Asymptotic study of a locally periodic oscillating boundary

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(Joint work with Prof. Klas Pettersson)

Outline

- Rough Domain

Outline

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- Problem description

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- Unfolding operator

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- Results

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Various Physical Domains

Figure: Heat Radiator

Various Physical Domains

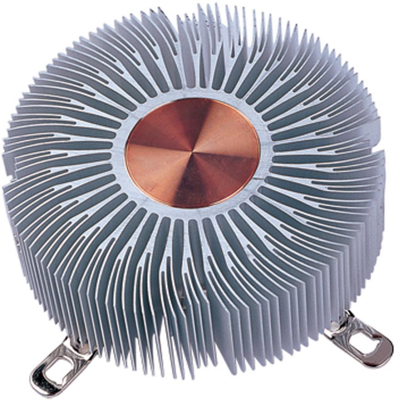


Figure: Heat Radiator

Various Physical Domains

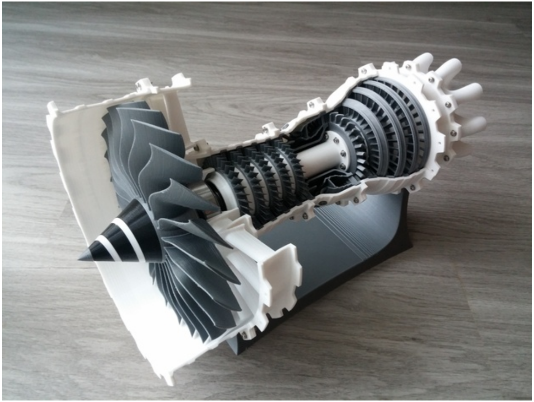


Figure: Jet Engine

Oscillatory Domain

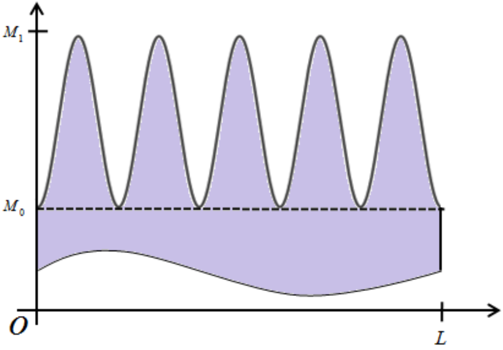


Figure: The domain Ω_ϵ $(\epsilon = \frac{L}{N}, N = 5)$,

Oscillatory Domain

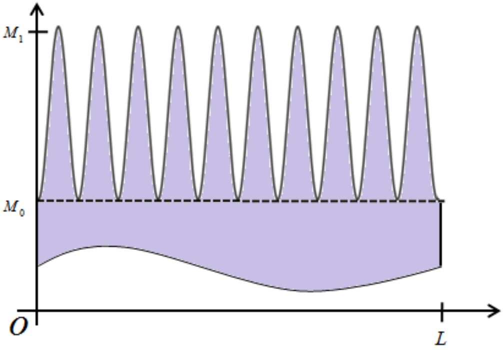


Figure: The domain Ω_ϵ $(\epsilon = \frac{L}{N}, N = 10)$,

Oscillatory Domain

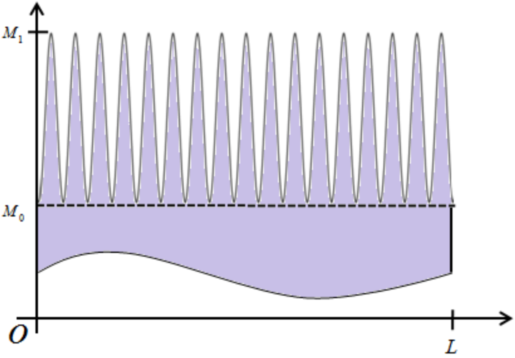


Figure: The domain Ω_ϵ $(\epsilon = \frac{L}{N}, N = 16)$,

Oscillatory Domain

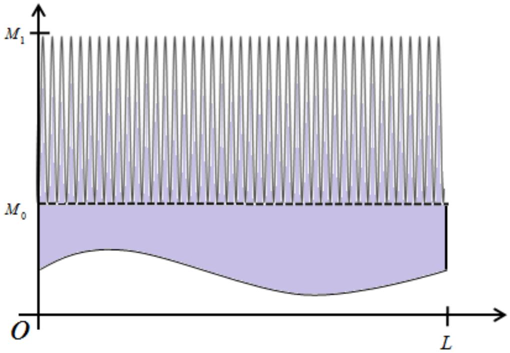


Figure: The domain Ω_ϵ $(\epsilon = \frac{L}{N}, N = 50)$,

Oscillatory Domain

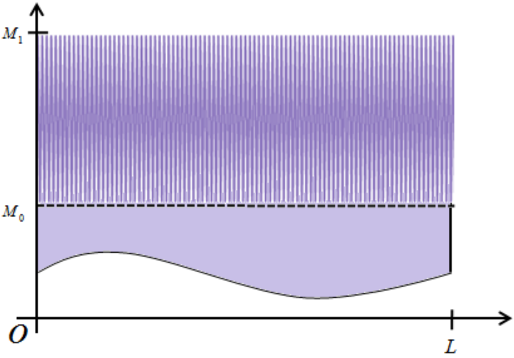
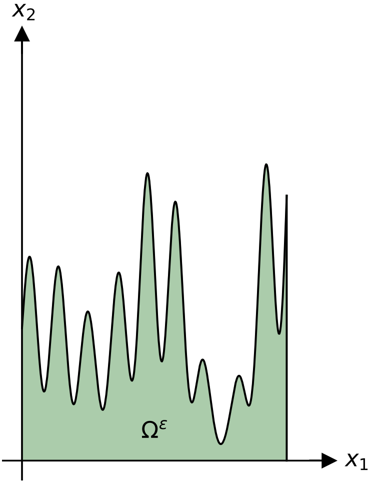


Figure: The domain Ω_ϵ $(\epsilon = \frac{L}{N}, N = 100)$,

Locally periodic boundary



Problem description

The oscillating boundary Γ_ε is given by

$$\Gamma_\varepsilon = \left\{ \left(x_1, \eta\left(x_1, \frac{x_1}{\varepsilon} \right) \right) : x_1 \in [0,1] \right\}$$

