A PDE approach to scaling limits of random interface models on $\mathbb{Z}^d$

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Random interface

A random interface is a probability measure on the space of functions $\Omega = \mathbb{R}^{\mathbb{Z}^d}$.

To each interface we associate an energy, which is given by an Hamiltonian $H(\varphi)$.

Given $\Lambda \Subset \mathbb{Z}^d$ *we define*

$$\mathbf{P}_\Lambda(d\varphi) = \frac{1}{Z_\Lambda} e^{-H(\varphi)} \prod_{x \in \Lambda} d\varphi_x \prod_{x \in \mathbb{Z}^d \setminus \Lambda} \delta_{\psi_x}(d\varphi_x).$$

$\varphi_x = \psi_x$ *when* $x \notin \Lambda$. *We shall assume* $\psi_x = 0$.

Some examples: DGFF

Discrete Gaussian free field arises out of discrete Dirichlet energy:
Favours flat configurations

$$H(\varphi) = \frac{1}{4d} \sum_{x \in \mathbb{Z}^d} |\nabla \varphi_x|^2 = \frac{1}{4d} \sum_{x \sim y} (\varphi_x - \varphi_y)^2$$

Here gradient: $\nabla \varphi_x = (\varphi_x - \varphi_{x+e_i})_{i=1}^d$.

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Here gradient: $\nabla \varphi_x = (\varphi_x - \varphi_{x+e_i})_{i=1}^d$.
Alternative form:

$$H(\varphi) = \sum_{x \in \mathbb{Z}^d} \varphi_x (-\Delta \varphi_x) = \langle \varphi, (-\Delta)\varphi \rangle_{\ell^2(\mathbb{Z}^d)}$$

where

$$\Delta \varphi_x = \frac{1}{2d} \sum_{y \sim x} (\varphi_y - \varphi_x).$$

Some examples: Membrane model

Used in modelling semiflexible membranes/polymers and takes curvatures into account. Favours flat hyper-surfaces so penalizes bending.

$$H(\varphi) = \frac{1}{2} \sum_{x \in \mathbb{Z}^d} |\Delta \varphi_x|^2 = \left\langle \varphi, \Delta^2 \varphi \right\rangle_{\ell^2(Z^d)}$$

Some examples: Mixed model

$$
\begin{aligned}
H(\varphi) &= \kappa_1 \sum_{x\in\mathbb{Z}^d} |\nabla\varphi_x|^2 + \kappa_2 \sum_{x\in\mathbb{Z}^d} |\Delta\varphi_x|^2 \\
&= \left\langle \varphi, \left(\kappa_1(-\Delta) + \kappa_2\Delta^2\right)\varphi \right\rangle_{\ell^2(\mathbb{Z}^d)}
\end{aligned}
$$

κ_1 is called lateral tension and κ_2 is called the bending rigidity.

We shall also consider the case where κ_1 and κ_2 depend on size of Λ.

Elliptic operator

In all the examples,

$$H(\varphi) = \langle \varphi, \mathcal{L}_d \varphi \rangle_{\ell^2(\mathbb{Z}^d)} .$$

where $\mathcal{L}_d : \Omega \to \Omega$ *is a nice operator of form*

$$\mathcal{L}_d \varphi_x = \sum_{\alpha} c_\alpha \varphi_{x+\alpha}.$$

Continuum elliptic operator:

$$\mathcal{L} f = \sum_{|\beta|,|\gamma| \leq m} a_{\beta\gamma} D^{\beta+\gamma} f, \quad a_\beta \in \mathbb{R}$$

where for $\beta = (\beta_1, \cdots, \beta_n)$

$$D^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n} .$$

Green's function

Under conditions of positive definiteness

- $\varphi_x = 0$, *for all* $x \in \mathbb{Z}^d \setminus \Lambda$, $\mathbf{P}_\Lambda$*-a. s.*

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- $\varphi_x = 0$, *for all* $x \in \mathbb{Z}^d \setminus \Lambda$, $\mathbf{P}_\Lambda$*-a. s.*

- $(\varphi_x)_{x\in\Lambda} \sim \mathcal{N}(\mathbf{0},\, G_\Lambda)$ *with*

$$\mathbf{E}_\Lambda[\varphi_x \varphi_y] = G_\Lambda(x, y), \quad x, y \in \Lambda.$$

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- *For all* $x \in \Lambda$

$$\mathcal{L}_d G_\Lambda(x,y) = \delta_x(y),\ \ y \in \Lambda$$
$$G_\Lambda(x,y) = 0, \quad y \notin \Lambda.$$

Lack of RW representations

DGFF

If P_x is the law of a SRW $(S_n)_{n\geq 0}$ started at $x \in \mathbb{Z}^d$, then

$$G_\Lambda(x,y) := \mathbf{E}_x \left[\sum_{n\geq 0} \mathbb{1}_{(S_n = y, n < H_{\Lambda^c})} \right]$$

where $H_{\Lambda^c} := \inf\{n \geq 0 : S_n \in \Lambda^c\}$.

