbert spaces

$$\eta \mapsto \mathcal{A}(\eta) := -(\nabla + i\eta) \cdot A(y)(\nabla + i\eta) \text{ in } L^2_\sharp(Y).$$

Second Step

- We look for the spectrum of $\mathcal{A}(\eta)$ in $L^2_\sharp(Y)$ as η varies in $Y^{'}$:

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- Thus we obtain the following further decomposition

$$L^2(\mathbb{R}^d) \cong L^2(Y^{'}; L^2_\sharp(Y)) \cong L^2(Y^{'}; \ell^2(\mathbb{N})).$$

Homogenization Step

- At the ϵ scale, the decomposition of spaces is

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- Under this mapping, the equation $\mathcal{A}^\epsilon u^\epsilon = f$ becomes a cascade of equations

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$$\lambda^\epsilon_m(\xi) u^{m,\epsilon}_b = f^{m,\epsilon}_b,\, m \in \mathbb{N},\, \xi \in Y^{'}/\epsilon.$$

- The homogenized equation is found by writing the first Bloch eigenvalue as a power series at $\xi = 0$ and passing to the limit.

Spectrum of Periodic Elliptic Operators

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- The spectrum of $\mathcal{A}$ can be written as a union of intervals[1].

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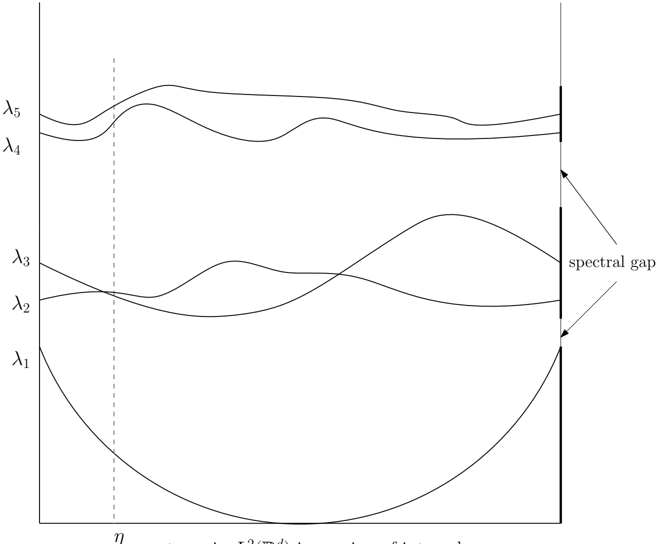

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- The endpoints of these intervals are called spectral edges or thresholds.
- Thus, homogenization may be interpreted as a phenomena which is governed by properties of the spectral edge.

Properties of bottom of the spectrum

- In Bloch wave method, we require at least C^2 differentiability of the first Bloch eigenvalue at $\eta = 0$.

[2] Rellich, F., *Perturbation theory of eigenvalue problems*, Gordon and Breach Science Publishers: 1969, pp x+127.

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- (We also require that first Bloch eigenvalue is strongly convex function of η, etc.)
- Differentiabilty properties of parametrized eigenvalues is a subject studied under the heading of "Perturbation Theory of Operators". Developed by Rellich[2], Kato[3], etc.

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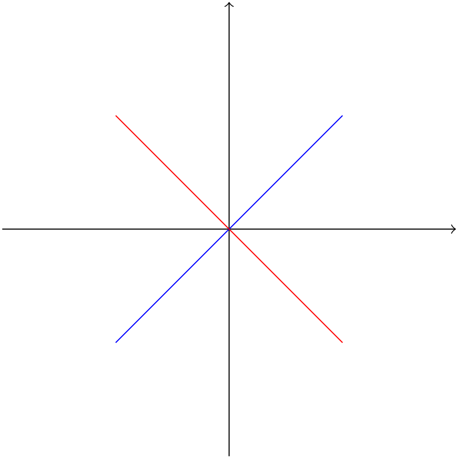
- Re-arranged eigenvalues $\pm x$ are analytic.



Examples : 2

- Another perturbation of the zero matrix is

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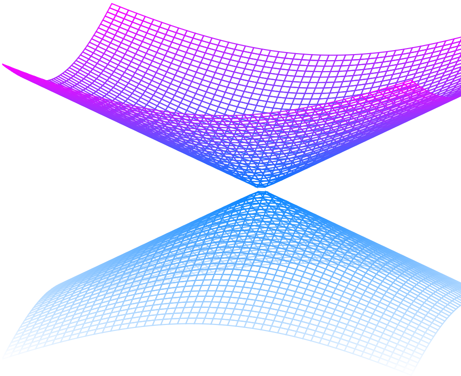
- Ordered eigenvalues $\pm\sqrt{x^2+y^2}$ are not analytic at 0 since there is a change of multiplicity at 0.
- There is no analytic re-arrangement of the eigenvalues.



