Yang–Mills for mathematicians

Infosys-ICTS Ramanujan Lectures: Lecture 1

Sourav Chatterjee

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- Physicists are generally familiar with most of what I'm going to say, but mathematicians are not. This talk is for mathematicians.

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- What is a QFT? — This is an open question, not only in mathematics, but also in physics.
- Remarkably, physicists can calculate and make surprisingly accurate predictions using QFTs, without really understanding what these objects are!
- The mathematical construction of quantum field theories — more specifically Yang–Mills theories — is one of the seven millennium problems posed by the Clay Institute.

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  - To a different observer, who is using a coordinate system obtained by the action of (a, Λ) on the coordinate system of the first observer, the state appears as $U(a, \Lambda)\psi$.

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- If a different observer was at (x,y,z) at time 0, and moves with constant velocity $\mathbf{v}$ for time t, he will observe the system to be in state $U((-t,x,y,z),\Lambda)\psi$, where Λ is a restricted Lorentz transformation depending on $\mathbf{v}$.

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- The field φ is used for calculating probabilities of events and expected values of various observables. In fact, it becomes the central object of interest in the study of the system.

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- It has been possible to construct certain simple QFTs, known as free fields, which satisfy the Wightman axioms.
- Free fields describe trivial systems of particles that do not interact with each other.
- *Unfortunately, no one has been able to rigorously construct a nontrivial (interacting) QFT satisfying the Wightman axioms.*

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- So it is not clear how one can justify such a perturbative expansion. *In fact, in most cases it is not clear what the new Hilbert space is!*
- And yet, in many cases, these calculations yield results that match experiments to remarkable degrees of accuracy.

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- The program, initiated in the 60s, was successful in constructing nontrivial QFTs when the dimension of spacetime was reduced from 4 to 2 or 3 — but not yet in dimension 4.
- The most notable achievements were the constructions of φ_2^4 and φ_3^4 theories (in spacetime dimensions 2 and 3, respectively).

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- However, the investigation was inconclusive and the question is still considered to be open.
- Even the first step in the probabilistic approach, namely, the construction of a random field, remains open. We will now talk about that.

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- For Yang–Mills theories, these random fields are called Euclidean Yang–Mills theories.
- These have not yet been constructed in spacetime dimensions 3 and 4.
- Euclidean Yang–Mills theories are supposed to be scaling limits of lattice gauge theories, which are well-defined discrete probabilistic objects, which I will now discuss.

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- Let $G(\Lambda)$ denote the set of all such configurations.

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- The Wilson action of U is defined as

$$S_{\mathrm{W}}(U) := \sum_{p \in P(\Lambda)} \mathrm{Re}(\mathrm{Tr}(I - U_p)).$$

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- An infinite volume limit of the theory is a weak limit of the above probability measures as $\Lambda \uparrow \mathbb{Z}^d$ (may not be unique).

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- Finally, one has to construct the actual QFT using this field, via the Osterwalder–Schrader axioms or otherwise.

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- Hairer's theory of regularity structures recently allowed a different construction of φ_3^4 theory via stochastic quantization. Ongoing work by Hairer and collaborators to extend this approach to 3D Yang–Mills theories.

Open problem #2: Mass gap

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- Various Yang–Mills theories — such as 4D Yang–Mills theory with gauge group $SU(3)$ — are supposed to have mass gaps.
- The first step to showing this is to show that the corresponding lattice gauge theories have exponential decay of correlations at large β.

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- There are a number of simpler models where exponential decay is conjectured at large β but not proved. Examples: 2D Ising type models with vector spins (the 2D $O(N)$ models with $N \geq 3$).
- A rare example where this has been proved recently: The loop $O(N)$ model, a toy version of the $O(N)$ model, by Duminil-Copin et al.

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- Quarks are always bound, and never occur freely in nature. This is known as quark confinement or color confinement.
- Wilson argued that this phenomenon occurs due to a mathematical feature of Yang–Mills theories, that is now called Wilson's area law.

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- Showing for rectangles is good enough.

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- Disproof at large β for 4D $U(1)$ theory by Guth (1980) and Fröhlich and Spencer (1982). *Therefore in 4D at large β, it is crucial that the gauge group is non-Abelian.*

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- Maldacena's discovery is known as AdS-CFT duality or gauge-string duality or gauge-gravity duality.

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- In 2015, I proved such a result for lattice gauge theories at small β — probably the first mathematical theorem in this area. This will be the topic of the next talk.
- However, this is a discrete result. It is an open problem to prove such a theorem when β is large. We need to consider large β for passing to the continuum limit.

Where to read about all this

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