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where $\eta : [0,1] \times \mathbb{T} \to \mathbb{R}$, is strictly positive Lipschitz function and $\mathbb{T}$ denotes the one-dimensional torus realized with unit measure. The domain Ω^ε is given by

$$\Omega^\varepsilon = \left\{ (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < \eta \left(x_1, \frac{x_1}{\varepsilon} \right) \right\}$$

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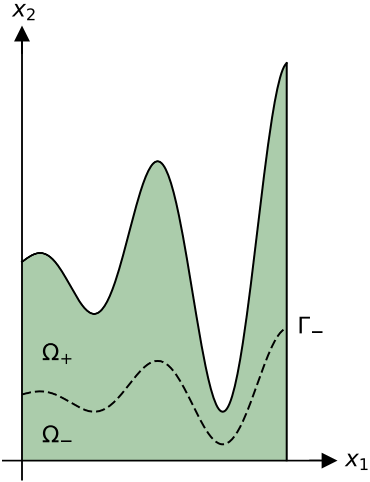
Model Problem: Elliptic Equation

$$\begin{aligned} -\Delta u^\varepsilon &= f \quad \text{in } \Omega^\varepsilon, \\ u^\varepsilon &= 0 \quad \text{on } \Gamma, \\ \nabla u^\varepsilon \cdot \nu &= 0 \quad \text{on } \partial\Omega^\varepsilon \setminus \Gamma. \end{aligned} \tag{1}$$

Here, the fixed boundary Γ is given by

$$\Gamma = \{(x_1, 0) : x_1 \in [0,1]\}$$

Approximate / limit problem



Limit Domain:

Define η_+ and η_- as

$$\eta_-(x_1) = \min_y \eta(x_1, y), \qquad \eta_+(x_1) = \max_y \eta(x_1, y).$$

Note that η_+ and η_- are Lipschitz functions.

The domain

$$\Omega = \{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < \eta_+(x_1)\}$$

is separated into the regions

$$\Omega_- = \{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < \eta_-(x_1)\},$$
$$\Omega_+ = \{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ \eta_-(x_1) < x_2 < \eta_+(x_1)\},$$

with interior interface $\Gamma_- = \partial\Omega_- \cap \partial\Omega_+$.

Limit space

Let h denote what we call the density of Ω^ε in Ω:

$$Y(x) = \{y : 0 < x_2 < \eta(x_1, y)\},$$
$$h(x) = |Y(x)|.$$

In Ω_- the density is $h = 1$.

