Entanglement entropy of highly excited eigenstates of many-body lattice Hamiltonians

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Outline

1. Introduction
   - Entanglement entropies and experiments
   - Integrability vs nonintegrability, energy levels
   - Integrability vs nonintegrability, entanglement entropy
2. Integrable systems
   - S_{vN} in eigenstates of quadratic fermionic Hamiltonians
3. Nonintegrable systems
   - Eigenstate thermalization and quantum chaos
   - Random matrix theory
   - S_{vN} in eigenstates of quantum chaotic Hamiltonians
4. Summary

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von Neumann and Renyi entanglement entropies

Let $|m\rangle$ be a state ket of a many-body system. The reduced density matrix

$$\hat{\rho}_A(m) = \mathrm{Tr}_{\bar{A}}(|m\rangle\langle m|),\ A \text{ is the subsystem of interest and } \bar{A} \text{ is its complement}$$

The von Neumann and n-Renyi entanglement entropies are defined as:

$$S_{\mathsf{vN}}(m) = -\mathrm{Tr}[\hat{\rho}_A(m) \ln \hat{\rho}_A(m)] \quad \text{and} \quad S_n(m) = \frac{1}{1-n} \ln \mathrm{Tr}[\hat{\rho}_A(m)^n]$$

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Ground-state entanglement entropies (area laws, quantum phase transitions)

Srednicki'93; Osterloh, Amico, Falci & Fazio'02; Osborne & Nielsen'02;
Vidal, Latorre, Rico & Kitaev'03; Calabrese & Cardy'04, . . . , Eisert, Cramer & Plenio, RMP'10

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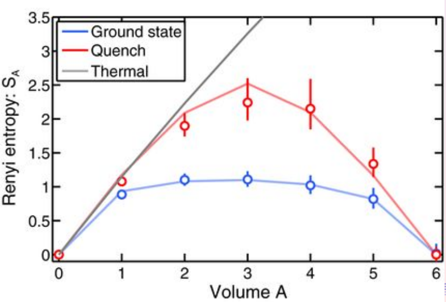
S_2 has been measured in experiments with ultracold quantum gases



Kaufman et al. (Greiner's group), Science **353**, 794 (2016).

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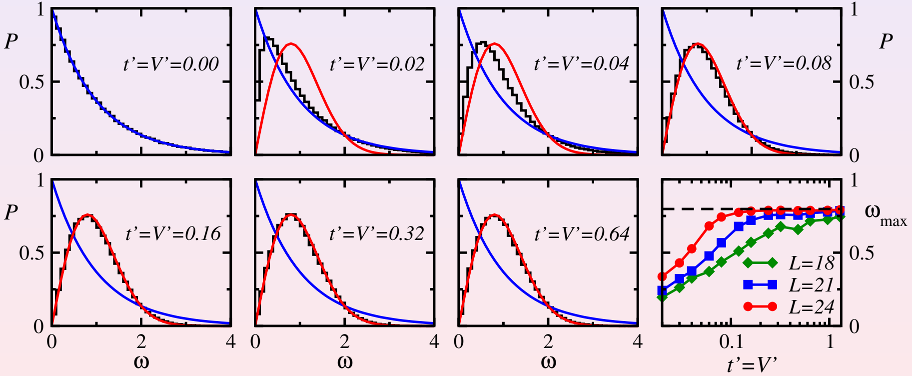
Integrability to quantum chaos transition

