A (2+1) dimensional anisotropic growth model with a smooth phase

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Joint work with Fabio Toninelli (Lyon).

Framework: 2D stochastic growth models

Consider stochastic growth modeled by a Markov chains with a local rule. Typical questions include

- stationary states (for interface gradients)
- space-time correlations of height fluctuations
- hydrodynamic limit

In this talk we will consider a random tiling model with a specific dynamics.

Warm up Example

Consider a Markov chain formed by stacking boxes (e.g. a 3D partition). Stacks of boxes must be nonincreasing on each row and column. A box is added at an available position with probability p and a box is removed (provided it can be) with probability q.

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(Video). We focus on a measures on the plane.

Overview picture in 2D

Let h be the height of an interface. For some cases, there exists a stationary measure π_ρ such that

$$\pi_\rho(h(x) - h(y)) = \rho.(x - y)$$

where ρ is called the slope.

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One expects that at large distances between x and y

$$\sqrt{\mathrm{Var}_{\pi_\rho}(h(x) - h(y))} \sim c_1 + c_2 |x - y|^\alpha$$

and for large times

$$\sqrt{\mathrm{Var}(h(x,t) - h(x,0))} \sim c_3 + c_4 t^\beta.$$

α is the roughness exponent while β is the growth exponent.

Expected fluctuations

On expected large space-time scales, fluctuations are expected to be described by a stochastic PDE of the form:

$$\partial_t h(x,t) = \Delta h(x,t) + \lambda \langle \nabla h(x,t), H_\rho \nabla h(x,t) \rangle + \xi(x,t)$$

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3. $d = 2$: Predicted irrelevance of non-linearity at transition λ_c when replacing $\langle \nabla h(x,t), H_\rho \nabla h(x,t) \rangle$ by $|\nabla h(x,t)|^2$: Chatterjee-Dunlop(18), Caravenna-Sun-Zygouras(18).

Wolf's Conjecture in 2D

Wolf found that there are two cases in 2D

- if $\det(H_\rho) > 0$, non-linearity is relevant, $\alpha \neq \alpha_{EW}$, $\beta \neq \beta_{EW}$, *Isotropic KPZ*.
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Some known results 1

Borodin-Ferrari(08) introduced a set of dynamics for lozenge tilings and obtained a hydodynamic limit for a specified initial condition

$$\lim_{L\to\infty} \frac{1}{L} h(xL, \tau L) = \psi(x,t)$$

where $\partial_t \psi + v(\nabla \psi) = 0$. Here, v is the speed of growth and is given by

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They also found that

$$\frac{1}{\sqrt{\log L}}[h(xL, \tau L) - \mathbb{E}[h(xL, \tau L)]]$$

converges to $N(0, 1/(2\pi)^2)$ and that the model belongs to the AKPZ class.

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Toninelli(15) generalized these dynamics to the plane, with particles being able to jump forward at rate p and backwards with rate q and showed

- dynamics are well-defined and stationary
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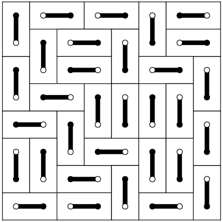
- Hydrodynamic limit for these dynamics proved in Laslier-Toninelli(17).
- Shocks and hydrodynamic limit proved in Legras-Toninelli(17).
- Borodin-Corwin-Toninelli(15): q-Whittaker model

Dimers on $\mathbb{Z}^2$

A dimer is an edge. A dimer covering is a subset of edges so that each vertex is incident to exactly one edge.



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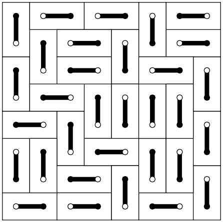

Assign $w : E \to \mathbb{R}_{>0}$, edge weights. The dimer model is

$$\mathbb{P}[M] = \frac{1}{Z} \prod_{e \in M} w(e),$$

where Z is the partition function.

Height function and phases

An idea dating back to Thurston(90) gives a surface representation to dimer models on bipartite graphs. Heights are defined on faces, and the height change is ± 3 between faces with a dimer covering a shared edge and ∓ 1 if there is no dimer covering the shared edge.

