Ergodicity of the KPZ Fixed Point

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The KPZ Fixed Point and Stochastic Integrability

It is the unique Markov process $(\mathfrak{h}_t(\cdot;\mathfrak{h}_0)\,,t\geq 0)$ taking place on UC (upper semicontinuous functions plus growth control) with transition probabilities on cylindrical sets given by

$$\mathbb{P}^{\mathfrak{h}_0}\Big(\cap_{i=1}^{m}\{\mathfrak{h}_t(x_i)\leq y_i\}\Big)=\det\left(I-K\right)_{L^2(\{x_1,\ldots,x_m\}\times\mathbb{R})}\,, \tag{1}$$

where $K=K(\mathfrak{h}_0,y,t)$ is the Brownian Scattering operator as introduced by Matetski, Quastel and Remenik (2017). The time evolution of the transition probabilities can be linearized through K (stochastic integrability).

Properties [Matetski, Quastel and Remenik (2017)]

KPZ 1:2:3 Scaling Invariance

Let $\mathfrak{h}_0^\alpha(x) := \alpha^{-1}\mathfrak{h}_0(\alpha^2 x)$. Then

$$\alpha\mathfrak{h}_{\alpha^{-3}t}(\alpha^{-2}x;\mathfrak{h}_0^\alpha) \stackrel{dist.}{=} \mathfrak{h}_t(x;\mathfrak{h}_0)\,,\forall\,\alpha>0\ \ \text{(as processes)}\,.$$

Stationarity: $\mathfrak{b} \equiv$ two-sided BM with $\sigma = 2$

Take $\mathfrak{b}_t(x) \equiv \mathfrak{h}_t(x;\mathfrak{b})$. Then,

$$\Delta\mathfrak{b}_t(x) := \mathfrak{b}_t(x) - \mathfrak{b}_t(0) \stackrel{dist.}{=} \mathfrak{b}(x)\,,\forall\,t\geq 0\ \ \text{(as processes in } x)\,.$$

Properties [Matetski, Quastel and Remenik (2017)]

Local Behaviour

Using (1) and kernel estimates for K one can get

$$\epsilon^{-1/2}\left(\mathfrak{h}_1(\epsilon x) - \mathfrak{h}_1(0)\right) \stackrel{dist.}{\to} \mathfrak{b}(x)\,, \text{as } \epsilon \to 0\,,$$

in terms of finite dimensional distributions.

Ergodicity

The long time behaviour of the KPZ fixed point is essentially equivalent to its local behaviour (1:2:3 scaling). If $\epsilon^{1/2}\mathfrak{h}_0(\epsilon^{-1})$ is convergent in distribution (in UC) as $\epsilon \to 0$, then

$$\Delta\mathfrak{h}_t(x) := \mathfrak{h}_t(x) - \mathfrak{h}_t(0) \stackrel{dist.}{\to} \mathfrak{b}(x)\,, \text{as } t \to \infty\,,$$

in terms of finite dimensional distributions.

Properties [Matetski, Quastel and Remenik (2017)]

Airy Sheet

Start with $\mathfrak{h}_0 = \mathfrak{d}^z$, where $\mathfrak{d}^z(z) = 0$ and $\mathfrak{d}^z(x) = -\infty$ for $x \neq z$, and define (uniqueness is still open)

$$A(z, x) := \mathfrak{h}_1(x; \mathfrak{d}_z) + (z - x)^2 .$$

Airy Sheet Variational Formula

For each $t > 0$,

$$\mathfrak{h}_t(x; \mathfrak{h}_0) = \sup_z \left\{ \mathfrak{h}_0(z) + t^{1/3} A(zt^{-2/3}, xt^{-2/3}) - \frac{(z - x)^2}{t} \right\} .$$

Variational Formula and Coupling

For each $t > 0$,

$$\mathfrak{h}_t(x) := \sup_z \left\{ \mathfrak{h}_0(z) + t^{1/3} A(zt^{-2/3}, xt^{-2/3}) - \frac{(z-x)^2}{t} \right\},$$

and

$$\mathfrak{b}_t(x) := \sup_z \left\{ \mathfrak{b}(z) + t^{1/3} A(zt^{-2/3}, xt^{-2/3}) - \frac{(z-x)^2}{t} \right\}.$$

Questions

Can we get ergodicity through variational formula? Do $\Delta\mathfrak{h}_t$ and $\Delta\mathfrak{b}_t$ get close together as $t \to \infty$?

Convergence in Probability

Denote
$$||f-g||_a := \sup_{x\in[-a,a]} |f(x)-g(x)|\,.$$

Theorem [P. '17]

Assume that $\epsilon^{1/2}\mathfrak{h}_0(\epsilon^{-1}x)$ is convergent in distribution (in UC) as $\epsilon \to 0$. Let $a>0$ and $t \geq a^{3/2}$. There exists a (variational) coupling $(\mathfrak{h}_t, \mathfrak{b}_t)$ such that for every $\eta > 0$

$$\mathbb{P}\Big(||\Delta\mathfrak{h}_t - \Delta\mathfrak{b}_t||_a > \eta\sqrt{a}\Big) \leq \theta(at^{-2/3}) + C\frac{(at^{-2/3})^{1/4}}{\eta}\,,$$

where $C>0$ is a universal constant and $\lim_{\delta\to 0}\theta(\delta) = 0$.

Convergence in Probability

Remark

In the first version of [P. '17] (arXiv:1708.06006) it was only considered initial profiles $\mathfrak{h}_0 \in UC$ satisfying $\epsilon^{1/2}\mathfrak{h}_0(\epsilon^{-1}x) = \mathfrak{h}_0(x)$. In a upcoming version, the result is extend to initial profiles $\mathfrak{h}_0 \in UC$ such that $\epsilon^{1/2}\mathfrak{h}_0(\epsilon^{-1}x)$ is convergent.