Limit space

Let h denote what we call the density of Ω^ε in Ω:

$$\begin{aligned} Y(x) &= \{y : 0 < x_2 < \eta(x_1, y)\}, \\ h(x) &= |Y(x)|. \end{aligned}$$

In Ω_- the density is $h = 1$.

Denote by $L^2(\Omega, h)$ the Lebesgue space $\{v : \int_\Omega v^2 h\, dx < \infty\}$, and $W(\Omega, \Gamma)$ the Sobolev space

$$W(\Omega, \Gamma) = \left\{v \in L^2(\Omega, h) : \frac{\partial v}{\partial x_1} \in L^2(\Omega_-, h),\ \frac{\partial v}{\partial x_2} \in L^2(\Omega, h),\ v = 0 \text{ on } \Gamma\right\},$$

with the inner product is given by

$$\langle \phi, \psi \rangle_W =: \langle \phi, \psi \rangle_{L^2(\Omega,h)} + \langle \partial_{x_2}\phi, \partial_{x_2}\psi \rangle_{L^2(\Omega,h)} + \langle \partial_{x_1}\phi, \partial_{x_1}\psi \rangle_{L^2(\Omega_-)}.$$

Limit Problem

Limit Equation

$$\begin{aligned} -\operatorname{div}(A^0\nabla u) = hf \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \Gamma, \\ A^0\nabla u \cdot \nu = 0 \quad & \text{on } \partial\Omega \setminus \Gamma, \\ [u] = 0 \quad & \text{on } \Gamma_-, \\ [A^0\nabla u \cdot \nu] = 0 \quad & \text{on } \Gamma_-, \end{aligned} \tag{2}$$

The effective matrix is

$$A^0 = \begin{pmatrix} \chi_{\Omega_-} h & 0 \\ 0 & h \end{pmatrix}.$$

The weak form of the limit equation is: Find $u^0 \in W(\Omega, \Gamma)$ such that

$$\int_\Omega A^0 \nabla u^0 \cdot \nabla \psi \, dx = \int_\Omega f \psi h \, dx, \tag{3}$$

for all $\psi \in W(\Omega, \Gamma)$.

The weak form of the limit equation is: Find $u^0 \in W(\Omega, \Gamma)$ such that

$$\int_{\Omega} A^0 \nabla u^0 \cdot \nabla \psi \, dx = \int_{\Omega} f \psi h \, dx, \tag{3}$$

for all $\psi \in W(\Omega, \Gamma)$.

In other words, find $u^0 \in W(\Omega, \Gamma)$ such that

$$\int_{\Omega_-} \nabla u^0 \cdot \nabla \psi \, dx + \int_{\Omega_+} \partial_{x_2} u^0 \partial_{x_2} \psi \, h \, dx = \int_{\Omega} f \psi h \, dx, \tag{4}$$

for all $\psi \in W(\Omega, \Gamma)$.

Homogenization

Theorem

Let $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ be the solutions to (1), and let $u^0 \in W(\Omega, \Gamma)$ be the solution to (2). Then

$$
\begin{aligned}
&i) \quad \widetilde{u^\varepsilon} \rightharpoonup hu^0 \quad \text{weakly in } L^2(\Omega),\\
&ii) \quad \widetilde{\nabla u^\varepsilon} \rightharpoonup A^0 \nabla u^0 \quad \text{weakly in } L^2(\Omega),
\end{aligned}
$$

as ε tends to zero, where $\sim$ denotes extension by zero.

Weak formulation

The variational form of (1) is: Find $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ such that

$$\int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi \, dx = \int_{\Omega^\varepsilon} f \psi \, dx, \tag{5}$$

for all $\psi \in H^1(\Omega^\varepsilon, \Gamma)$.

Weak formulation

The variational form of (1) is: Find $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ such that

$$\int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi \, dx = \int_{\Omega^\varepsilon} f \psi \, dx, \tag{5}$$

for all $\psi \in H^1(\Omega^\varepsilon, \Gamma)$.

For the solutions u^ε, the following a priori estimate holds:

$$\|u^\varepsilon\|_{H^1(\Omega^\varepsilon, \Gamma)} \leq C,$$

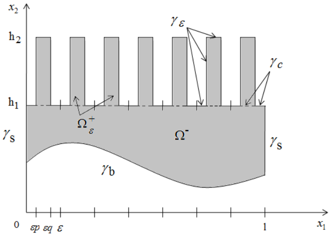
where C is independent of ε.