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- *frozen* or *solid* -configurations are deterministic
- *rough* or *liquid* - polynomial decay of correlations
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Disclaimer: These are not physical states of matter and we use the frozen, rough and smooth terminology to avoid any confusion.

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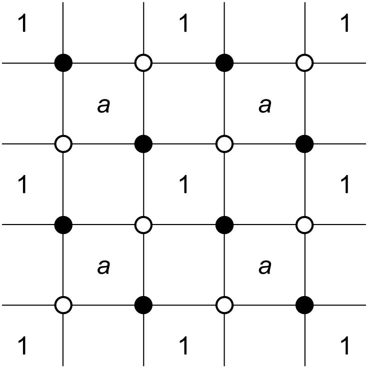
Measures depend on the height profiles and the underlying graph.

Two-periodic weighting on $\mathbb{Z}^2$

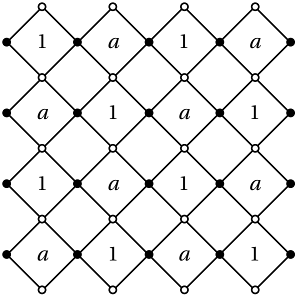
Not all dimer models give a smooth phase, but one which does is the two-periodic weighting on $\mathbb{Z}^2$

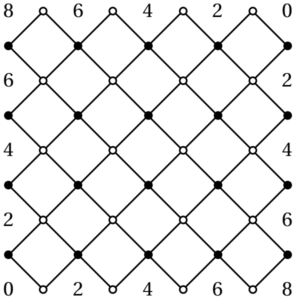


Here, edges incident to a face labelled a have edge weight $a \in (0, 1)$.

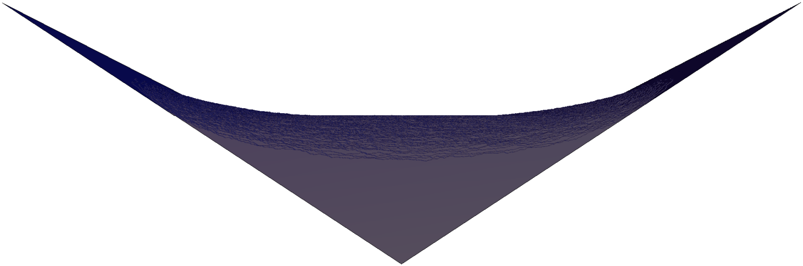
Aztec diamond

The same weighting can be considered on an Aztec diamond which a specific region of $\mathbb{Z}^2$ (purely for illustration):





Aztec diamond Pictures



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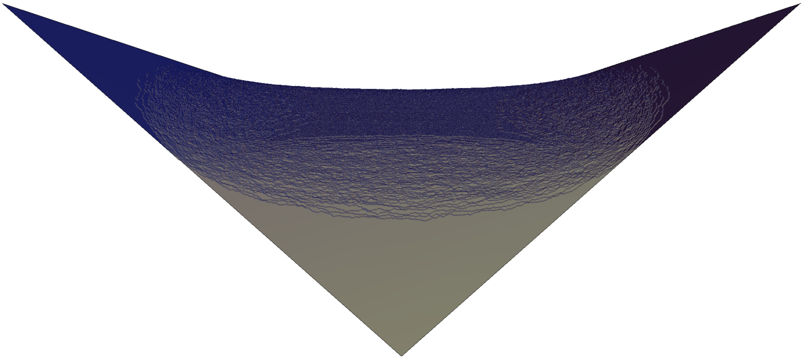