Spinless fermions (hard-core bosons, spin-1/2) in one dimension

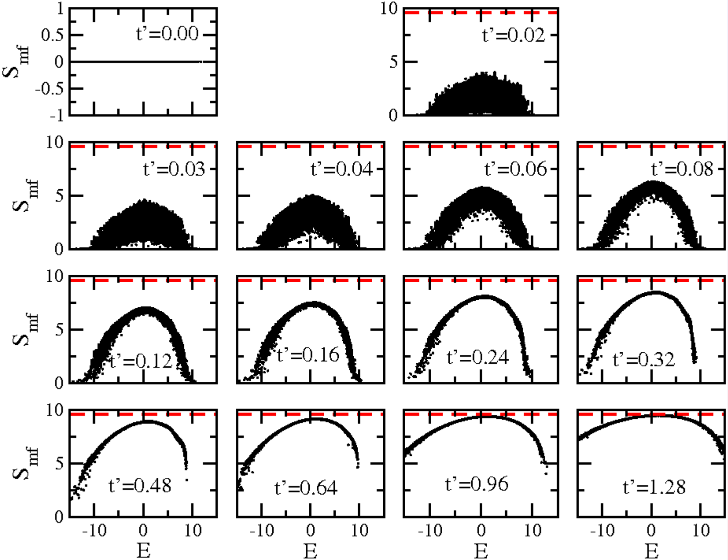
$$\hat{H} = \sum_{i=1}^{L} \left\{ -t \left(\hat{f}_i^\dagger \hat{f}_{i+1} + \text{H.c.} \right) + V \hat{n}_i \hat{n}_{i+1} - t' \left(\hat{f}_i^\dagger \hat{f}_{i+2} + \text{H.c.} \right) + V' \hat{n}_i \hat{n}_{i+2} \right\}$$

Level spacing distribution ($N_f = L/3$)



L. F. Santos and MR, PRE **81**, 036206 (2010); PRE **82**, 031130 (2010).

Information entropy ($S_j = -\sum_{k=1}^{D} |c_j^k|^2 \ln |c_j^k|^2$)



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Typicality (uniform distribution in the unit sphere)

Average (vN) entanglement entropy of subsystems of random pure states

$$S_{\mathrm{ave}} \simeq \ln \mathcal{D}_{\mathrm{A}} - (1/2)\mathcal{D}_{\mathrm{A}}^2/\mathcal{D},$$

for $1 \ll \mathcal{D}_{\mathrm{A}} \leq \sqrt{\mathcal{D}}$.
D. N. Page, PRL **71**, 1291 (1993).

S_{vN} for typical pure states exhibits a volume law ($\ln \mathcal{D}_{\mathrm{A}} \propto V_{\mathrm{A}}$ and $\ln \mathcal{D} \propto V$).
Typical pure states are (nearly) maximally entangled
(the correction is exponentially small for $V_{\mathrm{A}} < V/2$. It is $1/2$ for $V_{\mathrm{A}} = V/2$).

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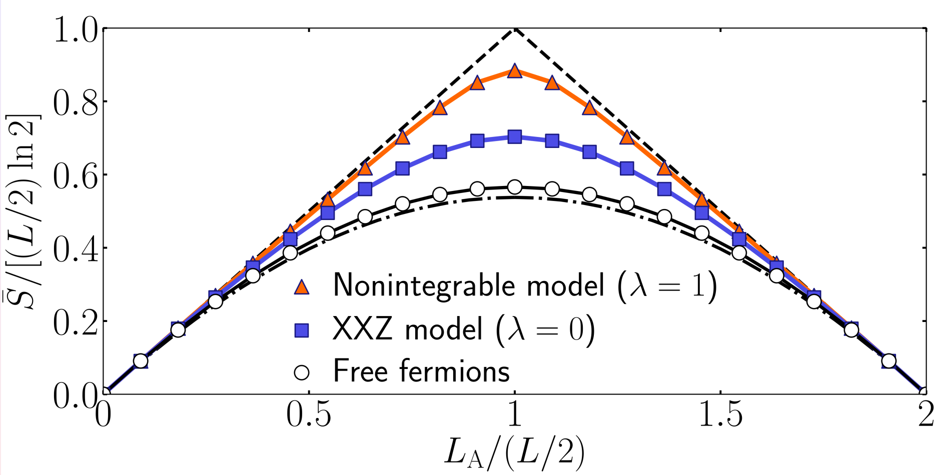
Spin-1/2 XXZ chain with next nearest neighbors interactions:

$$\hat{H} = \sum_{i=1}^{L} \left[\frac{1}{2}\left(\hat{S}_i^+ \hat{S}_{i+1}^- + \text{H.c.}\right) + \Delta \hat{S}_i^z \hat{S}_{i+1}^z \right] + \lambda \sum_{i=1}^{L} \left[\frac{1}{2}\left(\hat{S}_i^+ \hat{S}_{i+2}^- + \text{H.c.}\right) + \frac{1}{2} \hat{S}_i^z \hat{S}_{i+2}^z \right].$$

$\Delta = 0$, $\lambda = 0$: Mappable to free fermions
$\Delta \neq 0$, $\lambda = 0$: Interacting integrable model
$\Delta \neq 0$, $\lambda \neq 0$: Nonintegrable model