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In case of Membrane or Mixed there is No random walk representation known.

Infinite Volume measure for $\mathcal{L}_d = \left(\kappa_1(-\Delta) + \kappa_2\Delta^2\right)$

Does there exists a probability measure P on $\mathbb{R}^{\mathbb{Z}^d}$ such that $P_\Lambda \Rightarrow P$ as $\Lambda \uparrow \mathbb{Z}^d$?

κ_1	κ_2	*dim*	*Green's function*
1	0	$d \geq 3$	$G(x,y) = \Gamma_0(x,y)$
0	1	$d \geq 5$	$G(x,y) = \sum_{z\in\mathbb{Z}^d} \Gamma_0(x,z)\Gamma_0(z,y)$
$\kappa > 0$	1	$d \geq 3$	$G(x,y) = \sum_{z\in\mathbb{Z}^d} \Gamma_\kappa(x,z)\Gamma_0(z,y)$

$$\Gamma_\kappa(x,y) = \sum_{m=0}^{\infty} \frac{1}{(1+\kappa)^{m+1}} P_x(S_m = y)$$

Scaling limit in d=1

In all three cases (DGFF+MM+ Mixed) the limit turns out to have continuous paths. Let $\Lambda_N = [1, N-1] \cap \mathbb{Z}$. *Consider the linear interpolation of the interface model.*
For $0 \leq t \leq 1$,

$$\widehat{\varphi}_N(t) = \varphi_{\lfloor Nt \rfloor} + (Nt - \lfloor Nt \rfloor)\left(\varphi_{\lfloor Nt \rfloor + 1} - \varphi_{\lfloor Nt \rfloor}\right).$$

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For DGFF & Mixed model ($\kappa_1 = 1$, $\kappa_2 = 1$)

Theorem (d=1, Cipriani, Dan, H. (2018))

In $C[0,1]$,

$$(N^{-1/2}\widehat{\varphi}_N(t))_{t\in[0,1]} \Rightarrow (B_t^\circ)_{t\in[0,1]}$$

where $(B_t^\circ)_{t\in[0,1]}$ *is the Brownian Bridge.*

Scaling limit in d=1: Membrane

Let $X_i \overset{i.i.d}{\sim} N(0,1)$.

$$Y_n = X_1 + \cdots + X_n \text{ (Random walk)}$$

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$$\{\varphi_i\}_{1 \le i \le N} \overset{d}{=} (Z_i)_{1 \le i \le N} \text{ conditionally on } (Y_N, Z_N) = (0,0).$$

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Let $(B_t)_{t \in [0,1]}$ *be the standard Brownian motion and* $I_t = \int_0^t B_s ds$.

$$(\widehat{B}_t, \widehat{I}_t)_{t \in [0,1]} := \{(B_t, I_t)_{t \in [0,1]} \text{ Conditioned on } (B_1, I_1) = (0,0)\}.$$

Scaling limit in d=1: Membrane (contd.)

For $0 \leq t \leq 1$,

$$\widehat{\varphi}_N(t) = \varphi_{\lfloor Nt \rfloor} + (Nt - \lfloor Nt \rfloor)\left(\varphi_{\lfloor Nt \rfloor + 1} - \varphi_{\lfloor Nt \rfloor}\right).$$

Theorem (Caravenna and Deuschel (2009))

On $C[0,1]$,

$$(N^{-3/2}\widehat{\varphi}_N(t))_{t\in[0,1]} \Rightarrow (\widehat{I}_t)_{t\in[0,1]}.$$

Scaling limit in $d = 2, 3$: Membrane

Membrane Model is still in subcritical regime.
In these cases it turns out the limiting process has still continuous paths.
Let $\Lambda_N = (-N, N) \cap \mathbb{Z}^d$.

$$\Psi_N(t) = N^{\frac{d-4}{2}} \varphi_{Nt} \quad t \in \frac{1}{N} \mathbb{Z}^d \, .$$

Interpolate continuously on $[-1, 1]^d$.