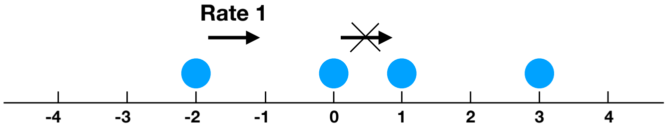
Totally Asymmetric Simple Exclusion Process

TASEP

- Markov process $(\eta_t\,,\ t \geq 0\,)$ with state space $\{0,1\}^{\mathbb{Z}}$.
- When $\eta_t(x) = 1$, we say that site x is occupied by a particle at time t, and it is empty if $\eta_t(x) = 0$.
- Particles jump to the neighbouring right site with rate 1 provided that the site is empty (the exclusion rule).

TASEP

Particles jump to the right with rate 1 provided the site is empty.



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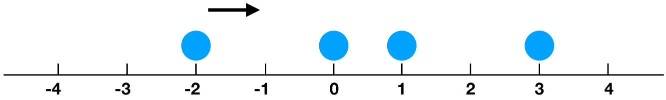

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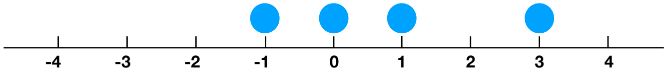

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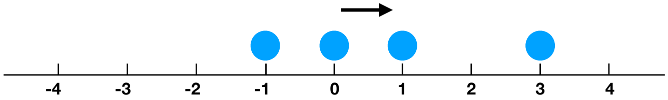

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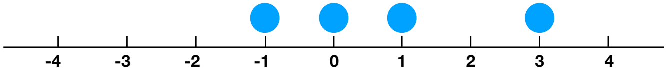

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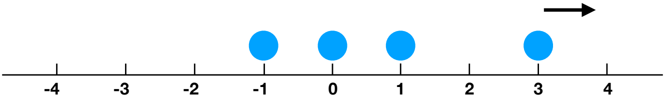

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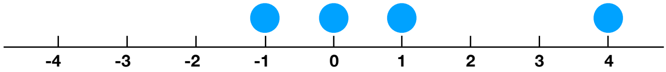


Interface Growth Model

TASEP Growth

Let N_t denote the total number of particles which jumped from site 0 to site 1 during the time interval $[0, t]$, and define

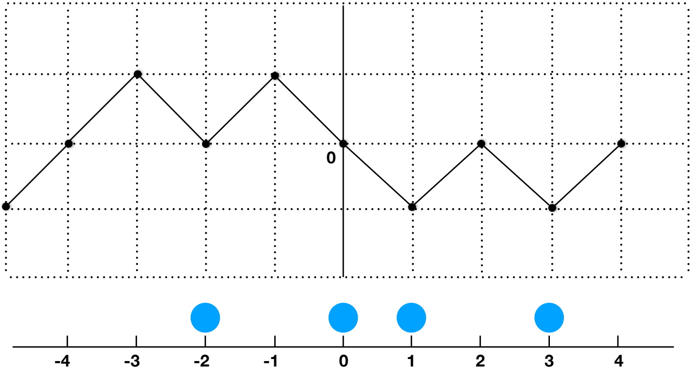
$$h_t(k) = \begin{cases} 2N_t + \sum_{j=1}^{k}(1 - 2\eta_t(j)) & \text{for } k \geq 1 \\ 2N_t & \text{for } k = 0 \\ 2N_t - \sum_{j=k+1}^{0}(1 - 2\eta_t(j)) & \text{for } k \leq -1 \,. \end{cases}$$

Interface Growth Model

TASEP Growth

- Markov process $(h_t\,,\; t \geq 0\,)$ with state space $\mathbb{Z}^{\mathbb{Z}}$.
- $h_t(k)$ is the value of height function at position $k \in \mathbb{Z}$ at time t.
- Local minimum becomes local maximum with rate 1.

TASEP Growth



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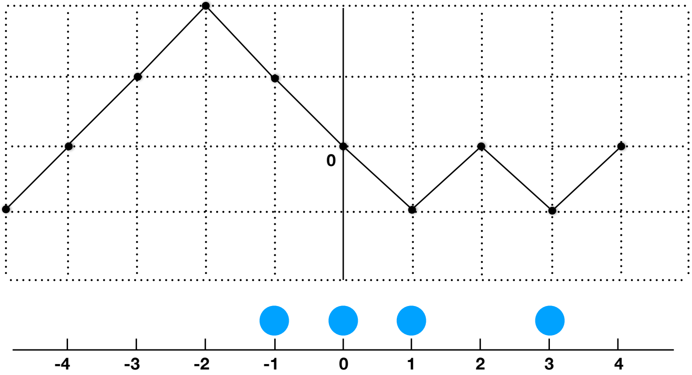

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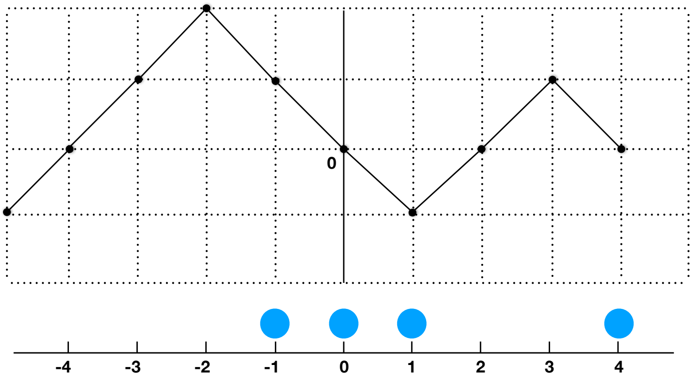

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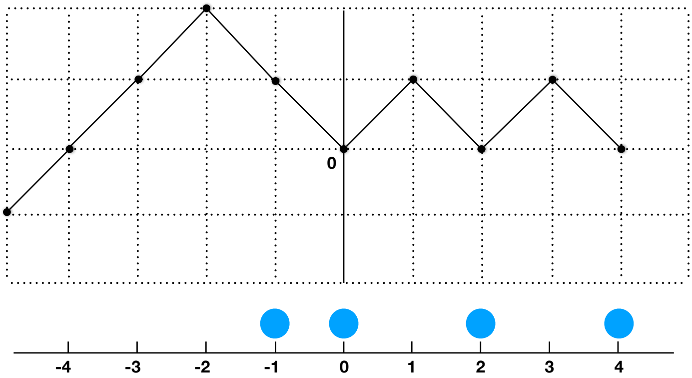