Integrability vs nonintegrability and entanglement

Average entanglement entropy ($L_A = L/2$, $L = 20$, $S^z_{\text{total}} = 0$)



T. LeBlond, K. Mallayya, L. Vidmar, and MR, arXiv:1909.09654.

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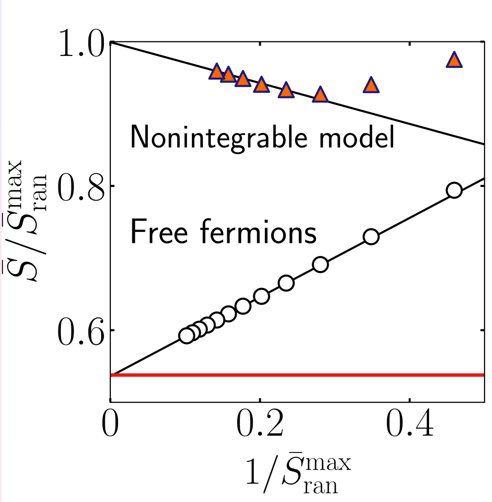
Average entanglement entropy ($L_A = L/2$, $L = 22$, $S^z_{\text{total}} = 0$)



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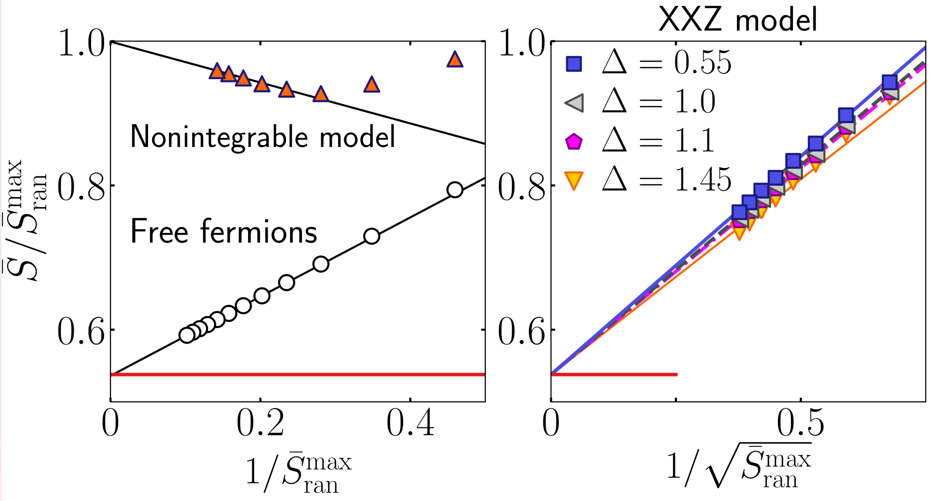

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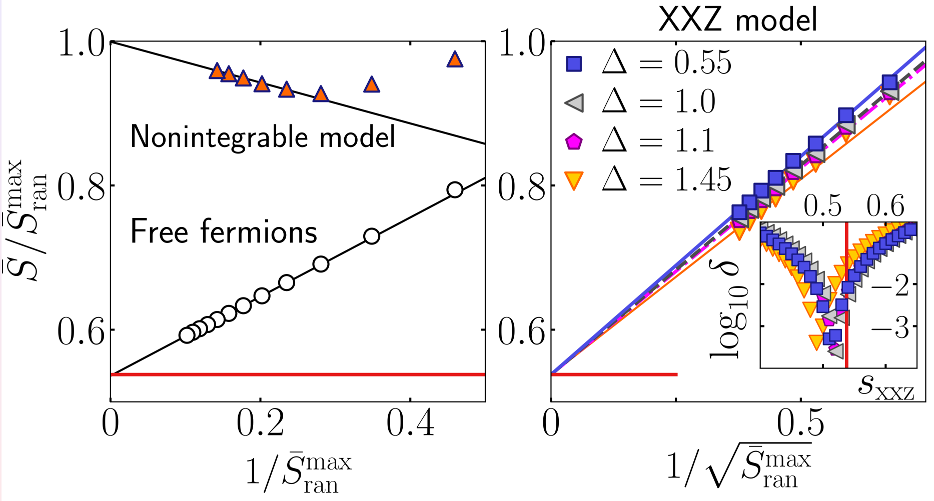