Theorem (Cipriani, Dan, H. (2018))

Suppose $d = 2$ or 3. In $C([-1,1]^d)$

$$\Psi_N \Rightarrow \Psi$$

where $\Psi = (\Psi_t)_{t\in[-1,1]^d}$ is a Gaussian process with continuous paths and

$$\mathbf{E}[\Psi_t \Psi_s] = G_D(t,s)$$

and G_D is the Green's function on $D = [-1,1]^d$ satisfying the following Dirichlet problem:

$$\begin{aligned} \Delta_c^2 G_D(x,y) &= \delta_x(y), \ \ y \in D \\ G_D(x,y) &= 0, \quad y \in \partial D \\ \mathbf{D} G_D(x,y) &= 0, \quad y \in \partial D \end{aligned}$$

Consequences

- *A consequence of the proof is that the process Ψ is almost surely Hölder continuous with exponent η, for every $\eta \in (0, 1)$ resp. $\eta \in (0, 1/2)$ in $d = 2$ resp. $d = 3$.*

- *One can get the extremes in $d = 2, 3$,*

$$N^{\frac{d-4}{2}} \max_{x \in (-N,N)^d} \varphi_x \xrightarrow{d} \sup_{x \in [-1,1]} \Psi_x$$

- *The extremes of Membrane in $\mathbb{Z}^d$ for $d \geq 5$ was resolved in Chiarini, Cipriani, Hazra (2017). Recentered tightness in $d = 4$ was derived in the Thesis of Roy (2016).*

- *OPEN: Maxima and point process behaviour of membrane in $d = 4$ should correspond to "log-correlated" models.*

Brief idea of the proof

Finite dimensional convergence follows from Green's function convergence.

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Checking Kolmogorov criteria for tightness :

$$\mathbf{E}\left[|\Psi_N(t) - \Psi_N(s)|^2\right] \leq C\|t-s\|^{1+b}$$

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If t and s are neighbours then we use some gradient bounds by Müller and Schweiger (2017).

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$$\mathbf{E}\left[|\Psi_N(t) - \Psi_N(t + e_1)|^2\right]$$

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$$\mathbf{E}\left[|\Psi_N(t) - \Psi_N(t+e_1)|^2\right]$$
$$= G_N(t,t) + G_N(t+e_1, t+e_1) - 2G_N(t, t+e_1)$$

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\begin{aligned}
&\mathbf{E}\left[|\Psi_N(t) - \Psi_N(t+e_1)|^2\right] \\
&= G_N(t,t) + G_N(t+e_1, t+e_1) - 2G_N(t, t+e_1) \\
&= \nabla^1_{-e_1} G_N(t+e_1, t) - \nabla^1_{-e_1} G_N(t+e_1, t+e_1)
\end{aligned}
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$$= \nabla^1_{-e_1} G_N(t+e_1, t) - \nabla^1_{-e_1} G_N(t+e_1, t+e_1)$$
$$= -\nabla^2_{-e_1} \nabla^1_{-e_1} G_N(t+e_1, t)$$

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&= \nabla^1_{-e_1} G_N(t+e_1, t) - \nabla^1_{-e_1} G_N(t+e_1, t+e_1) \\
&= -\nabla^2_{-e_1}\nabla^1_{-e_1} G_N(t+e_1, t) \leq \begin{cases} C\log N & \text{if } d=2 \\ C & \text{if } d=3 \end{cases}.
\end{aligned}
$$

Scaling limit in critical or super critical dimension

We specialise to the case when $(\varphi_x)_{x\in\mathbb{Z}^d}$ *is the Membrane model.*
Let D be a bounded domain with smooth boundary.