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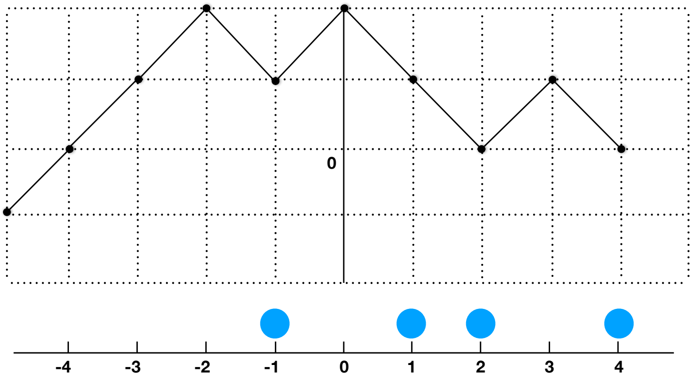

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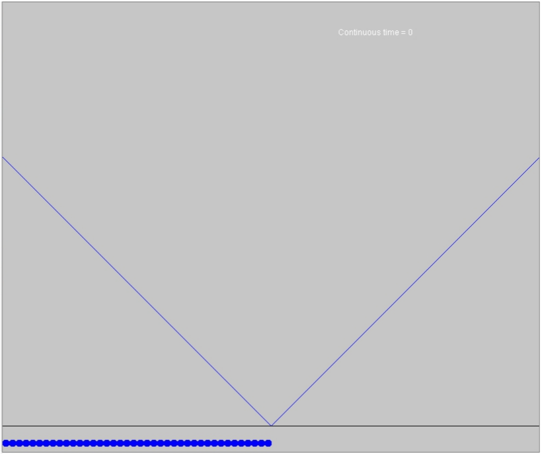

Figure: Narrow Wedge Initial Profile (Patrick Ferrari, Univ. Bonn).

TASEP Growth

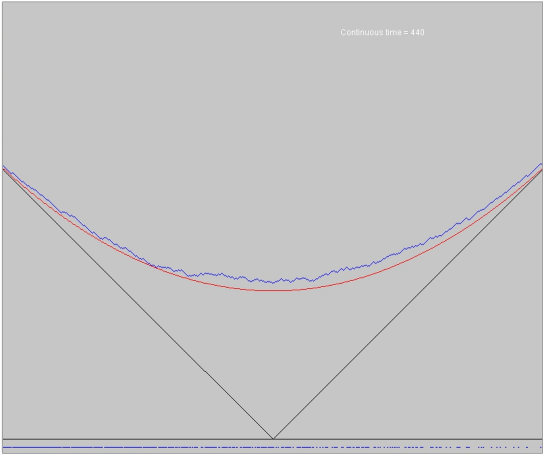

Figure: Narrow Wedge Initial Profile (Patrick Ferrari, Univ. Bonn).

TASEP Growth

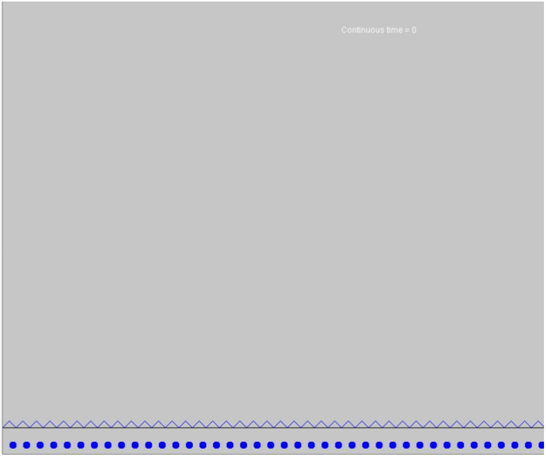

Figure: Flat Initial Profile (Patrick Ferrari, Univ. Bonn).

TASEP Growth

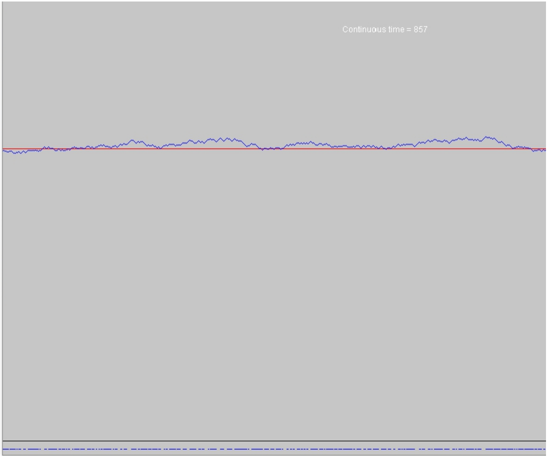

Figure: Flat Initial Profile (Patrick Ferrari, Univ. Bonn).

TASEP Growth

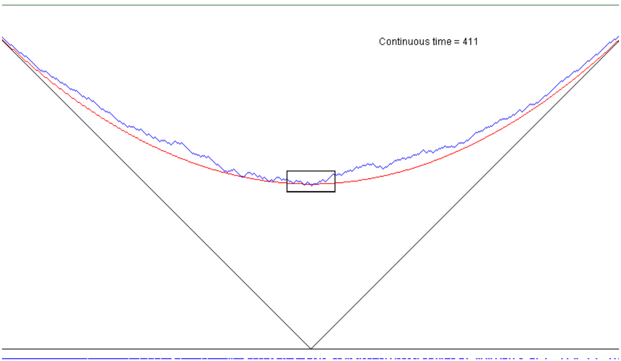

Figure: Scaling in a $n^{2/3} \times n^{1/3}$ rectangle.

TASEP Growth and the KPZ Fixed Point

Let

$$\mathfrak{h}_{n,t}(x) := \frac{tn - h_{2tn}^{(n)}\left(\lfloor 2xn^{2/3} \rfloor\right)}{n^{1/3}},$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$.