Unfolding Operator

Unfolding operators are introduced by D. Cioranescu, A. Damlamian and G. Griso, (SIAM J. Math. Anal. 2008) to study homogenization boundary value problems with oscillating coefficients.

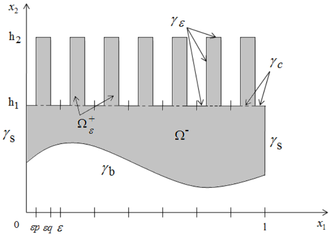
$$
\begin{aligned}
-\mathrm{div}(A(x/\varepsilon)\nabla u^{\varepsilon}) = f \quad & \text{in } \Omega, \\
u = 0 \quad & \text{on } \partial\Omega.
\end{aligned}
$$

Pillar type domain



- D Blanchard, A Gaudiello, and G Griso, (Journal de mathématiques pures et appliquées 2007)

Pillar type domain



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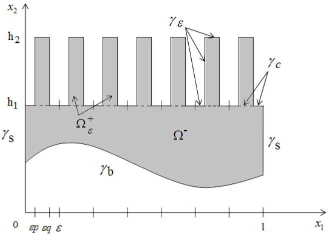

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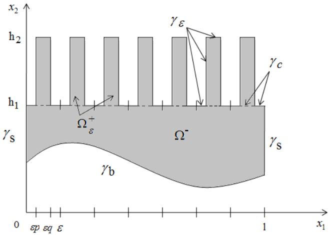

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- S.A., and B. C. Sardar, (Applicable Analysis 2017).

Proposition

Let $u \in L^1(\Omega_\varepsilon^+)$. Then,

$$\int_{\Omega_U = \Omega^+ \times Y} T^\varepsilon u \; dxdy = \int_{\Omega_\varepsilon^+} u \; dx.$$

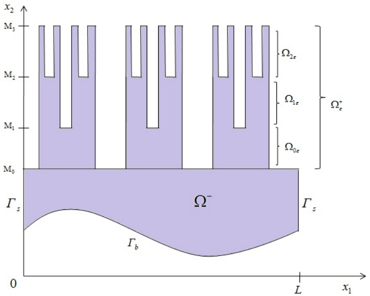
Here $Y = (p, q)$.

Proof:

$$\begin{aligned}
\int_{\Omega_U} T^\varepsilon u \; dxdy &= \int_{x_2=h_1}^{h_2} \int_{y \in Y} \int_{x_1=0}^{1} u\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right] + \varepsilon y,\; x_2\right)\; dx_1 dy dx_2 \\
&= \int_{x_2=h_1}^{h_2} \int_{y \in Y} \sum_{k=0}^{N-1} \int_{x_1 \in \varepsilon(k,k+1)} u(k\varepsilon + \varepsilon y, x_2)\; dx_1 dy dx_2 \\
&= \sum_{k=0}^{N-1} \int_{x_1 \in \varepsilon(k,k+1)} dx_1 \int_{x_2=h_1}^{h_2} \int_{y \in Y} u(k\varepsilon + \varepsilon y, x_2)\; dy dx_2
\end{aligned}$$

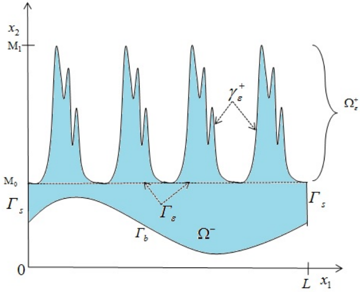
$$
\begin{aligned}
&= \varepsilon \sum_{k=0}^{N-1} \int_{x_2=h_1}^{h_2} \int_{y\in Y} u(k\varepsilon + \varepsilon y, x_2)\, dy dx_2 \\
&= \sum_{k=0}^{N-1} \int_{x_2=h_1}^{h_2} \int_{z\in k\varepsilon+\varepsilon Y} u(z, x_2)\, dz dx_2 \\
&= \int_{\Omega_\varepsilon^+} u(x)\, dx.
\end{aligned}
$$

Branched Structure Domain



- S. Aiyappan, A. K. Nandakumaran, (Math. Meth. Appl. Sci. 2017)

General Oscillating Domain



- S.A., A. K. Nandakumaran, and R. Prakash, (Calc. Var. Partial Differential Equations 2018)