$$D_h = \overline{D} \cap h\,\mathbb{Z}^d \ \text{ with } h = \frac{1}{N}.$$

Consider

$$R_h = \{x \in D_h : \ N_2(x) \subset D_h\},$$

$N_2(x)$= *neighbours at distance 2 from* x.
Consider

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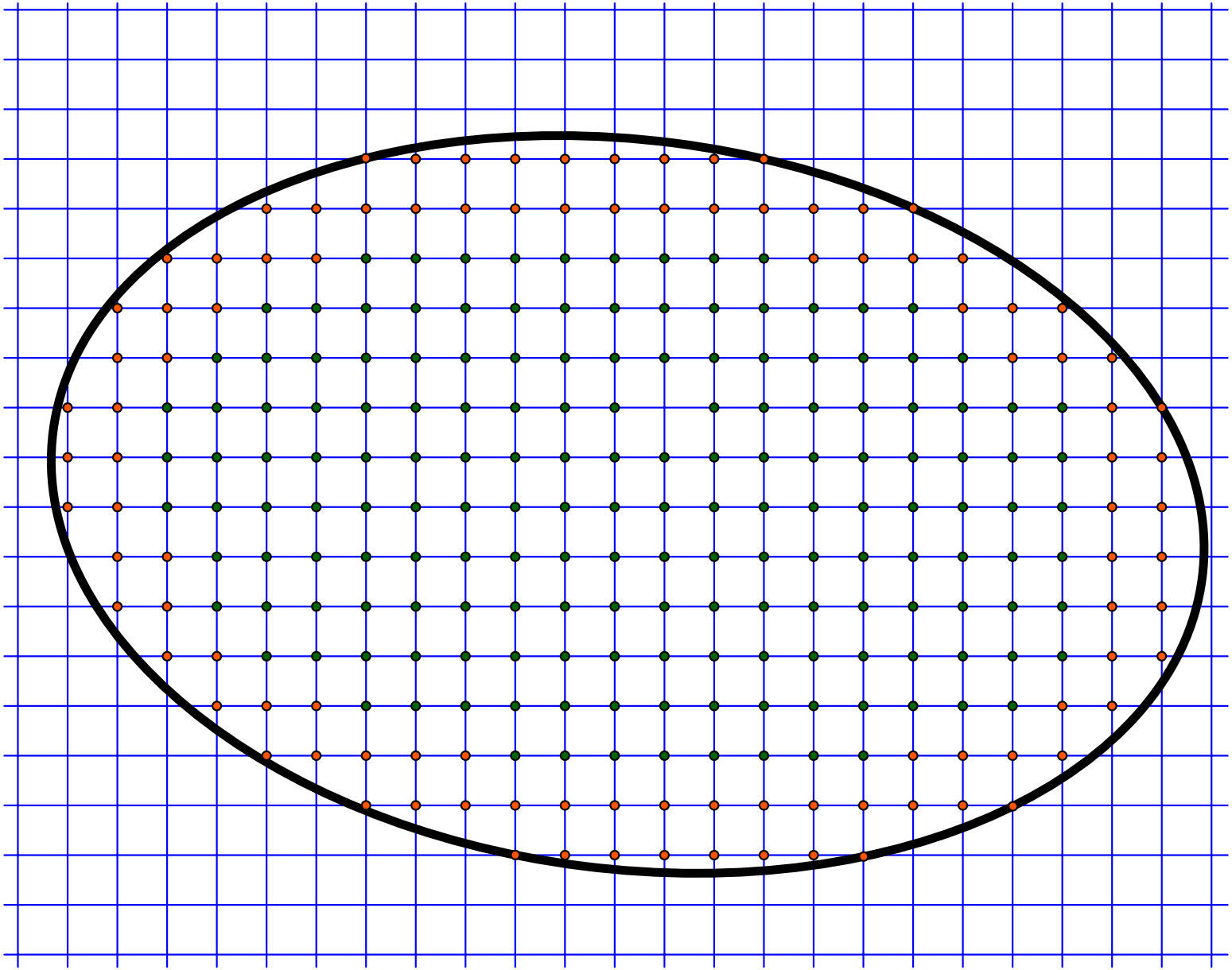

Let (φ_x) be the MM with zero boundary conditions on Λ_N.
Consider $f \in C_c^\infty(D)$

$$(\Psi_h, f) := h^{\frac{d+4}{2}} \sum_{x \in R_h} \varphi_{x/h} f(x).$$

Where does Ψ_h converge to?