Generic Simplicity

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- It may be proved that under various perturbations of domain, coefficients, boundary data the eigenvalues can be made simple.

Regularity Properties of Spectral edges

A spectral edge is expected to have the following regularity properties generically[4].

- Spectral edge is simple, i.e., it is attained by a single Bloch eigenvalue.

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- Spectral edge is non-degenerate, i.e., The Bloch eigenvalue is strongly convex when it is close to the spectral edge.

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What is known?

- Klopp & Ralston[5] proved that a spectral edge can be made simple through a small perturbation of the potential term in the operator $-\Delta + V$.

[5] Klopp, F.; Ralston, J. *Methods Appl. Anal.* **2000**, *7*, 459–463.
[6] Filonov, N.; Kachkovskiy, I. *Acta Math.* **2018**, *221*, 59–80.
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- Parnovski and Shterenberg[7] proved that in dimension 2, a perturbation by a potential of a larger period makes a spectral edge non-degenerate.

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Theorem (S. Sivaji Ganesh & V.T., 2019)

Let $A = (a_{ij})$ be a positive definite real symmetric matrix. Let (λ_-, λ_+) be a spectral gap of the operator $-\nabla \cdot (A\nabla)$. Suppose that either

- $a_{ij} \in W_\sharp^{1,\infty}(Y)$, or
- $a_{ij} \in L_\sharp^\infty(Y)$ and the spectral edge is attained at finitely many points.

Then, there exists a matrix $B = (b_{ij}) \in L_\sharp^\infty(Y)$ such that the new spectral edge $\tilde{\lambda}_+$ of the perturbed operator $\nabla \cdot (A + tB)\nabla$ is simple for small $t > 0$.

Applications to homogenization

Internal edge homogenization

- Hessian of the lowest Bloch eigenvalue is a measure of the convexity of the lowest spectral edge. Hence, homogenization may be interpreted as a phenomena governed by the regularity properties of the lowest spectral edge.

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- Birman and Suslina[8] extended the notion of homogenization to internal edges of the spectrum by proposing an effective operator to the highly oscillating operator at internal edges.

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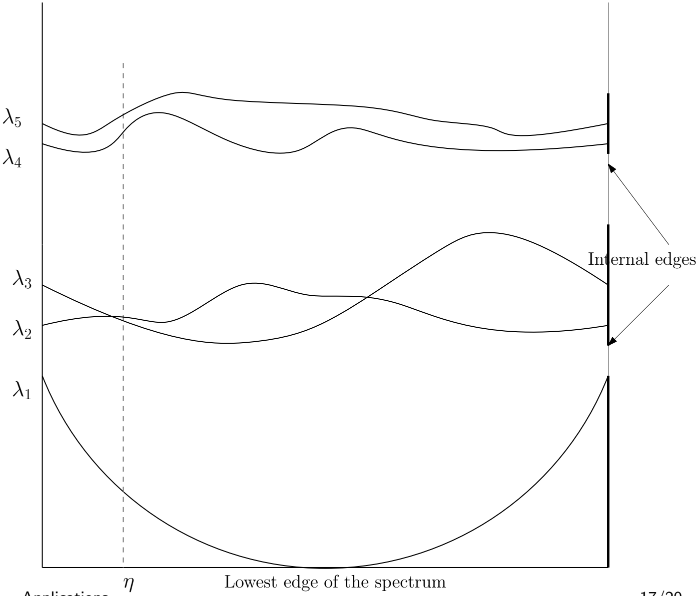
λ_5
λ_4
λ_3
λ_2
λ_1
Internal edges
η
Lowest edge of the spectrum

Internal edge homogenization

- Convergence to the effective operator depends on the regularity properties of the spectral edge. In particular, Birman and Suslina assume that the internal spectral edge is simple, isolated and non-degenerate.

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- Convergence to the effective operator depends on the regularity properties of the spectral edge. In particular, Birman and Suslina assume that the internal spectral edge is simple, isolated and non-degenerate.

- We have extended the theorem of Birman and Suslina to spectral edges with multiplicity greater than 1 by using a perturbation of the operator which renders the multiple edge simple.

Generic Simplicity of Spectral Edges and Applications to Homogenization. arXiv:1807.00917

Thank you