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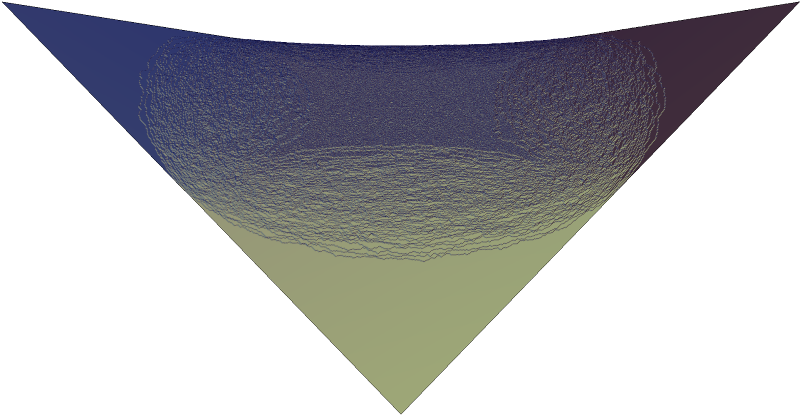

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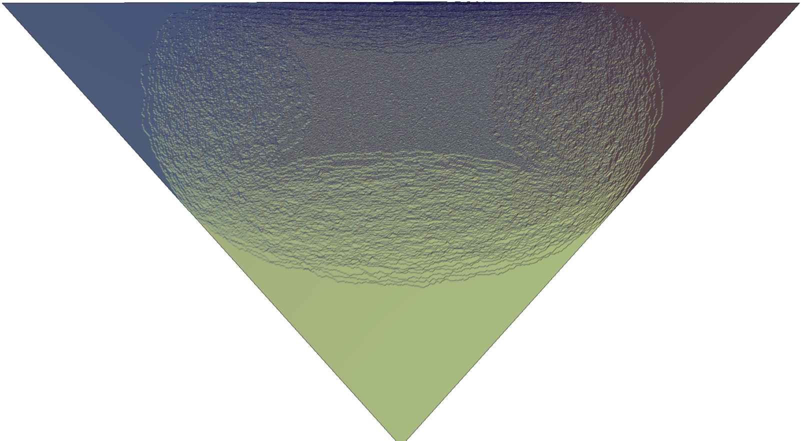
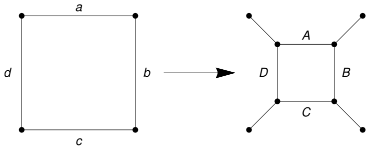


Dimer moves

The dynamics rely on two dimer moves that change the underlying graph.
Square move:



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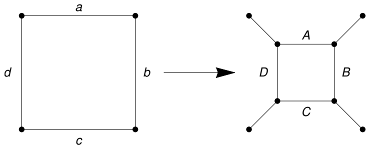

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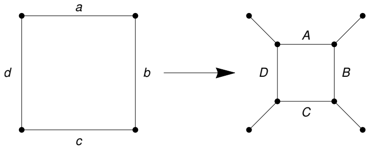

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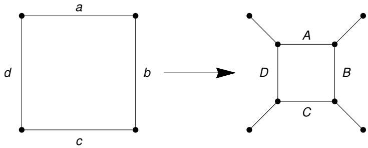

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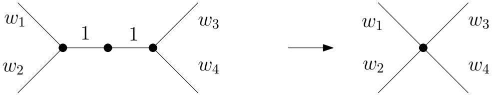
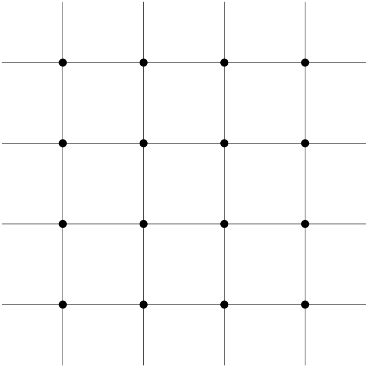


Application to the torus

We can apply these dimer moves to the torus (ignore weights for now). Apply the square move on the even faces and then on the odd faces, etc.