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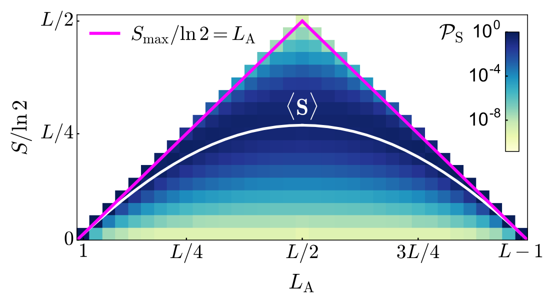
S_{vN} in eigenstates of quadratic fermionic Hamiltonians

Free fermions in 1D ($\hat{H} = -\sum_{\langle i,j \rangle}^{L} (\hat{f}_i^\dagger \hat{f}_j + \mathrm{H.c.})$, $L = 36$ sites)



L. Vidmar, L. F. Hackl, E. Bianchi, and MR, PRL **119**, 020601 (2017).
Earlier results in 2D, M. Storms and R. R. P. Singh, PRE **89**, 012125 (2014).

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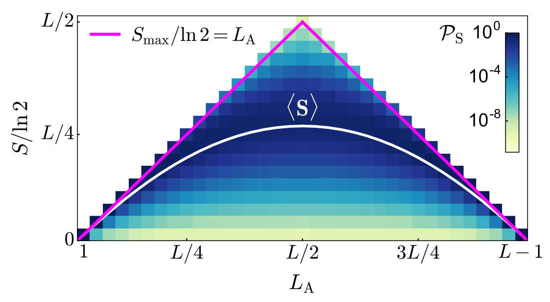

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Translational invariant quadratic Hamiltonian

$$\hat{H} = -\sum_{i,j=1}^{V}(t_{ij}\hat{f}_i^\dagger \hat{f}_j + \Delta_{ij}\hat{f}_i^\dagger \hat{f}_j^\dagger + \Delta_{ij}^* \hat{f}_j \hat{f}_i), \quad \Delta_{ij} = -\Delta_{ji} \text{ and } t_{ij} = t_{ji}^*$$

It is diagonalizable via a Bogoliubov transformation: $\hat{f}_i = \sum_{l=1}^{V}(\alpha_{il}\hat{c}_l + \beta_{il}\hat{c}_l^\dagger)$.
It is straightforward find all many-body eigenstates $|m\rangle$.

S_{vN} in eigenstates of quadratic fermionic Hamiltonians

The $(2V \times 2V)$ one-body correlation matrix fully characterizes the state

$$F = \left(\begin{array}{c|c} \langle m|\hat{f}_i^\dagger \hat{f}_j - \hat{f}_j \hat{f}_i^\dagger |m\rangle & \langle m|\hat{f}_i^\dagger \hat{f}_j^\dagger - \hat{f}_j^\dagger \hat{f}_i^\dagger |m\rangle \\ \hline \langle m|\hat{f}_i \hat{f}_j - \hat{f}_j \hat{f}_i |m\rangle & \langle m|\hat{f}_i \hat{f}_j^\dagger - \hat{f}_j^\dagger \hat{f}_i |m\rangle \end{array} \right)$$

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The entanglement entropy

$$S_{\text{vN}} = -\mathrm{Tr}\left\{\left(\frac{\mathbb{1} + [F]_A}{2}\right) \ln \left(\frac{\mathbb{1} + [F]_A}{2}\right)\right\},$$

where $[F]_A$ is the $2V_{\mathrm{A}} \times 2V_{\mathrm{A}}$ matrix with $i, j \in A$.

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One can prove that the eigenvalues λ_j of $[F]_A$ are real with $|\lambda_j| \leq 1$ so

$$0 \leq \text{Tr}[F]_\text{A}^{2(m+1)} \leq \text{Tr}[F]_\text{A}^{2m} \leq 2V_\text{A}$$

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Hence, the expansion

$$S_{\text{vN}} = V_\text{A} \ln 2 - \sum_{n=1}^{\infty} \frac{\text{Tr}[F]_\text{A}^{2n}}{4n(2n-1)}$$

is convergent. Higher-order terms lower the average entanglement entropy.