Theorem [Matetski, Quastel and Remenik '17]

If

$$\lim_{n\to\infty} \mathfrak{h}_{n,0} \stackrel{dist.}{=} \mathfrak{h}_0,$$

then

$$\lim_{n\to\infty} \mathfrak{h}_{n,t} \stackrel{dist.}{=} \mathfrak{h}_t,$$

where $(\mathfrak{h}_t, t \geq 0)$ is given by the KPZ fixed point.

Examples

- Narrow Wedge: $h_{n,0} \to \mathfrak{d}$, where $\mathfrak{d}(0) = 0$ and $\mathfrak{d}(x) = -\infty$ for $x \neq 0$.
- Flat: $h_{n,0} \to 0$.
- Stationary: $h_{n,0} \to \mathfrak{b}$ a two-sided BM with $\sigma = 2$.

Remark

The initial profile of particles $h^{(n)}$ might depende on n, in such way that for any $\mathfrak{h}_0 \in UC$ one can build a sequence of initial particle profiles $h^{(n)}$ such that $\mathfrak{h}_{n,0} \to \mathfrak{h}_0$.

The Corner Growth Model

Start with an initial profile $\mathrm{h} = (\mathrm{h}(k))_{k\in\mathbb{Z}} \in (\mathbb{Z}^2)^{\mathbb{Z}}$ constructed from the particle configuration η_0 as follows: $\mathrm{h}(0) = (0,0)$ and

$$\mathrm{h}(k+1) = \mathrm{h}(k) + \eta_0(k+1)(0,-1) + (1-\eta_0(k+1))(1,0)\,.$$

The path h splits $\mathbb{Z}^2$ into two regions and we denote Γ_0 the left-hand side one (including h).

The Corner Growth Model

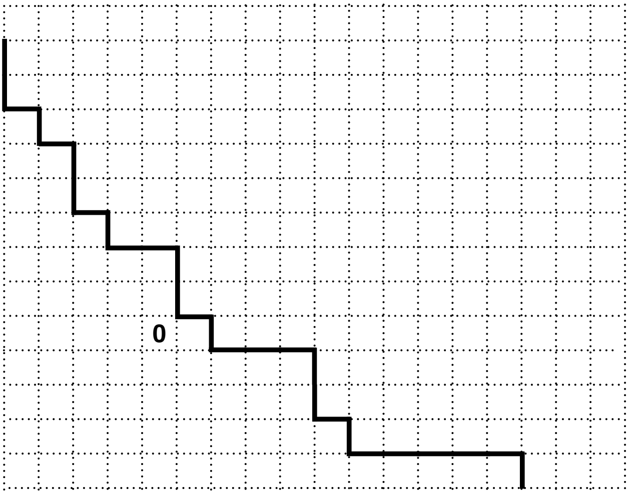

Figure: Initial interface profile h, $\mathrm{h}(0) = \mathbf{0}$.

The Corner Growth

The corner growth model $(\Gamma_t)_{t\geq 0}$ is described by the set Γ_t of occupied vertices at time t:

- Each site $(k, l) \in \Gamma_t^c$ becomes occupied at rate 1, once the sites $(k-1, l)$ and $(k, l-1)$ are both occupied.

The boundary of Γ_t and the TASEP height function h_t (local minima becomes local maxima at rate 1) have similar evolution rules, up to a 45^o rotation.

Corner Growth and the TASEP Height Function

For each $(k,l) \in \Gamma_0$ set $L^{\mathrm{h}}(k,l) := 0$ and for $(k,l) \in \Gamma_0^c$ let

$$L^{\mathrm{h}}(k,l) = \text{ time when } (k,l) \text{ becomes occupied}\,.$$

Thus,

$$\Gamma_t = \left\{(k,l) \in \mathbb{Z}^2 \,:\, L^{\mathrm{h}}(k,l) \le t\right\}\,.$$

If we construct h_0 and h using the same initial particle configuration η_0, and match the transition rates, the TASEP height function and the occupation time are related as follows: for all $t \ge 0$ and $\mathbf{x} = (k,l) \in \Gamma_0$,

$$L^{\mathrm{h}}(k,l) \le t \iff h(k-l,t) \ge k+l\,. \tag{2}$$

LPP Time and the KPZ Fixed Point

For $x \in \mathbb{R}$ and $t \geq 0$, we denote

$$[x]_t \equiv (\lfloor t \rfloor + \lfloor x \rfloor, \lfloor t \rfloor - \lfloor x \rfloor) .$$

For $n, t \geq 0$ fixed, define the processes (for $x \in \mathbb{R}$)

$$H^{\mathrm{h}}_{n,t}(x) = \frac{L^{\mathrm{h}}[2^{2/3} x n^{2/3}]_{tn} - 4tn}{2^{4/3} n^{1/3}} .$$

Using convergence of h_n to the KPZ fixed point and (2) one can show finite dimensional convergence of H^{h}_n to the KPZ fixed point, and it can extend it to functional convergence:

$$\text{If } \lim_{n\to\infty} \mathfrak{h}_{n,0} \stackrel{dist.}{=} \mathfrak{h}_0 \;\Rightarrow\; \lim_{n\to\infty} H^{\mathrm{h}}_{n,t} \stackrel{dist.}{=} \mathfrak{h}_t.$$

Last-Passage Percolation Time

The quantities $L^{\mathrm{h}}(k,l)$ satisfies a recurrence relation:

$$L^{\mathrm{h}}(k,l) = \omega_{k,l} + \max\{L^{\mathrm{h}}(k-1,l)\,,\; L^{\mathrm{h}}(k,l-1)\}\,,$$

where $\left\{\omega_{k,l}\ :\in \mathbb{Z}^2\right\}$ is a collection of i.i.d. random variables with exponential distribution of parameter 1.