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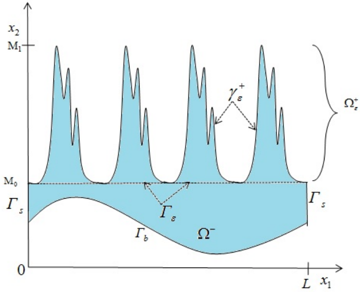

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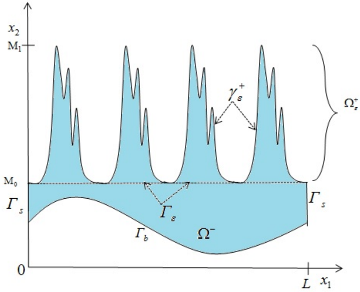

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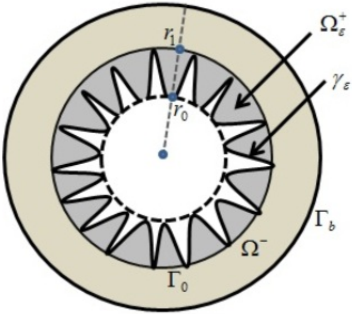
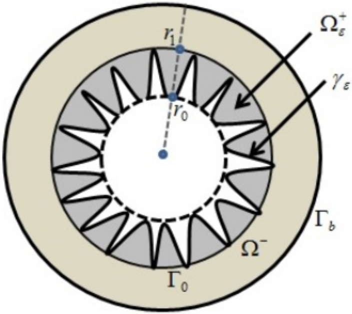
- R. Mahadevan, A. K. Nandakumaran, and R. Prakash, (Applied Mathematics & Optimization 2018)
- S.A., A K Nandakumaran, and Ravi Prakash, (Communications in Contemporary Mathematics 2019)
- S.A., A K Nandakumaran, and Abu Sufian, (Math. Meth. Appl. Sci. 2019)

Circular Oscillating Domain



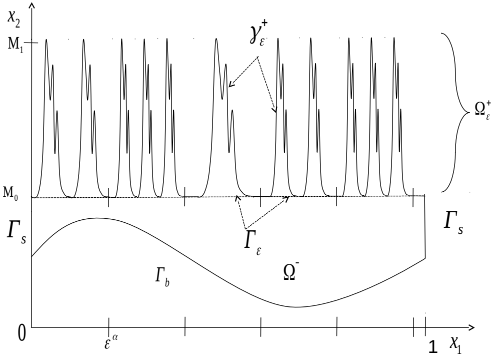
- S.A., A. K. Nandakumaran, and R. Prakash, (Calc. Var. Partial Differential Equations 2018)

Circular Oscillating Domain



- S.A., A. K. Nandakumaran, and R. Prakash, (Calc. Var. Partial Differential Equations 2018)
- S.A., Editha C. Jose, Ivy Carol B. Lomerio, and A. K. Nandakumaran, (Asymptotic Analysis 2019)

Locally periodic boundary-Meso scale



- S.A., A K Nandakumaran, and Ravi Prakash. (Annali di Matematica Pura ed Applicata 2019)

Unfolding Operator

The periodic unfolding at rate ε of a function $v : \mathbb{R}^2 \to \mathbb{R}$ along x_1 is

$$(T^\varepsilon v)(x, y) = v\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big] + \varepsilon y, x_2\Big),$$

where $[\cdot]$ denotes the integer part, where v is extended by zero when necessary.

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where $[\cdot]$ denotes the integer part, where v is extended by zero when necessary.

$$\Omega_+^\varepsilon = \Big\{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ \eta_-(x_1) < x_2 < \eta\Big(x_1, \frac{x_1}{\varepsilon}\Big)\Big\},$$

and

$$\Omega_u^\varepsilon \quad = \Big\{(x, y) \in \mathbb{R}^2 \times \mathbb{T} : 0 < x_1 < 1,\ \eta_-\Big(\varepsilon\Big[\tfrac{x_1}{\varepsilon}\Big] + \varepsilon y\Big) < x_2 < \eta\Big(\varepsilon\Big[\tfrac{x_1}{\varepsilon}\Big] + \varepsilon y, y\Big)\Big\}.$$