S_{vN} in eigenstates of quadratic fermionic Hamiltonians

We are interested in the average over all eigenstates

$$\langle S_{\mathsf{vN}} \rangle = V_{\mathrm{A}} \ln 2 - \sum_{n=1}^{\infty} \frac{\langle \mathrm{Tr}[F]_{\mathrm{A}}^{2n} \rangle}{4n(2n-1)}$$

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$$V_{\mathrm{A}} \ln 2 - \frac{\langle \mathrm{Tr}[F]_{\mathrm{A}}^{2} \rangle}{2} \ln 2 \leq \langle S_{\mathsf{vN}} \rangle \leq V_{\mathrm{A}} \ln 2 - \frac{\langle \mathrm{Tr}[F]_{\mathrm{A}}^{2} \rangle}{4}$$

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The "first order" (results for $n = 1$) bounds are universal in the thermodynamic limit

$$V_{\text{A}} \left(\ln 2 - \ln 2 \frac{V_{\text{A}}}{V} \right) \leq \langle S_{\text{vN}} \rangle \leq V_{\text{A}} \left(\ln 2 - \frac{1}{2} \frac{V_{\text{A}}}{V} \right)$$

Typical eigenstates have maximal entanglement entropy for $V_{\text{A}}/V \to 0$.

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The “second order” bounds (results for up to $n = 2$) for free fermions in 1D are

$$L_{\text{A}} \left[\ln 2 - \frac{1}{2} \frac{L_{\text{A}}}{L} - (2 \ln 2 - 1) \left(\frac{4}{3} \frac{L_{\text{A}}^2}{L^2} - \frac{L_{\text{A}}^3}{L^3} \right) \right] \le \langle S_{\text{vN}} \rangle \le L_{\text{A}} \left[\ln 2 - \frac{1}{2} \frac{L_{\text{A}}}{L} - \frac{2}{9} \frac{L_{\text{A}}^2}{L^2} + \frac{1}{6} \frac{L_{\text{A}}^3}{L^3} \right]$$

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$S_{\rm vN}$ in eigenstates of quadratic fermionic Hamiltonians

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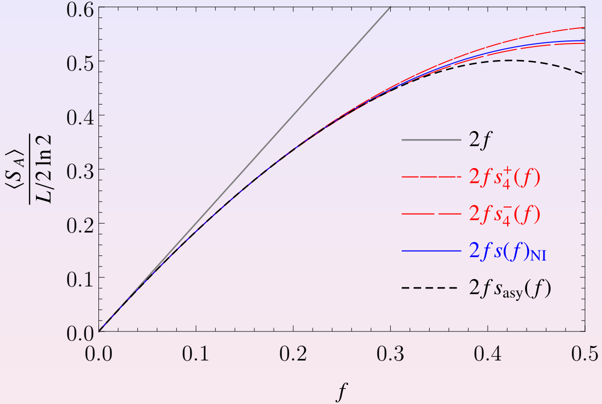

L. Hackl, L. Vidmar, MR, and E. Bianchi, Phys. Rev. B **99**, 075123 (2019).

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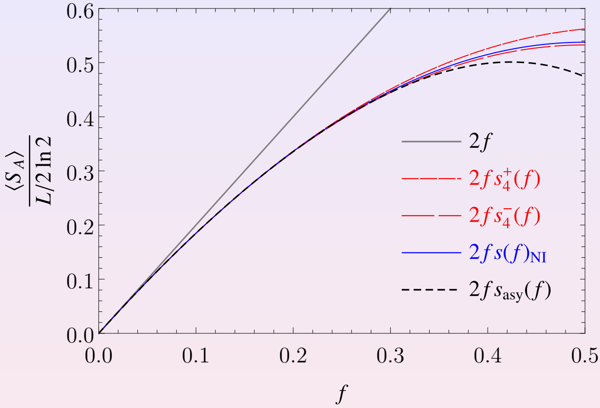

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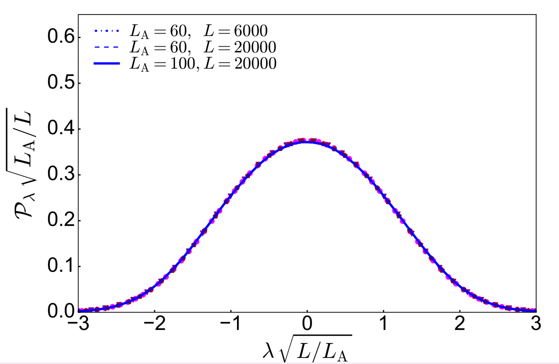
Normalized width of the distribution of S_{vN} for eigenstates

$$\Delta = \frac{\sqrt{\langle S^2 \rangle - \langle S \rangle^2}}{L_{\mathrm{A}} \ln 2}$$

vanishes as $1/\sqrt{L}$ or faster.