Last-Passage Percolation Time

Point-to-point last-passage percolation (LPP) time:

- $\omega = \{\omega_{\mathbf{z}} : \mathbf{z} \in \mathbb{Z}\}$ i.i.d. $\mathrm{Exp}(1)$ r.v.;
- For $\mathbf{y} \geq \mathbf{x}$, $\Pi_{\mathbf{x}}(\mathbf{y}) = \{$ up-right paths from $\mathbf{x}$ to $\mathbf{y}$ $\}$;
- $L_{\mathbf{x}}(\mathbf{y}) := \max\left\{\sum_{\mathbf{z}\in\pi} \omega_{\mathbf{z}} : \pi \in \Pi_{\mathbf{x}}(\mathbf{y})\right\}$.

Then

$$L_{\mathbf{x}}(k,l) = \omega_{k,l} + \max\{L_{\mathbf{x}}(k-1,l)\,,\, L_{\mathbf{x}}(k,l-1)\}\,.$$

Last-Passage Percolation Time

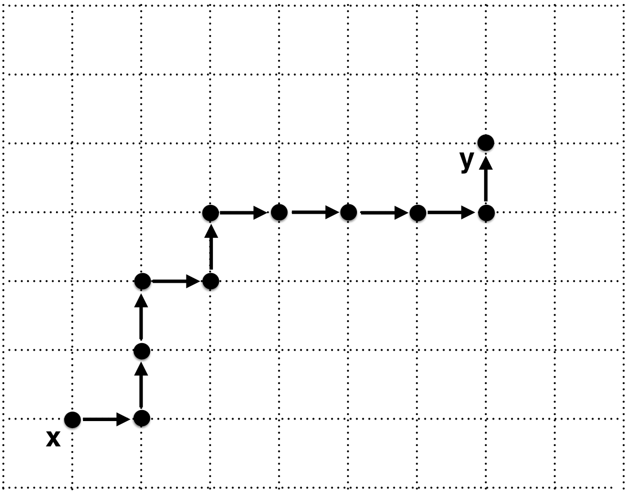

Figure: Up-right path.

Last-Passage Percolation Model

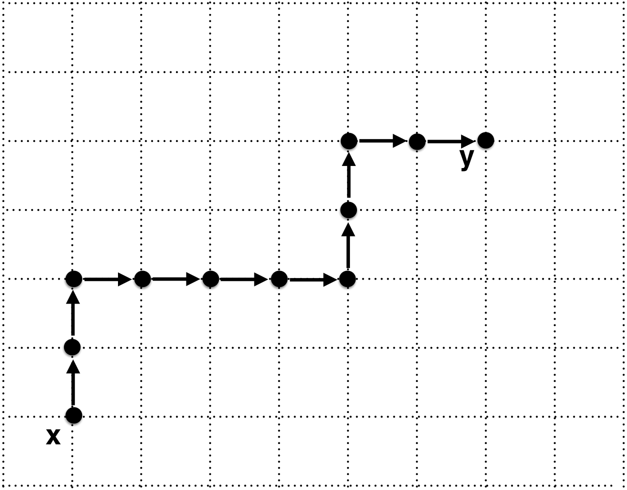

Figure: Up-right path.

Last-Passage Percolation Time

Curve-to-point LPP time:

- $\omega = \{\omega_{\mathbf{z}} : \mathbf{z} \in \mathbb{Z}\}$ i.i.d. $\mathrm{Exp}(1)$ r.v.;
- Down-right path $\mathrm{h} = (\mathrm{h}(k))_{k\in\mathbb{Z}}$;
- For $\mathbf{y} \geq \mathrm{h}$, $\Pi^{\mathrm{h}}(\mathbf{y}) = \{$ up-right paths from h to $\mathbf{y}$ $\}$;
- $L^{\mathrm{h}}(\mathbf{y}) := \max\left\{\sum_{\mathbf{z}\in\pi} \omega_{\mathbf{z}} : \pi \in \Pi^{\mathrm{h}}(\mathbf{y})\right\}$.

Then

$$L^{\mathrm{h}}(k,l) = \omega_{k,l} + \max\{L^{\mathrm{h}}(k-1,l)\,,\ L^{\mathrm{h}}(k,l-1)\}\,.$$

Last-Passage Percolation Time

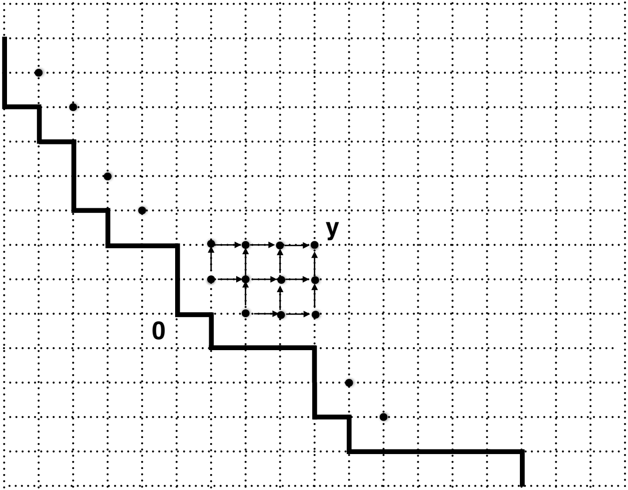

Figure: Curve to point paths.

Last-Passage Percolation Time

Point-to-point $L_{\mathbf{x}}(\mathbf{y}) = L^{\mathrm{h}_{\mathbf{x}}}(\mathbf{y})$ corresponds to curve-to-point with initial profile: $\mathrm{h}_{\mathbf{x}}(0) = \mathbf{x} - (1,1)$ and

$$\mathrm{h}_{\mathbf{x}}(k+1) - \mathrm{h}_{\mathbf{x}}(k) = \begin{cases} (0,-1) & \text{for } k < 0\,, \\ (1,0) & \text{for } k \geq 0\,, \end{cases}$$

Last-Passage Percolation Time

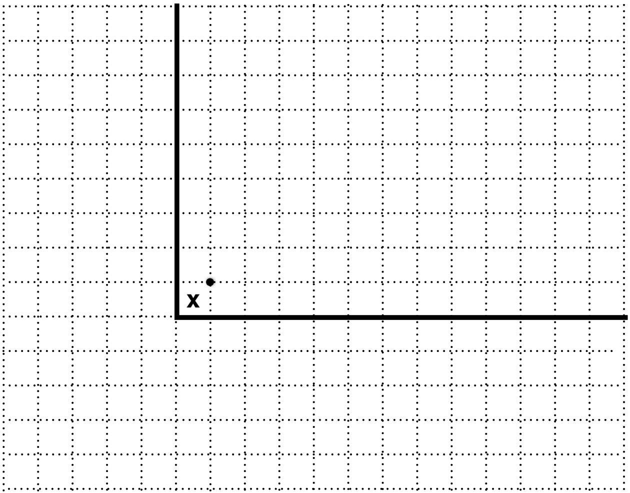

Figure: Curve to point and point to point.

