Polymer geometry in the large deviation regime via eigenvalue rigidity

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UNIVERSALITY IN RANDOM STRUCTURES: INTERFACES, MATRICES, SANDPILES,
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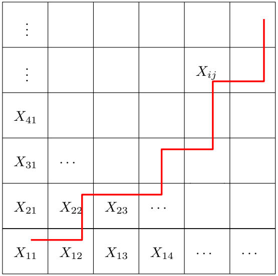

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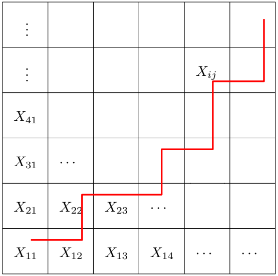

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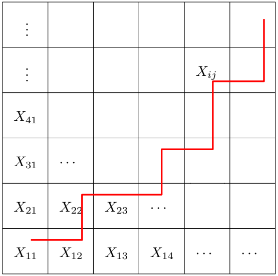

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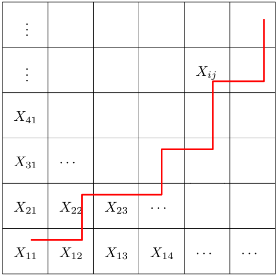

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Connections to particle systems and growth process

- Totally asymmetric exclusion process on $\mathbb{Z}$ (TASEP): particles at rate 1 jump to the right provided the site is empty.

Exponential LPP is equivalent to this model, where the passage times between $(0,0)$ and (N,N) denote the time taken by the N^{th} particle to reach 0 starting from wedge initial conditions.

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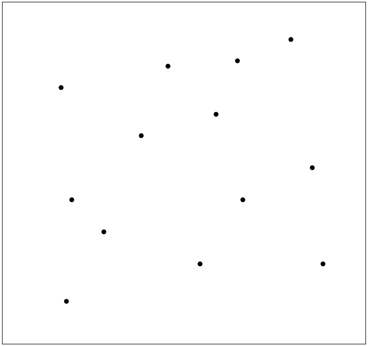
- Equivalent to Poly-nuclear growth model (PNG).

Maximal increasing subsequence/ Poissonian LPP



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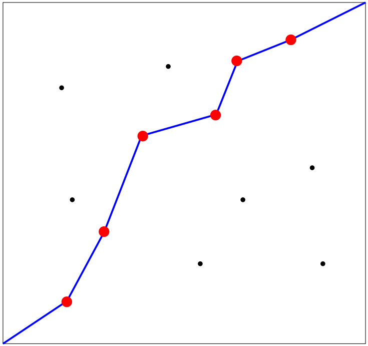

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- Under mild conditions, Poincáre inequality ensures that $\mathrm{Var}\ T_n = O(n)$.
- Γ_n w.h.p. has deviation $o(n)$ from the straight line joining $(0,0)$ to (n,n) under strict convexity of the limit shape boundary at $(1,1)$.

KPZ universality predictions

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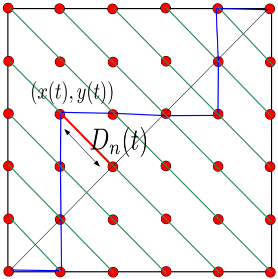
- and much more...

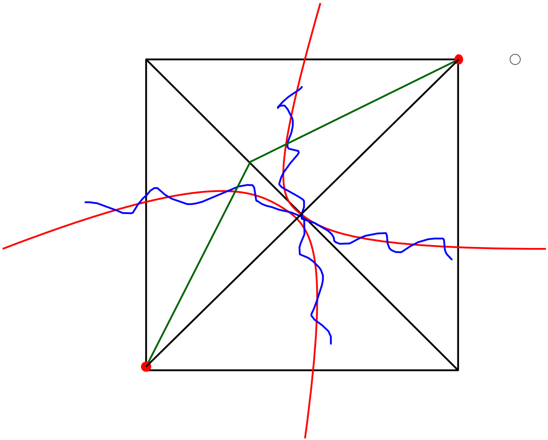
Transversal fluctuation

$$D_n(t) := |x(t) - y(t)|$$



• For the anti-diagonal line $\{x + y = t\}$, let the polymer intersect it at $(x(t), y(t))$.

