The Slow Bond Model with Small Perturbations

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Joint work with
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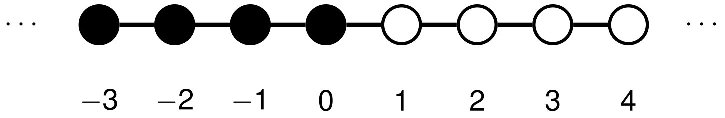
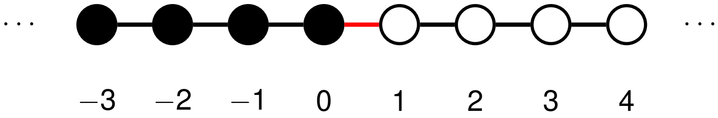
TASEP with Step Initial Condition



TASEP with step initial condition

- At time 0 there is one particle at every site of $\mathbb{Z}_-$ and the sites of $\mathbb{Z}_+$ are empty.

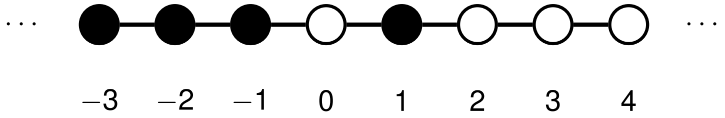
TASEP with Step Initial Condition



TASEP with step initial condition

- Each edge rings at rate 1, and the particle at the left of the edge moves one step to the right.

TASEP with Step Initial Condition



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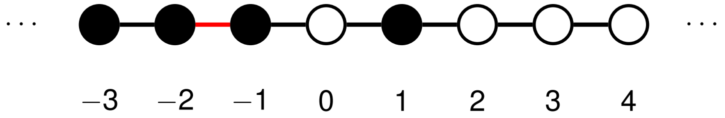
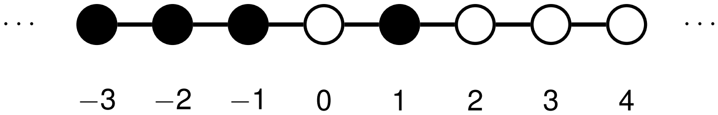


TASEP with step initial condition

• Except when the site at the right is occupied in which case nothing happens.

TASEP with Step Initial Condition



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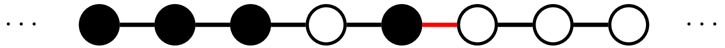

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

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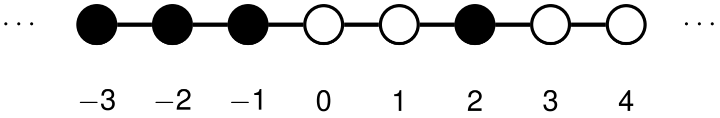

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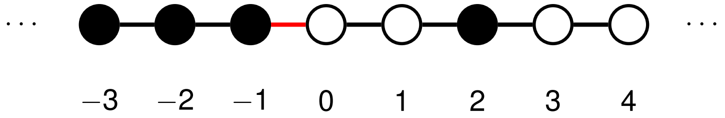

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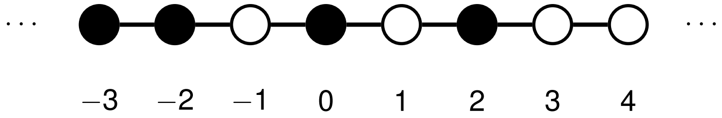

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TASEP and Last Passage Percolation

Variable of interest

- $T_n :=$ time till the n-th particle moves out of site 0.

- T_n can be represented as a passage time in an oriented last Passage percolation model with exponential passage times.

Connection with directed last passage percolation

- $S_n :=$ last passage time from $(1,1)$ to (n,n).
- γ : an oriented path from $(1,1)$ to (n,n).
- $S_n = \max_\gamma \sum_{i=1}^n X_{\gamma(i)}$.

Couple with TASEP so that X_{ij} is the waiting time for the i-th particle jumping out of site $j-i$.

$$T_n \overset{d}{=} S_n.$$

	$\cdots$		X_{44}
$\cdots$	X_{23}	X_{33}	X_{43}
X_{21}	X_{22}	X_{32}	X_{24}
X_{11}	X_{12}	X_{13}	X_{14}

$X_{ij} \sim$ i.i.d. $\mathrm{Exp}(1)$

Asymptotics of T_n

Theorem (Rost(1981))

As $n \to \infty$,

$$\frac{1}{n}\mathbb{E}[T_n] \to 4.$$

• This corresponds to the current of $\frac{1}{4}$ in the system, which is the maximum possible value of stationary current.