LPP Model with Boundary

To construct the coupling and handle the proofs it is more convenient to work with a slightly different LPP model where, instead of having an initial interface profile, we work with a boundary time profile along the coordinate axis.

LPP Model with Boundary Times

Denote $\mathbb{Z}^* := \{z \in \mathbb{Z} \,:\, z \geq 0\}$ and

$$\mathrm{b} := \{\omega^{\mathrm{b}}_{\mathbf{x}} \,:\, \mathbf{x} \in \mathbb{Z}^* \times \{0\} \cup \{0\} \times \mathbb{Z}^*\}$$

will be a collection of non-negative real numbers representing the profile of passage times along the non-negative coordinate axis, and we will always assume that $\omega^{\mathrm{b}}_{(0,0)} = 0$. To construct the passage time environment $\omega^{\mathrm{b}} = \left\{\omega^{\mathrm{b}}_{\mathbf{x}} \,:\, \mathbf{x} \in (\mathbb{Z}^*)^2\right\}$, take the same i.i.d. environment ω as before, given by exponential random variables of parameter 1, and set

$$\omega^{\mathrm{b}}_{\mathbf{x}} = \begin{cases} \omega_{\mathbf{x}} & \text{for } \mathbf{x} \geq (1,1) \\ \omega^{\mathrm{b}}_{\mathbf{x}} & \text{for } \mathbf{x} \in \mathbb{Z}^* \times \{0\} \cup \{0\} \times \mathbb{Z}^* \,. \end{cases}$$

LPP Model with Boundary

Given a time profile b we define

$$\mathrm{b}(z) = \begin{cases} \sum_{i=1}^{-z} \omega_{0,i}^{\mathrm{b}} & \text{for } z < 0 \\ 0 & \text{for } z = 0 \\ \sum_{i=1}^{z} \omega_{i,0}^{\mathrm{b}} & \text{for } z > 0\,, \end{cases}$$

and for $\mathbf{x} = (k,l) > (0,0)$ and $z \in [-l,k]$,

$$\bar{L}_z(\mathbf{x}) = \begin{cases} L_{(1,-z)}(\mathbf{x}) & \text{for } z \in [-l,0] \\ L_{(z,1)}(\mathbf{x}) & \text{for } z \in (0,k]\,. \end{cases}$$

The last-passage percolation time to $\mathbf{x} = (k,l) > (0,0)$, with time profile b, is defined as

$$L^{\mathrm{b}}(\mathbf{x}) := \max_{z \in [-l,k]} \left\{ \mathrm{b}(z) + \bar{L}_z(\mathbf{x}) \right\}\,.$$

Last-Passage Percolation Time

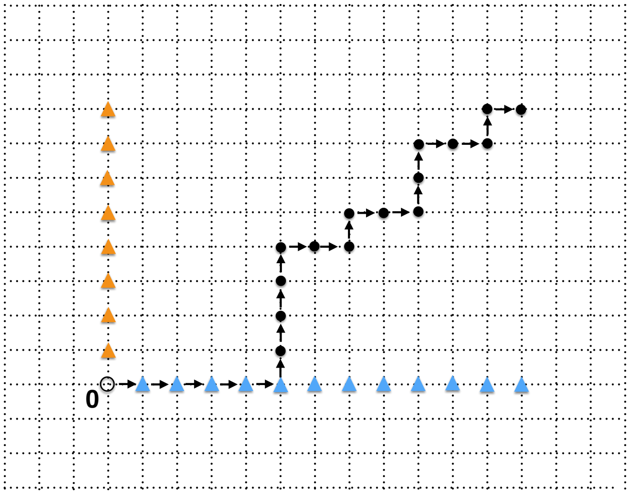

Figure: LPP with boundary.

Examples of Boundaries

Recovering L^{h}

If we set $\omega^{\mathrm{bh}}(0,0) = 0$, and for $k \geq 1$ let

$$\begin{cases} \omega^{\mathrm{bh}}(k,0) := L^{\mathrm{h}}(k,0) - L^{\mathrm{h}}(k-1,0) \\ \omega^{\mathrm{bh}}(0,k) := L^{\mathrm{h}}(0,k) - L^{\mathrm{h}}(0,k-1)\,, \end{cases}$$

then $L^{\mathrm{bh}}(\mathbf{x}) = L^{\mathrm{h}}(\mathbf{x})$, for all $\mathbf{x} > \mathbf{0}$.

Last-Passage Percolation Time

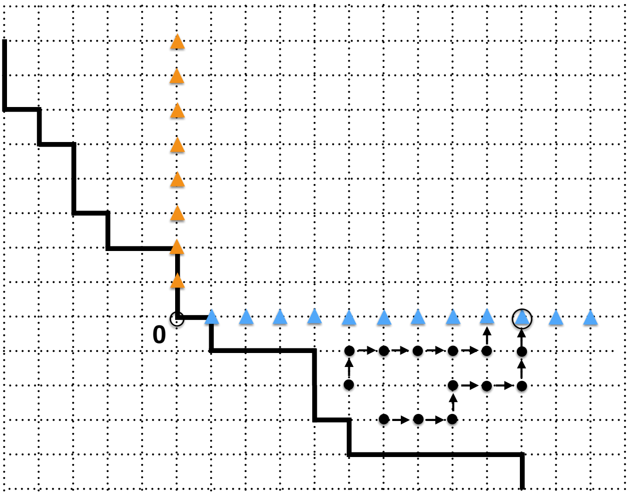

Figure: Substrate and LPP with boundary.