Globally Parabolic vs Locally Brownian



• The polymer passes through the points where the parabolic loss matches with Brownian fluctuation: $\frac{x^2}{n} \approx \sqrt{x}$.

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Based on bijections, exact formulae and connections to algebraic combinatorics, representation theory, determinantal processes, random matrix theory.

Integrable/Exactly Solvable Models

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Poissonian LPP

- RSK correspondence to Young Tableaux.
- $\frac{\mathbb{E}T_{nx,ny}}{n} = 2\sqrt{xy}$.
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- Transversal fluctuation exponent of $2/3$ is also rigorously known using moderate deviations.

Large Deviations for polymer weights

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Lower tail large deviation: $T_n \leq (\mu - \delta)n$

- Large deviation speed is n^2 under minimal conditions.

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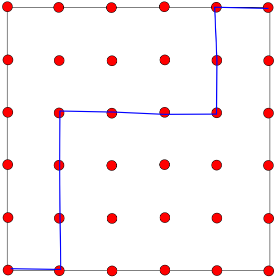

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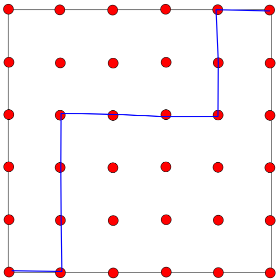

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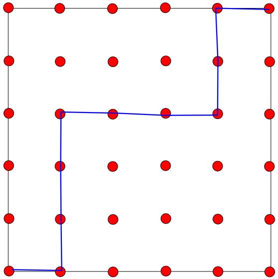

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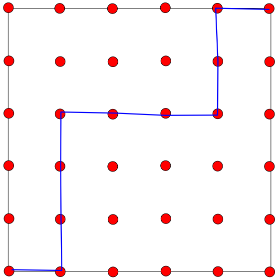

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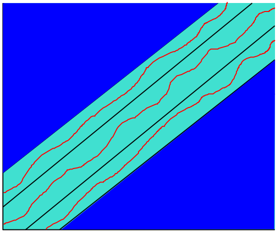
- Sub-additivity implies existence of rate function.

n^2 speed for $\{T_n \le (\mu - \delta)n\}$.



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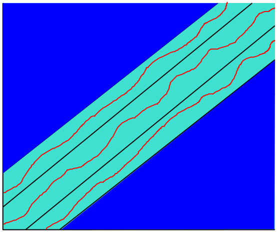

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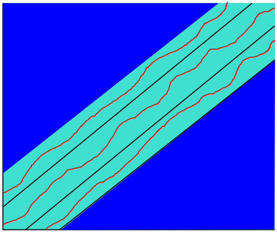

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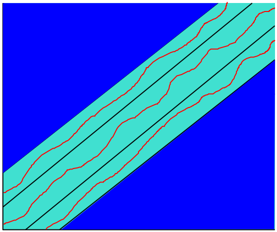

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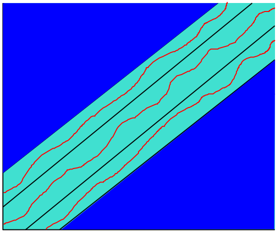

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The above argument already appeared in Kesten's work on First passage percolation.

Large deviation for exactly solvable models

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For Poissonian LPP

Deuschel-Zeitouni (1998), Seppäläinen (1998)

- For $\delta > 0$, $\lim_n \frac{\log \mathbb{P}(T_n \geq (2+\delta)n)}{n} = -I_u(\delta)$ where $I_u(\cdot)$ is an increasing convex function with $I(0) = 0$.

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- Via RSK, this is exactly the number of permutations with a given length for the longest increasing subsequence.

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- Deuschel and Zeitouni (1999) proved the lower bound using such variational results.

Young diagrams and Plancherel measure

1				
3	1			
4	2			
8	6	3	2	1

$$\mathbb{P}(T_n = k) = \sum_{\tau(0)=k} \frac{n!}{\pi(\tau)^2}.$$

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- Coupling with TASEP also has been exploited in analyzing the upper tail. Seppäläinen (1998)

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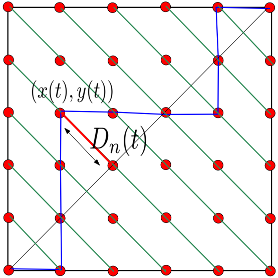
We will focus on Exponential LPP for the rest of the talk.