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There holds

$$T^\varepsilon \chi_{\Omega_+^\varepsilon} = \chi_{\Omega_u^\varepsilon} \to \chi_{\Omega_u} \quad \text{strongly in } L^p(\mathbb{R}^2 \times \mathbb{T}), \quad 1 \le p < \infty, \tag{6}$$

where

$$\begin{aligned}\Omega_u &= \{(x, y) \in \mathbb{R}^2 \times \mathbb{T} : 0 < x_1 < 1,\ \eta_-(x_1) < x_2 < \eta(x_1, y)\} \\ &= \{(x, y) \in \mathbb{R}^2 \times \mathbb{T} : x \in \Omega_+,\ y \in Y(x)\}.\end{aligned}$$

Main properties of Unfolding

Lemma (S.A., K. Pettersson)

Let $\Omega \supset \Omega_\varepsilon$. *Then*

1. *if* $v \in L^p(\Omega)$, $p > 1$,

$$\int_{\Omega_+^\varepsilon} v\,dx = \int_{\Omega_u} T^\varepsilon v\,dxdy + O(\varepsilon^{(p-1)/p}),$$

2. *if* $v \in L^1(\Omega)$,

$$\int_{\Omega_+^\varepsilon} v\,dx = \int_{\Omega_u} T^\varepsilon v\,dxdy + o(1),$$

as ε *tends to zero.*

Because $T^\varepsilon \chi_{\Omega_+^\varepsilon} = \chi_{\Omega_u^\varepsilon}$, the discrepancy can be computed as follows:

$$\int_{\Omega_+^\varepsilon} v\,dx - \int_{\Omega_u} T^\varepsilon v\,dxdy = \int_{\Omega\times\mathbb{T}} T^\varepsilon \chi_{\Omega_+^\varepsilon} T^\varepsilon v\,dxdy - \int_{\Omega_u} T^\varepsilon v\,dxdy$$
$$= \int_{\Omega\times\mathbb{T}} (\chi_{\Omega_u^\varepsilon} - \chi_{\Omega_u}) T^\varepsilon v\,dxdy.$$

The differences $\Omega_u^\varepsilon \setminus \Omega_u$ and $\Omega_u \setminus \Omega_u^\varepsilon$ are contained in some strip of measure $O(\varepsilon)$:

$$\{(x,y) \in \mathbb{R}^2 \times \mathbb{T} : \operatorname{dist}((x,y), \partial\Omega_u) < C\varepsilon\},$$

where C may be chosen independent of ε by the Lipschitz continuity of η and η_-,
Let $v \in L^p(\Omega)$, $p > 1$. Then by the Hölder inequality,

$$\Big| \int_{\Omega_+^\varepsilon} v\,dx - \int_{\Omega_u} T^\varepsilon v\,dxdy \Big| \le \|\chi_{\Omega_u^\varepsilon} - \chi_{\Omega_u}\|_{L^{p/(p-1)}(\Omega\times\mathbb{T})} \|v\|_{L^p(\Omega)} = O(\varepsilon^{(p-1)/p}),$$

which gives (i).
The estimate in (ii) follows from (i) and the density of $C^1(\overline{\Omega})$ in $L^1(\Omega)$.

Proof of the main Theorem:

The variational form (5) can be written as

$$\int_{\Omega_-} \nabla u^\varepsilon \cdot \nabla \psi \, dx + \int_{\Omega_+^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi \, dx = \int_{\Omega^\varepsilon} f \psi \, dx.$$

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After unfolding Ω_+^ε and using the properties of unfolding one arrives with $\psi \in H^1(\Omega^\varepsilon, \Gamma)$ at

$$\int_{\Omega_-} \nabla u^\varepsilon \cdot \nabla \psi \, dx + \int_{\Omega_u} T^\varepsilon \nabla u^\varepsilon \cdot T^\varepsilon \nabla \psi \, dxdy = \int_{\Omega^\varepsilon} f \psi \, dx + o(1). \tag{7}$$

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By choosing u^ε as a test function, and the Poincaré inequality, we get

$$\begin{aligned} \|u_-^\varepsilon\|_{L^2(\Omega_-)} &\leq C, \\ \|T^\varepsilon u^\varepsilon\|_{L^2(\Omega_u)} &\leq C, \\ \|T^\varepsilon \nabla u^\varepsilon\|_{L^2(\Omega_u)} &\leq C, \end{aligned}$$

where the constant $C > 0$ is independent of ε.