Facts

Fact

There exist eigenfunctions $u_1, u_2, \cdots$ *of* Δ_c^m *with corresponding eigenvalues* $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ *such that*

1. $\{u_j\}_{j \in \mathbb{N}}$ *is an orthonormal basis for* $L^2(D)$

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1. $\{u_j\}_{j \in \mathbb{N}}$ *is an orthonormal basis for* $L^2(D)$
2. *For each* $j \in \mathbb{N}$ *one has* $u_j \in C^\infty(D)$. *(elliptic regularity)*
3. $\lambda_j \sim c j^{2m/d}$ *(Weyl's asymptotics, Beals (1967)).*

Let $f \in C_c^\infty(D)$, define

$$\|f\|_s^2 = \sum_{j \geq 1} \frac{1}{\lambda_j^{s/2}} \langle f, u_j \rangle_{L^2}^2 .$$

$$\mathcal{H}_0^s = \overline{C_c^\infty(D)}^{\|\cdot\|_s}.$$

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<u>*Random series:*</u> *Let λ_j be the eigenvalues of Δ_c^m and u_j be the corresponding eigenfunctions. Define*

$$\Psi_D = \sum_{j\geq 1} \frac{X_j u_j}{\sqrt{\lambda_j}}, \quad X_j \overset{iid}{\sim} N(0,1).$$

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Theorem (Cipriani, Dan, H. (2018))

For $m = 1, 2$ and $s > \frac{d-2m}{2}$, Ψ_D exists in

$$\mathcal{H}_0^{-s} := \mathcal{H}_0^s(D)^*.$$

Scaling limit: Main result

Consider

$$\Lambda_N = NR_h \subset \mathbb{Z}^d \text{ the blow up of } R_h$$

For $f \in \mathcal{H}_0^s(D)$ *define.*

$$(\Psi_h, f) := h^{\frac{d+4}{2}} \sum_{x \in R_h} \varphi_{x/h} f(x).$$

Theorem (Cipriani, Dan, H. (2018))

Suppose $d \geq 4$*, then* Ψ_h *converges in distribution to* Ψ_D *as* $h \to 0$ *in the strong topology of* $\mathcal{H}^{-s}(D)$*,* $s > s_d$ *where*

$$s_d := \frac{d}{2} + 2\left(\left\lceil \frac{1}{4}\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right)\right\rceil + \left\lceil \frac{1}{4}\left(\left\lfloor \frac{d}{2} \right\rfloor + 6\right)\right\rceil - 1\right).$$

Extensions to $\mathcal{L}_d = \left(\kappa_1(-\Delta) + \kappa_2\Delta^2\right)$

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$\mathcal{L}_d = \left(\kappa_1(-\Delta) + \kappa_2\Delta^2\right)$ [Cipriani, Dan, H. (2018, 2019+)]

κ_1	κ_2	*scaling*(α)	$\mathcal{H}^{-s}$, $s > s_d$	*Limit*	*dim*

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1	0	$h^{-\frac{d+2}{2}}$	$\frac{3}{2}$	*GFF*	$d \geq 2$

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1	$\kappa_2 \ll N^{\frac{1}{2}}$	$h^{-\frac{d+2}{2}}$	$\frac{d}{2} + \lfloor \frac{d}{2} \rfloor + \frac{3}{2}$	*GFF*	$d \geq 2$
1	$\kappa_2 \sim N^2$	$h^{-\frac{d+2}{2}}$	s_d^{MM}	$(\Delta + \Delta^2)$	$d \geq 2$

$$\delta = \frac{\log \kappa}{\log N} + d - 4.$$

Idea of proof (Membrane Case)

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- $\Delta_h f(x) := \frac{1}{h^2} \sum_{i=1}^d (f(x + he_i) + f(x - he_i) - 2f(x))$
- *For all* $x \in R_h := \frac{1}{N}\Lambda_N$

$$\Delta_h^2 G_h(x, y) = \frac{4d^2}{h^4}\delta_x(y), \;\; y \in R_h$$
$$G_h(x, y) = 0 \;\; y \notin R_h.$$

Idea of proof

$$(\Psi_h, f) := \sum_{x \in R_h} h^{\frac{d+4}{2}} \varphi_{x/h} f(x).$$

$$var((\Psi_h, f)) = \sum_{x \in R_h} h^d \underbrace{\sum_{y \in R_h} h^4 G_h(x,y) f(y)}_{H_h(x)} f(x)$$

$$= \sum_{x \in R_h} h^d H_h(x) f(x)$$

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- *Discrete Dirichlet problem:*

$$\Delta_h^2 H_h(x) = f(x), \quad x \in R_h$$
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- *Using an extension of result by V. Thomée(1964)*

$$\|e_h\|_{L^2(R_h)} \leq C h^{\frac{1}{2}}.$$

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- $$var[(\Psi_D, f)] = \int_D \int_D G_D(x,y) f(x) f(y).$$

$\|e_h\|_{L^2(R_h)} \leq Ch^{1/2}$

- *The operator* Δ_h^2 *is consistent with the operator* Δ_c^2*:* $u \in C^5(W)$ *then*

$$\Delta_h^2 u(x) = \Delta_c^2 u(x) + h^{-4}\mathcal{R}_5(x)$$

where $|\mathcal{R}_5(x)| \leq C M_5 h^5$.