Transversal fluctuation for upper tail

$$D_n(t) := |x(t) - y(t)|; \;\; D_n = \sup_t D_n(t).$$



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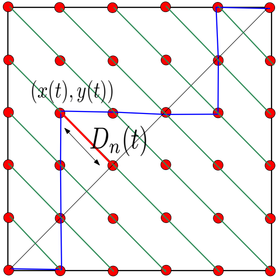

$$\bar{\xi} := \inf\{\xi' \leq 1 : \limsup_{n\to\infty} \mathbb{P}(D_n \geq n^{\xi'} \mid \mathcal{U}_\delta) = 0\},$$

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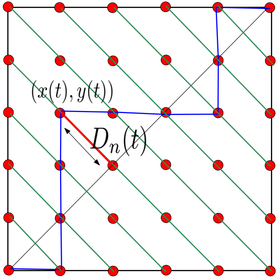

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If they agree we call the exponent ξ_δ.

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Fix $\delta > 0$.

$$\limsup_{n\to\infty} \mathbb{P}(D_n \leq h n^{1/2} \mid \mathcal{U}_\delta) \to 0$$

as $h \to 0$.

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• If time permits, towards the end, I will describe what happens for the lower tail.

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Theorem (Johansson)

$$\lambda_1 \stackrel{d}{=} T_{(1,1),(M,N)}.$$

Recall again

For Exponential LPP — Johansson (2000)

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• Similar results have been proved by Majumdar and Vergassola relying on Coulomb gas methods.

Refined LDP results

Theorem

Let $M = N$ and let $\delta > 0$ be fixed. Then

$$\log \mathbb{P}(\lambda_1 > 4 + \delta) = -NI(\delta) - \log N + O(1)$$

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- $I(0) = 0.$
- $I'(\delta), I''(\delta)$ converge to 1 and zero respectively as δ goes to infinity.

Eigenvalue density

$\Lambda_N := \{(\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{R}^N : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N\}$. The joint eigenvalue density of the scaled Wishart matrix $(\frac{1}{M} X^* X)$ is given by:

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- Partition function $Z_{M,N}$ is given by

$$Z_{M,N} = \frac{\prod_{j=0}^{N-1} j!(M-N+j)!}{M^{NM}}.$$

Asymptotics of Partition function

$$\log \frac{Z_{M-1,N-1}}{Z_{M,N}} = 2N + M - N \log \frac{N}{M} + O(1).$$

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where $\lambda^{(1)} = (\lambda_2 \geq \lambda_3 \geq \ldots, \geq \lambda_n)$ and the inside integral is restricted to $\lambda_2 < \lambda_1$.

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Along with the precise estimate for the partition function this yields that (for a fixed L):

$$\int_{L>\lambda_1\geq(4+\delta)}f_{n,n}(\underline{\lambda})d\underline{\lambda}=\int_{L>\lambda_1>(4+\delta)}e^{-nI(\lambda_1-4)+O(1)}d\lambda_1$$

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• The lower bound will follow by just considering the first term in the sum.

Key rigidity results used to make the previous discussion rigorous.

Concentration via Log-Sobolev inequality

X is an $N \times M$ $(N \leq M)$ Complex Gaussian Matrices;
$\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N$ are the eigenvalues of $\frac{1}{N} XX^*$.