Proof Ctd...:

By weak compactness, there exist $u_-^0 \in H^1(\Omega_-, \Gamma)$, $u_+^0 \in L^2(\Omega_u)$, $p \in L^2(\Omega_u)$, and a subsequence of ε which we still denoted by ε, such that

$$u^\varepsilon \rightharpoonup u_-^0 \qquad \text{weakly in } H^1(\Omega_-, \Gamma), \tag{8}$$

$$T^\varepsilon u^\varepsilon \rightharpoonup u_+^0 \qquad \text{weakly in } L^2(\Omega_u), \tag{9}$$

$$T^\varepsilon \nabla u^\varepsilon \rightharpoonup \big(p, \frac{\partial u_+^0}{\partial x_2}\big) \qquad \text{weakly in } L^2(\Omega_u). \tag{10}$$

Proof Ctd...:

By weak compactness, there exist $u_-^0 \in H^1(\Omega_-, \Gamma)$, $u_+^0 \in L^2(\Omega_u)$, $p \in L^2(\Omega_u)$, and a subsequence of ε which we still denoted by ε, such that

$$u^\varepsilon \rightharpoonup u_-^0 \qquad \text{weakly in } H^1(\Omega_-, \Gamma), \tag{8}$$

$$T^\varepsilon u^\varepsilon \rightharpoonup u_+^0 \qquad \text{weakly in } L^2(\Omega_u), \tag{9}$$

$$T^\varepsilon \nabla u^\varepsilon \rightharpoonup \left(p, \frac{\partial u_+^0}{\partial x_2}\right) \qquad \text{weakly in } L^2(\Omega_u). \tag{10}$$

Note that

$$\frac{\partial}{\partial x_2} T^\varepsilon u^\varepsilon = T^\varepsilon \frac{\partial u^\varepsilon}{\partial x_2}, \qquad \frac{\partial}{\partial y} T^\varepsilon u^\varepsilon = \varepsilon T^\varepsilon \frac{\partial u^\varepsilon}{\partial x_1}.$$

Identification of p:

Choose x_k^ε such that the graphs of $\eta(x_1, x_1/\varepsilon)$ and $\eta_-(x_1)$ are close:

$$x_k^\varepsilon \in \underset{x_1 \in \varepsilon[k,k+1]}{\arg\min}\, \eta\Big(x_1, \frac{x_1}{\varepsilon}\Big), \quad k = 0, \dots, \frac{1}{\varepsilon} - 1. \tag{11}$$

Let $\phi \in C_0^\infty(\Omega_+)$. Then

$$\varphi^\varepsilon(x) = (x_1 - x_k^\varepsilon)\phi(x), \quad \text{if } x_1 \in [x_k^\varepsilon, x_{k+1}^\varepsilon), \tag{12}$$

belongs to $C_0^\infty(\Omega_+^\varepsilon)$ for all small enough ε.

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belongs to $C_0^\infty(\Omega_+^\varepsilon)$ for all small enough ε. With φ^ε given by (11), (12) as test functions in the equation (7) for u^ε, and using that

$$T^\varepsilon \varphi^\varepsilon \to 0,$$
$$T^\varepsilon \nabla \varphi^\varepsilon \to (\phi, 0),$$

strongly in $L^2(\Omega_u)$ as ε tends to zero because $|x_1 - x_k^\varepsilon| \le \varepsilon$, one obtains in the limit

$$\int_{\Omega_u} p\phi \, dxdy = \int_{\Omega_+} \int_{Y(x)} p \, dy \, \phi \, dx = 0, \quad \phi \in C_0^\infty(\Omega_+).$$

which shows that $\int_{Y(x)} p(x,y)\, dy = 0$ a.e in Ω_+.

$$
\begin{aligned}
\int_{\Omega_+} \widetilde{u_+^\varepsilon} \phi \, dx = \int_{\Omega_+} \chi_{\Omega_+^\varepsilon} u_+^\varepsilon \phi \, dx &= \int_{\Omega_+^\varepsilon} u_+^\varepsilon \phi \, dx \\
&= \int_{\Omega_u} T^\varepsilon u_+^\varepsilon T^\varepsilon \phi \, dxdy + o(1), \\
&= \int_{\Omega_u} u_+^0 \phi \, dxdy, \\
&= \int_{\Omega_+} \int_{Y(x)} u_+^0 \phi \, dxdy, \\
&= \int_{\Omega_+} h(x) u_+^0 \phi \, dx
\end{aligned}
$$

This shows that $\widetilde{u^\varepsilon} \rightharpoonup hu^0$ weakly in $L^2(\Omega)$.