Toeplitz Gaussian ensemble (TGE)

Distribution of eigenvalues of $[F]_\mathrm{A}$ for $L_A/L \to 0$



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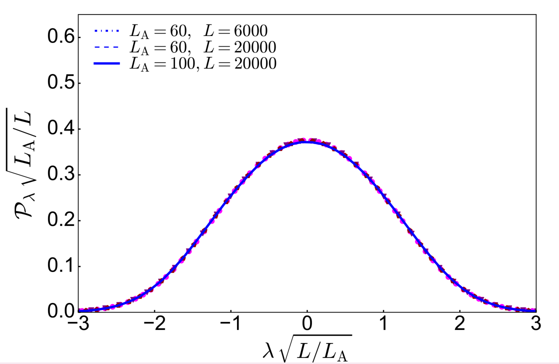

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- Replace the matrix elements of $[F]_{\mathrm{A}}$ by random complex numbers.
- Their absolute value is that of a normally distributed variable with zero mean and variance $1/L$
- Their phase is uniformly distributed between 0 and 2π

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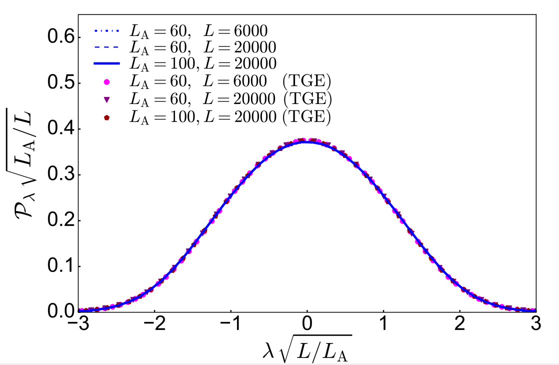

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- Their phase is uniformly distributed between 0 and 2π

Entanglement in the 1D TFIM

Hamiltonian: $\hat{H} = J \sum_{\langle \mathbf{i},\mathbf{j} \rangle} \hat{\sigma}^z_{\mathbf{i}} \hat{\sigma}^z_{\mathbf{j}} + g \sum_{\mathbf{i}} \hat{\sigma}^x_{\mathbf{i}}$

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Volume Law and Quantum Criticality in the Entanglement Entropy of Excited Eigenstates of the Quantum Ising Model

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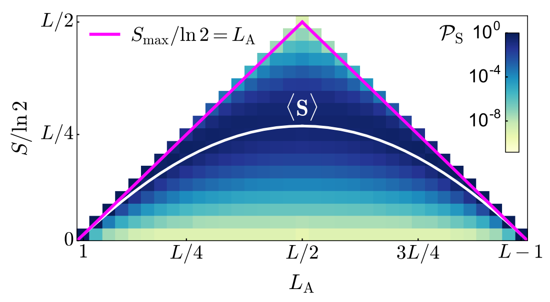
Much has been learned about universal properties of entanglement entropies in ground states of quantum many-body lattice systems. Here we unveil universal properties of the average bipartite entanglement entropy of eigenstates of the paradigmatic quantum Ising model in one dimension. The leading term exhibits a volume-law scaling that we argue is universal for translationally invariant quadratic models. The subleading term is constant at the critical field for the quantum phase transition and vanishes otherwise (in the thermodynamic limit); i.e., the critical field can be identified from subleading corrections to the average (over all eigenstates) entanglement entropy.

S_{vN} in eigenstates of quadratic fermionic Hamiltonians

Free fermions in 1D ($\hat{H} = -\sum_{\langle i,j \rangle}^{L} (\hat{f}_i^\dagger \hat{f}_j + \text{H.c.})$, $L = 36$ sites)



L. Vidmar, L. F. Hackl, E. Bianchi, and MR, PRL **119**, 020601 (2017).
Similar results in 2D, M. Storms and R. R. P. Singh, PRE **89**, 012125 (2014).