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Theorem (Guionnet, Zeitouni)

For any Lipschitz f, there exists $C > 0$ depending on the Lipschitz constant of f such that for all M, N and all $\delta > 0$ we have

$$\mathbb{P}\left(|\mathrm{tr}(f) - \mathbb{E}(\mathrm{tr}(f))| \geq \delta \frac{M+N}{N} \right) \leq e^{-C\delta^2 (M+N)^2}.$$

Square case

Theorem (Goetze-Tikhomirov '14)

Let $M = N$ and let ESM *denote the expected empirical spectral distribution of* $\frac{1}{M}XX^*$*. There exists an absolute constant* C *such that* $d_{\mathrm{KS}}(\mathsf{ESM}, \mathsf{MP}) \leq CN^{-1}$ *for all* N *where* $d_{\mathrm{KS}}(\cdot, \cdot)$ *denote the Kolmogorov-Smirnov distance between two distributions.*

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Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1 and $||f'||_1 < \infty$. Integration by parts implies that

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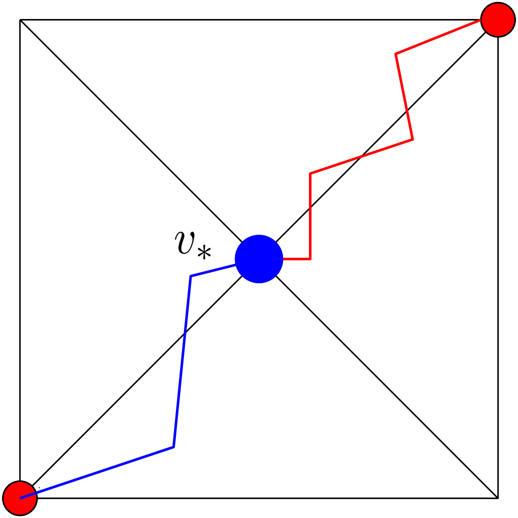
- There are similar results by Bai-Silverstein, Guionnet, Johansson.

Transversal fluctuation lower bound



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Lemma

Fix $\delta > 0$. There exists a constant $C = C(\delta) > 0$ such that we have for all n sufficiently large

$$\mathbb{P}(T_{\mathbf{0},v_*} + T'_{v_*,\mathbf{n}}) \geq (4+\delta)n \mid \mathcal{U}_\delta(n)) \leq \frac{C}{\sqrt{n}}$$

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•The LHS is bounded by sum over terms like

$$\mathbb{P}(T_{\mathbf{0},v_*} \geq (4+\delta_1)n/2)\mathbb{P}(T_{v_*,\mathbf{n}} \geq (4+\delta_2)n/2)$$

with $\delta_1 + \delta_2 \geq \delta$ and the precise LDP result for each of them along with convexity of $I(\delta)$.

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- This shows delocalization at scale $\sqrt{n}$.

Stochastic inequalities for determinantal process

Let Y be an $(M+1) \times (N-1)$ matrix with standard complex Gaussian entries, and let $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_{N-1}$ denote the eigenvalues of Y^*Y.

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There exists a coupling such that almost surely

$$(\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_{N-1}) \subset (\lambda_1, \lambda_2, \cdots, \lambda_N).$$

In particular we have $\lambda_1 \succeq \tilde{\lambda}_1$, where $\succeq$ denotes stochastic domination.

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- We thank Manjunath Krishnapur for showing us how to prove this result.

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- Thus we have to analyze λ_1 for non-square Wishart matrices ($M \times N$ where $M = N + o(N)$) and using Coulomb gas methods would need to rely on rigidity results in this context.

Upper bound

- To prove the upper bound one needs to understand precise LDP for passage times $T_{(1,1),(\frac{N}{2}-C,\frac{N}{2}+C)}$.
- Thus we have to analyze λ_1 for non-square Wishart matrices ($M \times N$ where $M = N + o(N)$) and using Coulomb gas methods would need to rely on rigidity results in this context.
- Luckily they are available.

For $j = 1, 2, \ldots, N$ let $\gamma_j = \gamma_{j,M,N}$ denote the classical location of the eigenvalues of $\frac{1}{M} XX^*$, i.e., $\gamma_{j,M,N}$ are the solutions of the equations

$$\int_{(1-\sqrt{y})^2}^{\gamma_{j,M,N}} d\mathsf{MP}_y(x) = 1 - \frac{j}{N}$$

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Theorem (B-Y-Y (2013))

For $c > 0$, let $\mathcal{E}_c$ denote the event that

$$\{\exists j \in [(\log N)^{c \log\log N}, N - (\log N)^{c \log\log N}] \text{ such that } |\lambda_j - \gamma_j| \geq \frac{c(\log N)^{c \log\log N}}{\min(j, N+1-j)^{\frac{1}{3}} N^{\frac{2}{3}}}\}.$$