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Theorem (Johansson(2000))

As $n \to \infty$,

$$\frac{T_n - 4n}{2^{4/3} n^{1/3}} \xrightarrow{d} F_{TW},$$

where F_{TW} denotes the Tracy-Widom distribution.

Transversal Fluctuations

- Let Γ^n be the maximal path from $(1,1)$ to (n,n).

Define the transversal fluctuation for Γ^n to be

$$F_n = \sup_{x\in[0,n]} |\Gamma^n_x - x|.$$

Theorem (Johansson(2000))

F_n is $O(n^{2/3+o(1)})$ with high probability.

Introducing Local Defects

- Introduce a single slow bond.
- Bond between sites 0 and 1 rings at rate $1-\varepsilon$, all other bonds ring at rate 1.
- In the DLPP representation, diagonal entries are changed to Exponentials with smaller rate.
- Identity: $\mathrm{Exp}(1-\varepsilon) \overset{\mathrm{d}}{=} \mathrm{Exp}(1) + \mathrm{Ber}(\varepsilon) \cdot \mathrm{Exp}(1-\varepsilon)$

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X_{ij} independent, $\sim \mathrm{Exp}(1)$ for $i \neq j$, $X_{ii} \sim \mathrm{Exp}(1-\varepsilon)$

The Slow Bond Problem

- $T_n^\varepsilon :=$ time till the n-th particle moves out of site 0.

Question

Is the law of large numbers for T_n^ε different from that of T_n? i.e.,

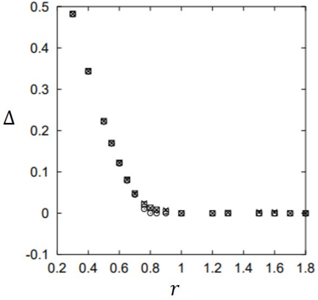
$$\kappa(1-\varepsilon) := \lim_{n\to\infty} \frac{\mathbb{E}T_n^\varepsilon}{n} \overset{??}{>} 4.$$

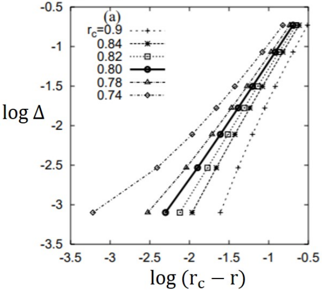
- Easy for ε sufficiently large.
- An affirmative answer for all $\varepsilon > 0$, implies that for any value of the slowness parameter, the maximal current in the system changes, i.e., the macroscopic behaviour is affected.

History

- Janowsky and Lebowitz (1992) introduced the slow bond model.
- Disagreement among physicists
  - Mean field prediction: $\varepsilon_c = 0$ (Janowsky and Lebowitz(1994)).
  - Ha, Timonem, den Nijs (2003): $\varepsilon_c \approx 0.20$. Based on numerical simulation and finite size scaling.

$r = 1 - \epsilon \qquad \kappa(r) = \frac{4}{1-\Delta^2}$





History

- Rigorous bounds
  - $\varepsilon_c < 0.49$ (Janowsky and Lebowitz(1994)).
  - Seppäläinen (2001):

$$\max\{4, \frac{(1-\varepsilon)^2 + 2(2-\varepsilon)}{2(1-\varepsilon)(2-\varepsilon)}\} \leq \kappa(1-\varepsilon) \leq 3 + \frac{1}{1-\varepsilon}.$$

Related work

- Covert and Rezakhanlou (1997): Hydrodynamic limits.
- Baik and Rains(2001): Longest Increasing Subsequence of Involutions with fixed points-non-trivial phase transition
- Georgiu, Kumar, Seppäläinen (2010).
- Beffara, Sidoravicius and Vares(2010): Polynuclear growth model with columnar defect-non-trivial phase transition.
- Costin, Lebowitz et al.(2012).

Our Results

Theorem (Basu, Sidoravicius, S. (2014))

For each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}T_n^{\varepsilon} > 4,$$

and so $\epsilon_c = 0$.

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The fluctuations of T_n^ε are order $n^{1/2}$ and Gaussian.

Proof Strategy

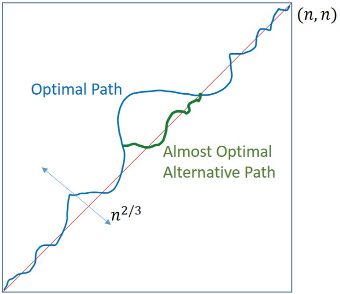
Observation

Fix $\epsilon > 0$. By superadditivity, it suffices to prove that for some n,

$$\mathbb{E}[T_n^\epsilon] > 4n.$$

- From Tracy-Widom fluctuations, $\mathbb{E}[T_n] = 4n - O(n^{1/3})$.
- It is enough to obtain an expected improvement of $Cn^{1/3}$ for some large constant C.
- Transversal fluctuations are order $n^{2/3}$ so the expected time on the diagonal is of order $n^{1/3}$. Thus we get an $\epsilon n^{1/3}$ improvement.