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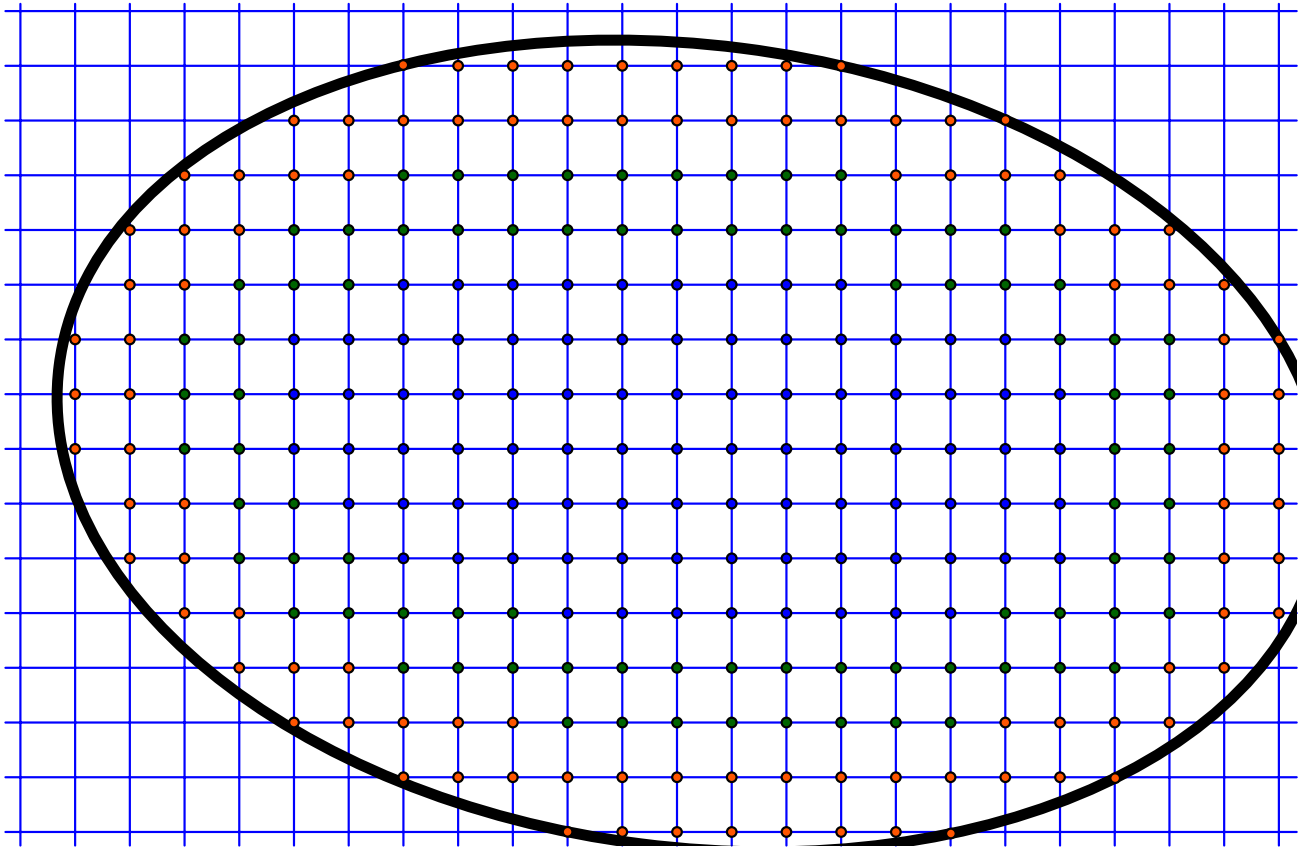
- *There are constants* $C > 0$ *independent of* f *and* h *such that*

$$\|f\|_{L^2(R_h)} \leq C\|f\|_{h,2} := \left(\sum_{|\beta|\leq 2} \|D^\beta f\|^2_{L^2(R_h)}\right)^{1/2}$$

  for any grid function f *vanishing outside* R_h.