There exists $c > 0$ such that for all sufficiently large N

$$\mathbb{P}(\mathcal{E}_c) \leq e^{-(\log N)^{c \log\log N}}.$$

Asymmetric case

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Theorem

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where for $y \in (0,1]$ and $\delta > 0$,

$$J_y(\delta) := \int \log(4+\delta-x) d\mathsf{MP}_y$$

$$I_y(\delta) := -(2+y^{-1}) + \log y + 1 + (4+\delta)y^{-1} - (y^{-1}-1)(\log(4+\delta)) - 2J_y(\delta),$$

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- One can replace the $O(N^\varepsilon)$ term by $-\log(N) + O(1)$ term with Bai-Silversteins' result.

Key comparison of rate functions

- $m_1 = n + c, n_1 = n - c.$
- $(4 + \hat{\delta})m_1 = (4 + \delta)n.$

Lemma

$$\mathbb{P}_{m_1,n_1}\left(\lambda_1 \geq (4 + \hat{\delta})\right) = \mathbb{P}_{n,n}\left(\lambda_1 \geq (4 + \delta)\right) e^{-\beta_\delta(\frac{c^2}{n}) + O(\frac{c^3}{n^2} + n^\varepsilon)},$$

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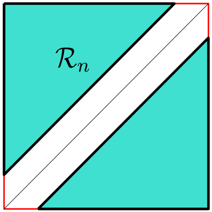

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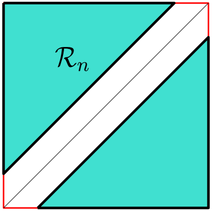
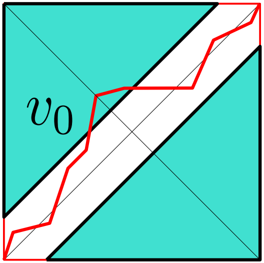


Clearly it suffices to show that

$$\sum_{v \in \mathcal{R}_n} \frac{\mathbb{P}(\ell(\Gamma_n(v)) \geq (4+\delta)n)}{\mathbb{P}(T_n \geq (4+\delta)n)} = o(1).$$

Localization at Midpoint



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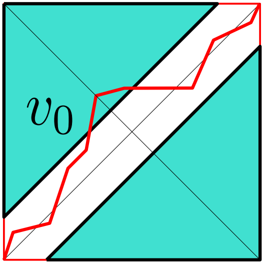

We will show for $v_0 \in \mathcal{R}_n$:

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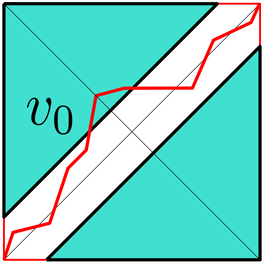

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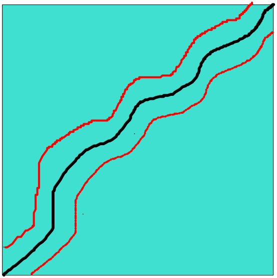
- We see that the whole thing is at most

$$e^{-\frac{c^2}{n}}\mathrm{Poly}(n)\mathbb{P}(T_{\mathbf{0},\mathbf{n}}\geq(4+\delta)n).$$

Transversal fluctuation for lower tail



- Let $\gamma : [0,1] \to [0,1]$ be a continuous increasing surjection.
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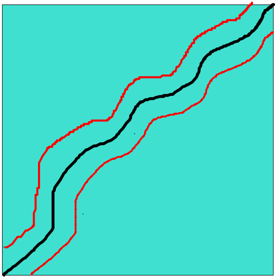

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Theorem (Basu, G., Sly (2017))

Fix $\delta \in (0,4)$. Given any γ as above, and $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that for all large enough n,

$$\mathbb{P}(\Gamma_n \subseteq \gamma_n^{\varepsilon'} \mid T_n \leq (4-\delta)n) \leq \varepsilon.$$

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4. Formalizing this heuristic requires two ingredients.