Translational invariant quadratic Hamiltonian

$$\hat{H} = -\sum_{i,j=1}^{V} (t_{ij} \hat{f}_i^\dagger \hat{f}_j + \Delta_{ij} \hat{f}_i^\dagger \hat{f}_j^\dagger + \Delta_{ij}^* \hat{f}_j \hat{f}_i), \quad \Delta_{ij} = -\Delta_{ji} \ \text{ and } \ t_{ij} = t_{ji}^*$$

It is diagonalizable via a Bogoliubov transformation: $\hat{f}_i = \sum_{l=1}^{V} (\alpha_{il} \hat{c}_l + \beta_{il} \hat{c}_l^\dagger)$.
It is straightforward find all many-body eigenstates $|m\rangle$.

Controlled numerical experiments

Hamiltonian of the form: $\hat{H}_{\text{nonint}}(\Lambda) = \hat{H}_{\text{int}} + \Lambda\hat{W}$

- $\hat{H}_{\text{int}}$: integrable (strongly interacting) Hamiltonian
- $\hat{H}_{\text{nonint}}$: nonintegrable (strongly interacting) Hamiltonian

Hard-core bosons in 1D:

$$\hat{H}_{\text{int}} = \sum_i \left[-\hat{b}_i^\dagger \hat{b}_{i+1} - \text{H.c.} + \left(\hat{n}_i - \frac{1}{2}\right)\left(\hat{n}_{i+1} - \frac{1}{2}\right) \right]$$

$$\hat{W} = \sum_i \left[-\hat{b}_i^\dagger \hat{b}_{i+2} - \text{H.c.} + \left(\hat{n}_i - \frac{1}{2}\right)\left(\hat{n}_{i+2} - \frac{1}{2}\right) \right]$$

MR, PRL **116**, 100601 (2016).

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Quenches:

- Initial state is a thermal equilibrium state (at a temperature T_I) of $\hat{H}_{\text{nonint}}$ with some Λ_I
- Evolve unitarily under $\hat{H}_{\text{int}}$ (quench $\Lambda \to 0$)

MR, PRL **116**, 100601 (2016).

Numerical linked cluster expansions (NLCEs)

Linked-cluster theorem: Extensive observables O per site $\mathcal{O}$ in a lattice

$$\mathcal{O} = \sum_c L(c) \times W_O(c),$$

where $L(c)$ is the multiplicity of cluster c (ways per site in which it can be embedded on the lattice),

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  $\hat{\rho}_c^{\text{GC}} = \frac{1}{Z_c^{\text{GC}}} \exp^{-\beta\left(\hat{H}_c - \mu \hat{N}_c\right)}$, expand $O(c)$ in powers of β

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- NLCEs for quantum quenches:
  Diagonal ensemble: $\hat{\rho}_c^{\mathrm{DE}} \equiv \lim_{\tau' \to \infty} \frac{1}{\tau'} \int_0^{\tau'} d\tau\, \hat{\rho}_c(\tau) = \sum_\alpha W_\alpha^c |\alpha_c\rangle\langle\alpha_c|$
  MR, PRL **112**, 170601 (2014).
  Quantum dynamics: $\hat{\rho}_c(\tau)$
  K. Mallayya and MR, PRL **120**, 070603 (2018).
  (2D) White *et al.*, arXiv:1710.07696; Guardado-Sanchez *et al.*, PRX **8**, 021069 (2018).

Lack of thermalization after equilibration

$$\hat{I} = \frac{1}{L}\sum_i \left(\hat{n}_i - \frac{1}{2}\right)\left(\hat{n}_{i+1} - \frac{1}{2}\right)$$

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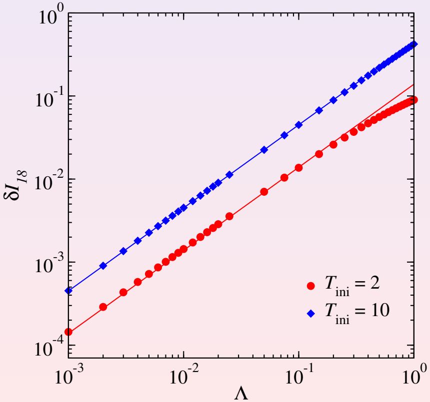