Additional for Improvements



- If the path deviates from the diagonal for a long time, then we try to get another $O(\epsilon n^{1/3})$ improvement by taking a path almost as long as the longest path.

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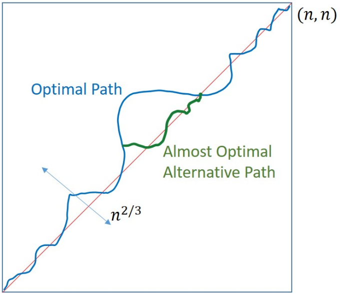

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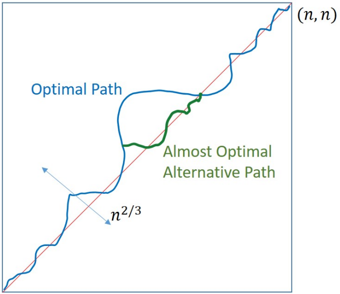

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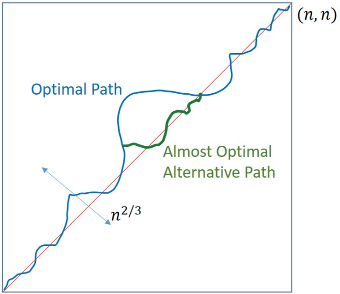

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- This will help if the cost of the alternative path is within $\delta n^{1/3}$ of the optimal path for some small $\delta < \epsilon$.
- **Idea:** Look for improvements on all scales.
- **Trick:** Do reinforcement on a random line parallel to the diagonal.

Why was this hard to predict with simulations?

Our proof requires a rather large value of n before $\mathbb{E}T_n^\epsilon > 4n$. The following theorem explains why understanding the behaviour around 0 proved challenging.

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Theorem (Sarkar, S., Zhang (2019+))

For every $C > 0$, as $\epsilon \to 0$,

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}T_n^\varepsilon - 4 = O(\epsilon^C).$$

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For every $C > 0$, as $\epsilon \to 0$,

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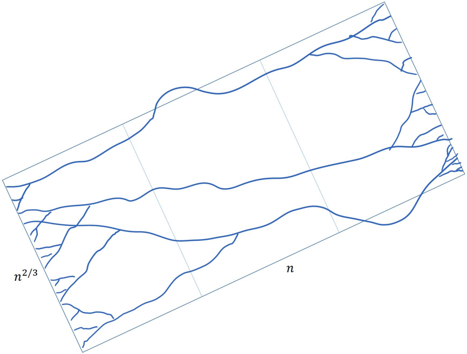
Predicted to be $e^{-c/\epsilon}$.

Coalescence into a few geodesics



Theorem (Basu, Hoffman, S. (2018))

On a $n \times n^{2/3}$ rectangle parallel to the diagonal, the expected number of distinct paths through the middle third is $O(1)$.

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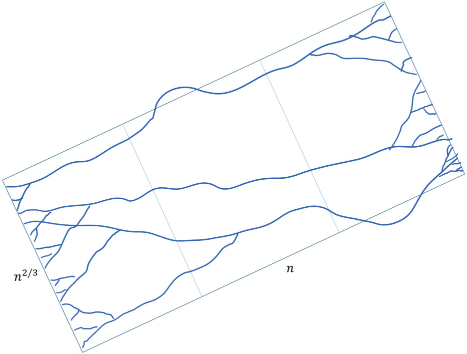
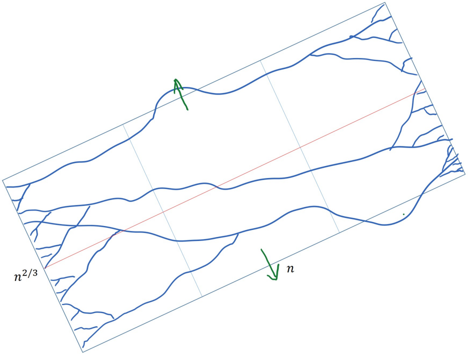


Theorem (Basu, Hoffman, S. (2018))

On a $n \times n^{2/3}$ rectangle parallel to the diagonal, the expected number of distinct paths through the middle third is $O(1)$.

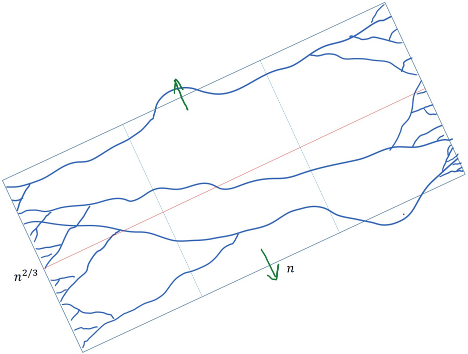
Corollary: No non-trivial infinite bi-geodesics.