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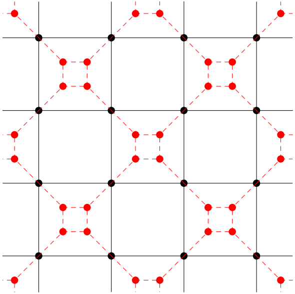

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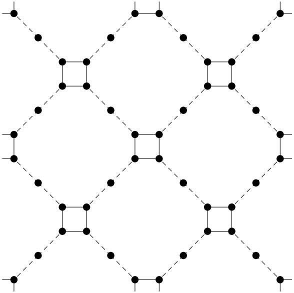

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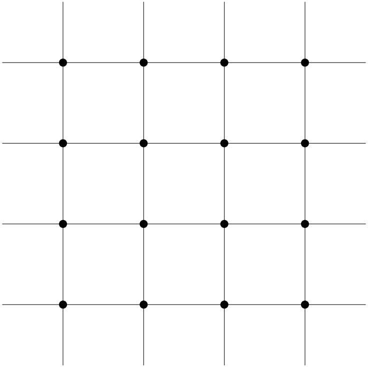

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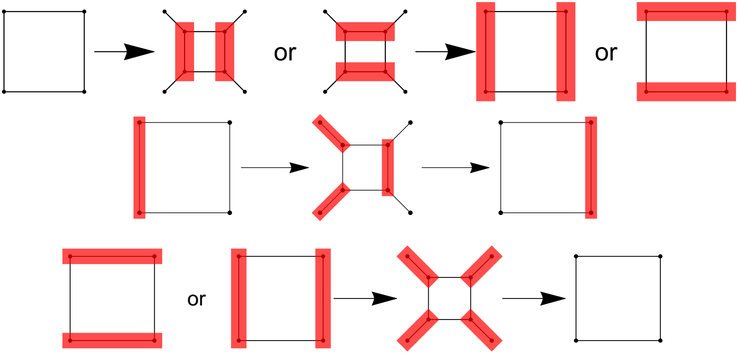

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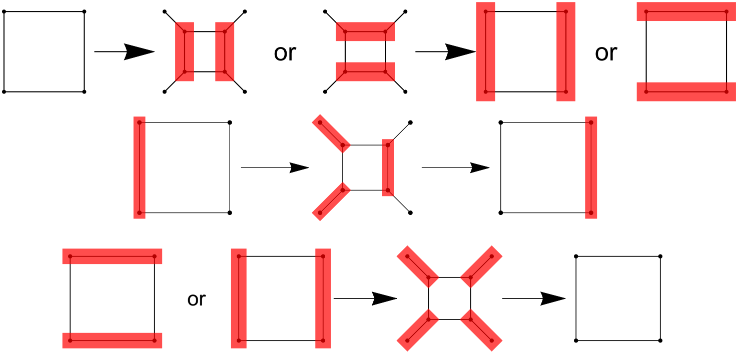

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From KOS, consider the torus with a slope and extend to the fullplane.

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- The dynamics are local,
- The dynamics preserve π_ρ in the two-periodic weighting.
- If $\rho \neq 0$, then π_ρ is rough, whereas if $\rho = 0$, π_ρ is smooth.

Main results

With Toninelli, for $\rho \neq 0$, we found an explicit formula for $v(\rho)$. Moreover

Theorem (C.-Toninelli(18))

For $\rho \neq 0$:

- *Logarithmic growth of fluctuations:*

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$$\det H_\rho < 0.$$

For $\rho = 0$:

- *Bounded flucutations:*

$$\mathrm{Var}_{\pi_\rho}(h(x,t) - h(0,t)) = O(1)$$

- *v is not twice-differentiable at $\rho = 0$.*

Main results

With Toninelli, for $\rho \neq 0$, we found an explicit formula for $v(\rho)$. Moreover

Theorem (C.-Toninelli(18))

For $\rho \neq 0$:

- *Logarithmic growth of fluctuations:*

$$\mathrm{Var}_{\pi_\rho}(h(x,t) - h(0,t)) = O(\log t)$$

- *Speed is twice differentiable with*

$$\det H_\rho < 0.$$

For $\rho = 0$:

- *Bounded flucutations:*

$$\mathrm{Var}_{\pi_\rho}(h(x,t) - h(0,t)) = O(1)$$

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- Using Kenyon-Okounkov-Sheffield(03) (and C.-Johansson(14)), we get good expressions for v.
- The rough phase results follow immediately, while it is not hard to show that the speed is discontinuous from the explicit formula.

Summary

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Thank you for listening!