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Similarly we can show that $\widetilde{\nabla u^\varepsilon} \rightharpoonup (\int_{Y(x)} p dy, \partial_{x_2} u^0)$ weakly in $L^2(\Omega)$.

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This shows that $\widetilde{u^\varepsilon} \rightharpoonup hu^0$ weakly in $L^2(\Omega)$.

Similarly we can show that $\widetilde{\nabla u^\varepsilon} \rightharpoonup (\int_{Y(x)} pdy, \partial_{x_2} u^0)$ weakly in $L^2(\Omega)$. In other words $\widetilde{\nabla u^\varepsilon} \rightharpoonup (0, \partial_{x_2} u^0)$ weakly in $L^2(\Omega)$.

Recall the variational form

$$\int_{\Omega_-} \nabla u^\varepsilon \cdot \nabla \psi \, dx + \int_{\Omega_u} T^\varepsilon \nabla u^\varepsilon \cdot T^\varepsilon \nabla \psi \, dxdy = \int_{\Omega^\varepsilon} f \psi \, dx + o(1). \qquad (13)$$

As $\varepsilon \to 0$,

$$\int_{\Omega_-} \nabla u^0 \cdot \nabla \psi \, dx + \int_{\Omega_+} (0, h\partial_{x_2} u^0) \cdot \nabla \psi \, dxdy = \int_{\Omega} hf\psi \, dx. \qquad (14)$$

In other words

$$\int_{\Omega} A^0 \nabla u^0 \cdot \nabla \psi \, dx = \int_{\Omega} f\psi h \, dx, \qquad (15)$$

for all $\psi \in W(\Omega, \Gamma)$.

Justification

Lemma

Let $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ *be the solutions to* (1), *and let* $u^0 \in W(\Omega, \Gamma)$ *be the solution to* (2). *Then*

$$(i) \quad T^\varepsilon u^\varepsilon \to u^0 \quad \text{strongly in } L^2(\Omega_u),$$

$$(ii) \quad T^\varepsilon \nabla u^\varepsilon \to \big(0, \frac{\partial u^0}{\partial x_2}\big) \quad \text{strongly in } L^2(\Omega_u),$$

as ε *tends to zero.*

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as ε *tends to zero.*

Theorem

Let $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ *be the solutions to* (1)*, and let* $u^0 \in W(\Omega, \Gamma)$ *be the solution to* (2)*. Then*

$$(i) \quad \|u^\varepsilon - u^0\|_{L^2(\Omega^\varepsilon,\, h)} \to 0,$$
$$(ii) \quad \|\nabla u^\varepsilon - h^{-1} A^0 \nabla u^0\|_{L^2(\Omega^\varepsilon,\, h)} \to 0,$$

as ε *tends to zero.*

By the weak convergence of u^ε, $T^\varepsilon u^\varepsilon$, $T^\varepsilon \nabla u^\varepsilon$ (8)–(10), the property that sum of lim inf is less than or equal to lim inf of sum, using that u^ε, u^0 solve (7), (2),

$$
\begin{aligned}
&\int_{\Omega_u} p^2 \, dxdy + \int_\Omega A^0 \nabla u^0 \cdot \nabla u^0 \, dxdy \\
&\quad \le \liminf_{\varepsilon \to 0} \int_{\Omega_u} |T^\varepsilon \nabla u^\varepsilon|^2 \, dxdy + \liminf_{\varepsilon \to 0} \int_{\Omega_-} |\nabla u^\varepsilon|^2 \, dxdy \\
&\quad \le \liminf_{\varepsilon \to 0} \Big(\int_{\Omega_u} |T^\varepsilon \nabla u^\varepsilon|^2 \, dxdy + \int_{\Omega_-} |\nabla u^\varepsilon|^2 \, dx \Big) \\
&\quad = \liminf_{\varepsilon \to 0} \Big(\int_{\Omega_u} T^\varepsilon f T^\varepsilon u^\varepsilon \, dxdy + \int_{\Omega_-} f u^\varepsilon \, dx \Big) \\
&\quad = \int_\Omega f u^0 h \, dx \\
&\quad = \int_\Omega A^0 \nabla u^0 \cdot \nabla u^0 \, dx.
\end{aligned}
$$

It follows that each weak convergence in (8)–(10) is strong, and $p = 0$. The convergence in (i) and (ii) then follow from Lemma 2(ii).

GRACIAS *por su atencion*

QUESTIONS ???