Splitting of domain to define the Truncated operator



Truncated operator

$$\Delta^2_{h,2} f(x) = \begin{cases} \Delta^2_h f(x) & x \in R^*_h \\ h^2 \Delta^2_h f(x) & x \in B^*_h \\ 0 & x \notin R_h. \end{cases}$$

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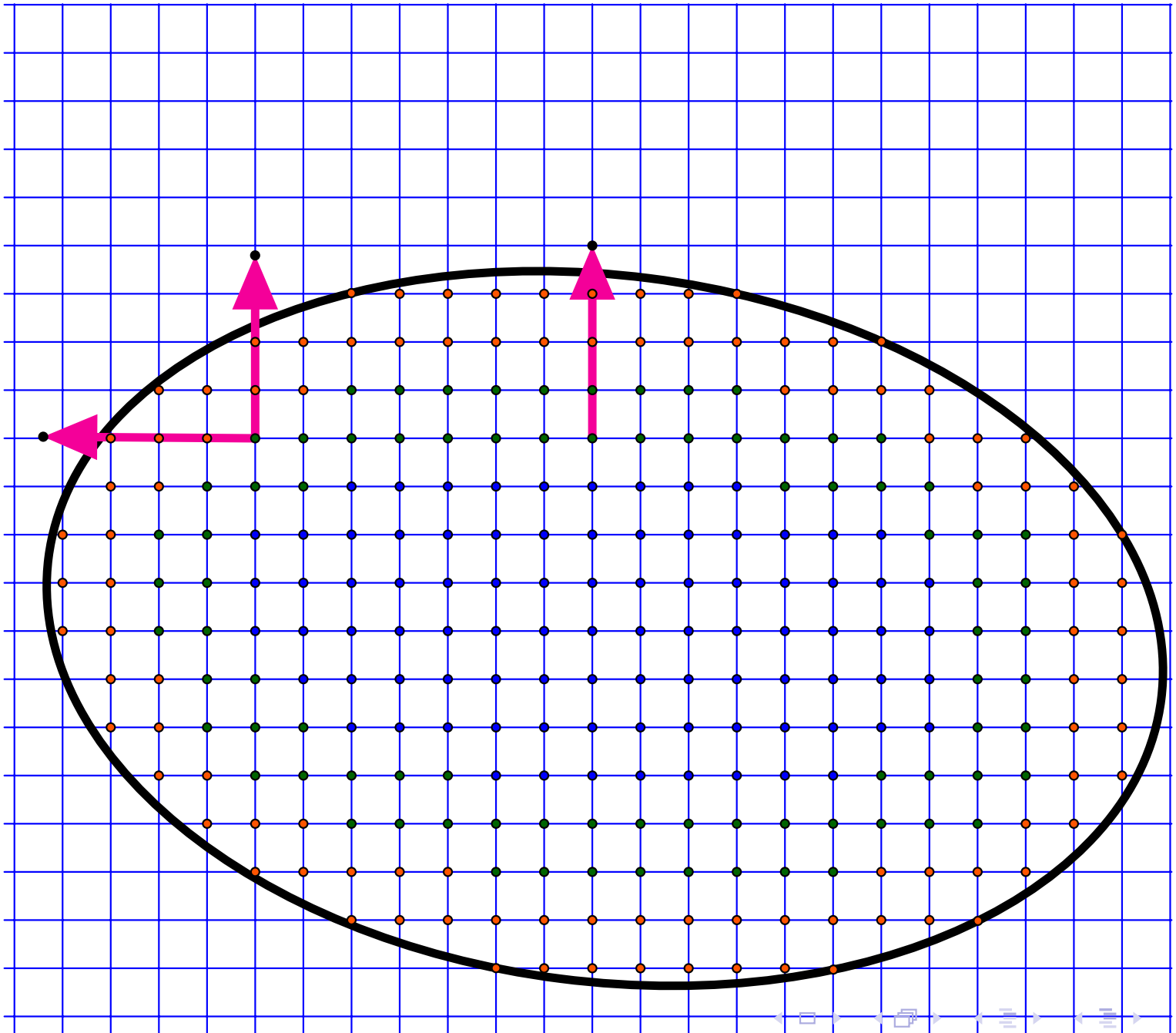
There exists a constant $C > 0$ such that for all grid functions f vanishing outside R_h

$$\|f\|_{h,2} \leq C \|\Delta_{h,2} f\|_{L^2(R_h)},$$

where C is independent of h as well.