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Proposition

Fix $\delta \in (0,4)$. Given any $\varepsilon > 0$ there exists $H > 0$ such that

$$\mathbb{P}\left(T_n \geq (4-\delta)n - \frac{H}{n} \mid T_n \leq (4-\delta)n\right) \geq 1-\varepsilon.$$

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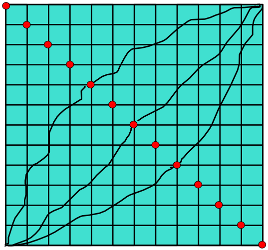

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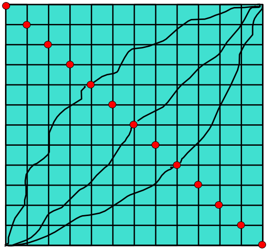

M_v is precisely the value that would make the longest path passing through v have weight $(4-\delta)n$.

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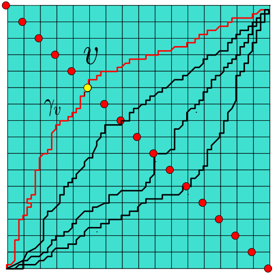

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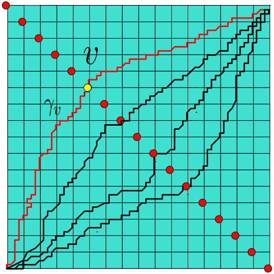

Now the theorem follows from an observation similar to the one which says maximum of n independent $U[0,1]$ variables is typically like $1 - O(\frac{1}{n})$.

Anti-concentration of the best path in a thin strip

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Proposition

Fix $\delta \in (0, 4)$. Given any H and $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that for every deterministic set $A \subseteq [0, n]^2 \cap \mathbb{Z}^2$, with $|A| \leq \varepsilon' n^2$ we have

$$\mathbb{P}\left(T_n(A) \geq (4 - \delta)n - \frac{H}{n} \mid T_n \leq (4 - \delta)n\right) \leq \varepsilon.$$

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Theorem (Basu, G., Sly (2017))

Fixing $\delta \in (0,2)$, for any increasing continuous $\gamma : [0,1] \to [0,1]$ with $\gamma(0) = 0$ and $\gamma(1) = 1$, there exists $\varepsilon > 0$, such that

$$\mathbb{P}(\mathcal{E}_{\gamma,n} \mid T_n \leq (2-\delta)n) \to 1$$

as $n \to \infty$, where $\mathcal{E}_{\gamma,n}$ denotes the event that there exists a polymer Γ_n between $(0,0)$ and (n,n) that is not contained in γ_n^{ε}.

Beyond integrable settings

- Proofs do not use any inputs from integrable probability.
- Properties of Exponential distribution makes the calculation easier and more transparent.
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Theorem (Basu, G., Sly (2017))

Let F be a probability measure on $[0, \infty)$ that has continuous and non-increasing density and enough moments (or log-concave density). For $\delta \in (0, \mu_F)$ and $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that for all $\gamma : [0,1] \to [0,1]$ surjective and increasing one has

$$\mathbb{P}(\Gamma_n \subseteq \gamma_n^{\varepsilon'} \mid T_n \leq (\mu_F - \delta)n) \leq \varepsilon.$$

- The key thing analyzed is the conditional distribution of the sum of a bunch of i.i.d. random variables conditioned on their projection on the unit L_1 ball.

Final remarks

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- There has been a lot of recent activity trying to understand LDP behavior for the KPZ equation, Corner growth height process etc. relying on integrable inputs.
- In a recent work with Basu and Sly, we prove existence of the rate function for various non-integrable models like First Passage Percolation.
- One open question: Does the polymer conditioned on the upper tail event converge to a Brownian bridge? A first step would be to show that the transversal fluctuation at the midpoint is given by a Gaussian.

Final remarks

- There has been a lot of recent activity trying to understand LDP behavior for the KPZ equation, Corner growth height process etc. relying on integrable inputs.
- In a recent work with Basu and Sly, we prove existence of the rate function for various non-integrable models like First Passage Percolation.
- One open question: Does the polymer conditioned on the upper tail event converge to a Brownian bridge? A first step would be to show that the transversal fluctuation at the midpoint is given by a Gaussian.

Thank You