Examples of Boundaries

Recovering L^{h}

To go from h to $\mathbf{x} = (k, l) > \mathbf{0}$ a maximal path must cross the non-negative coordinate axis for the last time at some point and, w.l.g., let us assume that this point is $(Z, 0)$. Thus,

$$\begin{aligned} L^{\mathrm{h}}(\mathbf{x}) &= L^{\mathrm{h}}(Z,0) + L((Z,1),\mathbf{x}) \\ &= b^{\mathrm{h}}(Z) + \bar{L}_Z(\mathbf{x}) \\ &\leq L^{b^{\mathrm{h}}}(\mathbf{x})\,. \end{aligned}$$

Since

$$b^{\mathrm{h}}(z) + \bar{L}_z(\mathbf{x}) \leq L^{\mathrm{h}}(\mathbf{x})\,,\ \forall\, z \in (-l, k)$$

we have the other inequality.

Examples of Boundaries

Stationary LPP

Let $\rho \in (0,1)$ and set $\omega^\rho(0,0) = 0$, and for $k \geq 1$ let $\omega^\rho(k,0) \sim \mathrm{Exp}_{1,k}(1-\rho)$ and $\omega^\rho(0,k) \sim \mathrm{Exp}_{2,k}(\rho)$ be a collection of independent variables. Recall $[k]_n \equiv (n+k, n-k)$ and define

$$\zeta^\rho_{n,k} := L^\rho[k]_n - L^\rho[k-1]_n\,, \text{ for } k = -n+1, \ldots, n\,.$$

Then $\left\{\zeta^\rho_{n,k} \,:\, k = -n+1, \ldots, n\right\}$ is a collection of i.i.d. random variables with

$$\zeta^\rho_{n,k} \stackrel{dist.}{=} \mathrm{Exp}_1(1-\rho) - \mathrm{Exp}_2(\rho)\,, \tag{3}$$

where $\mathrm{Exp}_1(1-\rho)$ and $\mathrm{Exp}_2(\rho)$ are independent.

Scaling Local LPP Increments

For $n, t \geq 0$ fixed, define the processes

$$\Delta^{\rho}_{n,t}(x) := \frac{L^{\rho}[xn^{2/3}]_{tn} - L^{\rho}[0]_{tn}}{2^{3/2}n^{1/3}} \, , x \in \mathbb{R} \, .$$

Then (Donsker's Theorem)

$$\lim_{n\to\infty} \Delta^{1/2}_{n,t} \overset{dist.}{=} \Delta^{1/2}_{t} \, ,$$

where $(\Delta^{1/2}_{t}(x) \, , x \in \mathbb{R})$ is a standard two-sided Brownian Motion (for all $t \geq 0$).

Scaling Local LPP Increments

For $n, t \geq 0$ fixed, define the processes

$$\Delta^{\mathrm{h}}_{n,t}(x) := \frac{L^{\mathrm{h}}[xn^{2/3}]_{tn} - L^{\mathrm{h}}[0]_{tn}}{2^{3/2}n^{1/3}}\, , x \in \mathbb{R}\, .$$

Thus,

$$H^{\mathrm{h}}_{n,t}(x) = H^{\mathrm{h}}_{n,t}(0) + 2^{1/6}\Delta^{\mathrm{h}}_{n,t}(2^{2/3}x)\, .$$

Then

$$\lim_{n\to\infty} \Delta^{\mathrm{h}}_{n,t} \stackrel{dist.}{=} \Delta^{\mathfrak{h}_0}_t\, , \quad \text{where} \quad 2^{1/6}\Delta^{\mathfrak{h}_0}_t(2^{2/3}\cdot) \stackrel{dist.}{=} \Delta\mathfrak{h}_t(\cdot)\, .$$

Then all we have to do is to show that $\Delta^{\mathfrak{h}_0}_t$ and $\Delta^{1/2}_t$ get close to each other as $t \to \infty$.

Coupling LPP with Different Boundaries

Given boundary profiles $\mathrm{b}^1, \ldots, \mathrm{b}^k$, the *basic coupling* is a joint realization $L^{\mathrm{b}_1}, \ldots, L^{\mathrm{b}_k}$ of the last-passage times that is defined by constructing the passage times $\omega^{\mathrm{b}_1}, \ldots, \omega^{\mathrm{b}_k}$ with the same ω. In the next lemmas we will assume that all the joint realisations are given by the basic coupling.

Local Comparison and Attractiveness

Lemma 1 [Local Comparison]

Let

$$Z^{\mathrm{b}}(\mathbf{x}) := \max \arg \max_{z \in [-l,k]} \left\{ \mathrm{b}(z) + \bar{L}_z(\mathbf{x}) \right\} .$$

For $i \leq j$ and $n \geq 1$, if $Z^{\mathrm{b}_1}[j]_n \leq Z^{\mathrm{b}_2}[i]_n$ then

$$L^{\mathrm{b}_1}[j]_n - L^{\mathrm{b}_1}[i]_n \leq L^{\mathrm{b}_2}[j]_n - L^{\mathrm{b}_2}[i]_n .$$

Lemma 2 [Attractiveness]

Assume that $\mathrm{b}_1(j) - \mathrm{b}_1(i) \leq \mathrm{b}_2(j) - \mathrm{b}_2(i)$ for all $i \leq j$. Then

$$L^{\mathrm{b}_1}[j]_n - L^{\mathrm{b}_1}[i]_n \leq L^{\mathrm{b}_2}[j]_n - L^{\mathrm{b}_2}[i]_n \, , \; \forall \, i \leq j \, .$$

Sandwiching Increments

In the basic coupling context, one can always take a joint realisation $(\Delta^{\mathrm{h}}_{n,t}, \Delta^{1/2}_{n,t})$, by taking the boundary b^{h} and the stationary one with parameter $1/2$. Thanks to fundamental properties of this coupling (local comparison and attractiveness), we will be able to show that they will stay uniformly close to each other, and then the same will be true for the limit processes.