Extensions to boundary



Works for domains with UEBC

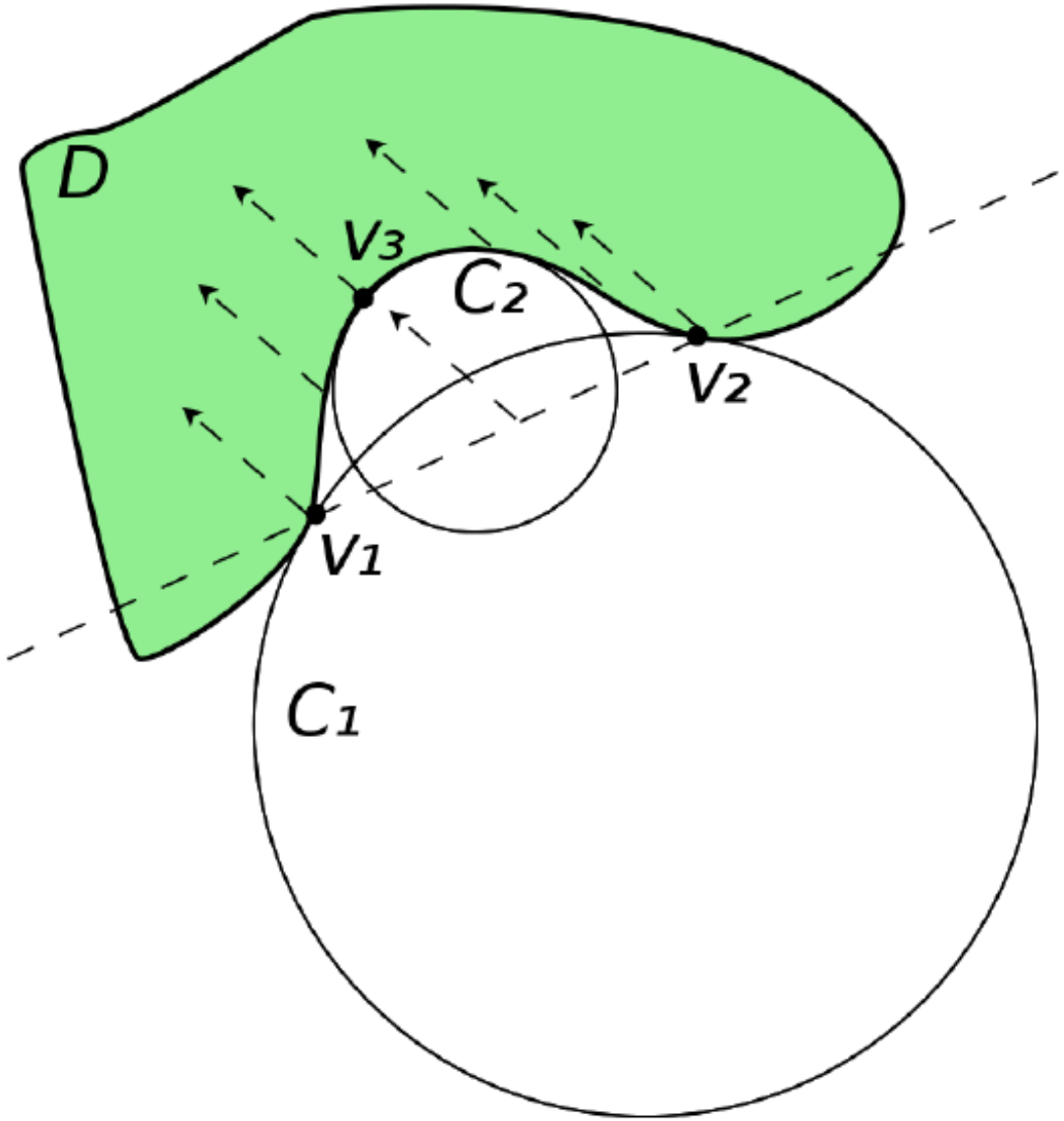


Image: Bramson et al. (2012)

Final Remarks

- *The method for getting the error in Membrane needs a change in mixed model. For κ_2 depending on N the inequalities come with various factors of N.*
- *Question: The level lines of mixed model converge to SLE_4?*
- *The case of $\kappa_2 \ll N^{1/2}$ is expected to be $\kappa_2 \ll N^2$.*
- *The Green's function asymptotics is not known for Membrane model in $d = 4$ and also*

$$G_{\mathbb{Z}^4 \setminus A}(0,0) < \infty.$$

  where A is a finite set ?
- *On-going works: Pinned membrane in $d = 4$, scaling limit of integer GFF.*