Local Time on the diagonal



Let L_n be the time the optimal geodesic spends on the diagonal. Then $\mathbb{E}L_n = O(n^{1/3})$.

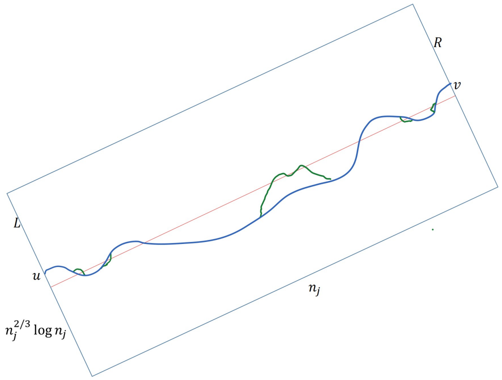
Local Time on the diagonal



Let L_n be the time the optimal geodesic spends on the diagonal. Then $\mathbb{E}L_n = O(n^{1/3})$.
Moreover it is concentrated, for some $\gamma > 0$,

$$\mathbb{P}[L_n n^{-1/3} > t] \leq C \exp(-ct^{\gamma}).$$

Inductive Statement

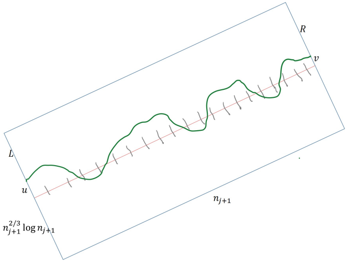


Let $n_j = (1/\epsilon)^{1+j/100}$. On a $n_j \times n_j^{2/3} \log n_j$ rectangle

$$\mathbb{P}[\max_{u \in L, v \in R} T^{\epsilon}_{u,v} - T_{u,v} > t\epsilon^{1/3} n_j^{1/3} \log^{C(j+1)} n_j] \leq \exp(-ct^{\gamma}),$$

for $j = 0, \ldots, J_\epsilon$ with $J_\epsilon \to \infty$ as $\epsilon \to 0$.

Local time of almost geodesics



On a $n_{j+1} \times n_{j+1}^{2/3} \log n_{j+1}$ rectangle, by induction maximal improvement is

$$\max_{u \in L, v \in R} T^{\epsilon}_{u,v} - T_{u,v} \leq \frac{n_{j+1}}{n_j} \epsilon^{1/3} n_j^{1/3} \log^{C(j+2)} n_j \leq \epsilon^{1/10} n_{j+1}^{1/3} =: Y_j$$

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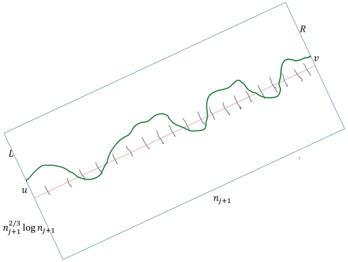

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Let W_γ be the number of segments of length n_j on the diagonal that γ intersects.

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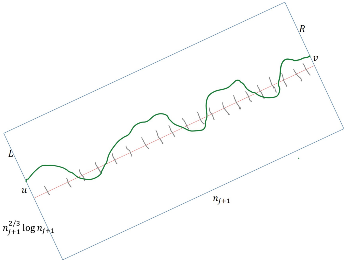

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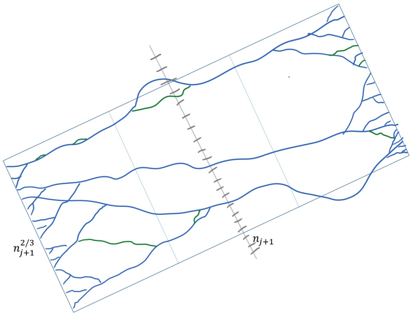
Let W_γ be the number of segments of length n_j on the diagonal that γ intersects.
For all γ from L to R that are within Y_j of the optimal path $W_\gamma \leq (n_{j+1}/n_j)^{1/3} \log^C n_j$.

Space of almost geodesics



On a $n_{j+1} \times n_{j+1}^{2/3} \log n_{j+1}$ rectangle let A be a line parallel to the sides in the middle third split into segments of length $n_j^{2/3} \log n_j$. Let N be the number of segments intersected by paths from L to R that are within Y_j of the optimal path. Then

$$\mathbb{P}[N > t \log^C n_j] \leq \exp(-ct^\gamma).$$

Thanks you for listening