Sandwiching Increments

- The equilibrium initial profile with $\rho = 1/2$ will be tilted slightly, $\rho^{\pm}_{n,t} = 1/2 \pm r_t (tn)^{-1/3}$, so that the sandwiching effect, between a given general profile and its counterpart at the tilted equilibrium, in Lemma 1 is forced, either on the left or the right, according to the sign of the tilt.

- The basic coupling will be construct in such way that the tilted profiles are initially ordered, and by attractiveness, they will remain ordered at all times, which will ensure uniform bounds for the distance between $\Delta^{\mathrm{h}}_{n,t}$ and $\Delta^{1/2}_{n,t}$.

Localisation

Lemma 3 [KPZ Localisation]

Recall $\delta_t := at^{-2/3}$ and set

$$\rho^{\pm}_{n,t} := \frac{1}{2} \pm \frac{\delta_t^{-1/4}}{(tn)^{1/3}} .$$

Let $E(n,t)$ denote the event that

$$Z^{\rho^{-}_{n,t}}[an^{2/3}]_{tn} \le Z^{\mathrm{h}}[-an^{2/3}]_{tn} \text{ and } Z^{\mathrm{h}}[an^{2/3}]_{tn} \le Z^{\rho^{+}_{n,t}}[-an^{2/3}]_{tn} .$$

There exists a function $\theta(\delta) \ge 0$ such that $\lim_{\delta\to 0} \theta(\delta) = 0$ and

$$\limsup_{n\to\infty} \mathbb{P}\left(E(n,t)^c\right) \le \theta(\delta_t) .$$

Proof of the Theorem

Given a profile $\mathrm{b}^{1/2}$ define $\mathrm{b}^{\pm}$ by setting

$$\omega_{\mathbf{x}}^{\pm} = \begin{cases} \frac{1}{2\rho_{n,t}^{\pm}}\omega_{(0,|z|)}^{1/2} & \text{for } z < 0 \\ \frac{1}{2(1-\rho_{n,t}^{\pm})}\omega_{(z,0)}^{1/2} & \text{for } z > 0\,. \end{cases}$$

(Ordering the initial profiles.) Thus, $\omega^{\pm} \overset{dist.}{=} \omega^{\rho_{n,t}^{\pm}}$, and

$$\mathrm{b}^{-}(j) - \mathrm{b}^{-}(i) \leq \mathrm{b}^{1/2}(j) - \mathrm{b}^{1/2}(i) \leq \mathrm{b}^{+}(j) - \mathrm{b}^{+}(i)\,, \text{ if } i < j\,.$$

Proof of the Theorem

By local comparison (Lemma 1), on the event $E(n,t)$,

$$\Delta^{-}_{n,t}(x) \le \Delta^{\mathrm{h}}_{n,t}(x) \le \Delta^{+}_{n,t}(x)\,, \text{ for } x \in [0,a]\,,$$

and

$$\Delta^{+}_{n,t}(x) \le \Delta^{\mathrm{h}}_{n,t}(x) \le \Delta^{-}_{n,t}(x)\,, \text{ for } x \in [-a,0]\,.$$

By attractiveness (Lemma 2)

$$\Delta^{-}_{n,t}(x) \le \Delta^{1/2}_{n,t}(x) \le \Delta^{+}_{n,t}(x)\,, \text{ for } x \ge 0 \quad,$$

and

$$\Delta^{+}_{n,t}(x) \le \Delta^{1/2}_{n,t}(x) \le \Delta^{-}_{n,t}(x)\,, \text{ for } x \le 0 \quad.$$

Proof of Theorem

Hence, on the event $E(n,t)$,

$$\begin{aligned} |\Delta^{\mathrm{h}}_{n,t}(x) - \Delta^{1/2}_{n,t}(x)| &\le \Big(\Delta^{+}_{n,t}(x) - \Delta^{-}_{n,t}(x)\Big)\,\mathbb{1}\{x \in [0,a]\} \\ &+ \Big(\Delta^{-}_{n,t}(x) - \Delta^{+}_{n,t}(x)\Big)\,\mathbb{1}\{x \in [-a,0]\} \\ &\le \Delta^{+}_{n,t}(a) - \Delta^{-}_{n,t}(a) \\ &+ \Delta^{-}_{n,t}(-a) - \Delta^{+}_{n,t}(-a)\,, \end{aligned}$$

for all $x \in [-a,a]$. By attractiveness, $\Delta^{+}_{n,t}(x) - \Delta^{-}_{n,t}(x)$ increases with x, which implies the second inequality.

Proof of Theorem

This shows that, on the event $E(n,t)$,

$$||\Delta^{\mathrm{h}}_{n,t} - \Delta^{1/2}_{n,t}||_a \leq I_{n,t}(a)\,,$$

where

$$I_{n,t}(a) := \Delta^{+}_{n,t}(a) - \Delta^{-}_{n,t}(a) + \Delta^{-}_{n,t}(-a) - \Delta^{+}_{n,t}(-a)\,.$$

Therefore (notice that $I_{n,t}(a) \geq 0$)

$$\mathbb{P}\left(||\Delta^{\mathrm{h}}_{n,t} - \Delta^{1/2}_{n,t}||_a > \eta\sqrt{a}\right) \leq \mathbb{P}\left(E(n,t)^c\right) + \frac{\mathbb{E}\left(I_{n,t}(a)\right)}{\eta\sqrt{a}}\,.$$

Proof of Theorem

By KPZ localisation (Lemma 3),

$$\limsup_{n\to\infty} \mathbb{P}\left(E(n,t)^c\right) \le \theta(\delta_t)\,.$$

Using (3), together with simple calculations, we get

$$\limsup_{n\to\infty} \mathbb{E}\left(I_{n,t}(a)\right) \le C\sqrt{a}\,\delta_t^{1/4}\,.$$