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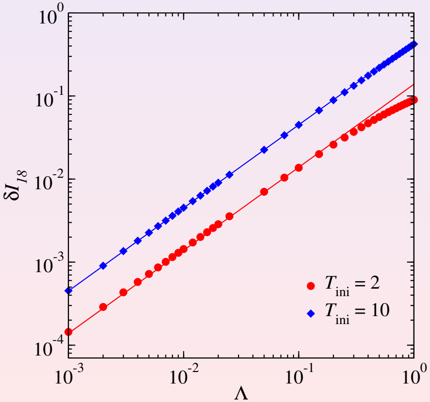

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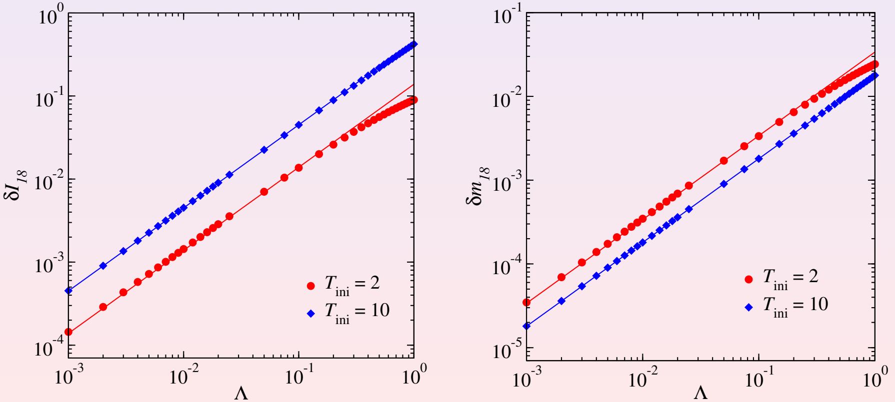

Lack of thermalization after equilibration

$$S^{\mathsf{GE}} = -\mathrm{Tr}[\hat{\rho}^{\mathsf{GE}} \ln \hat{\rho}^{\mathsf{GE}}] \quad \text{and} \quad S^{\mathsf{DE}} = -\mathrm{Tr}[\hat{\rho}^{\mathsf{DE}} \ln \hat{\rho}^{\mathsf{DE}}]$$

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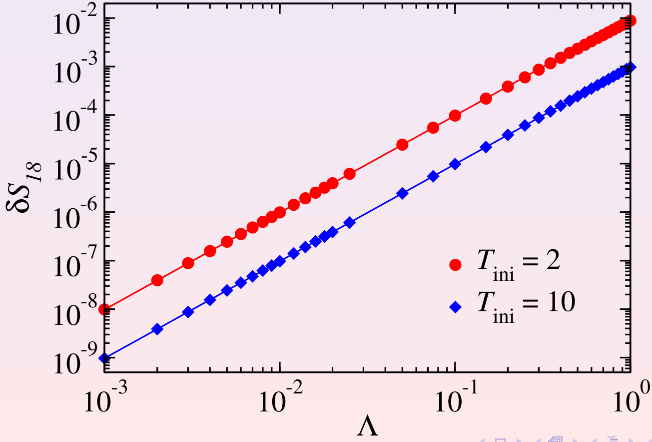

Outline

1. Introduction
   - Entanglement entropies and experiments
   - Integrability vs nonintegrability, energy levels
   - Integrability vs nonintegrability, entanglement entropy
2. Integrable systems
   - S_{vN} in eigenstates of quadratic fermionic Hamiltonians
3. Nonintegrable systems
   - Eigenstate thermalization and quantum chaos
   - Random matrix theory
   - S_{vN} in eigenstates of quantum chaotic Hamiltonians
4. Summary

Eigenstate thermalization hypothesis

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M. Srednicki, J. Phys. A **32**, 1163 (1999), L. D'Alessio *et al.*, Adv. Phys. **65**, 239 (2016).

$$O_{\alpha\beta} = O(E)\delta_{\alpha\beta} + e^{-S(E)/2} f_O(E,\omega) R_{\alpha\beta}$$

where $E \equiv (E_\alpha + E_\beta)/2$, $\omega \equiv E_\alpha - E_\beta$, $S(E)$ is the thermodynamic entropy at energy E, and $R_{\alpha\beta}$ is a random number with zero mean and unit variance.

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The t-V-t'-V' model for hard-core bosons in one dimension

$$\hat{H} = \sum_{i=1}^{L} \left\{ -t \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + V \hat{n}_i \hat{n}_{i+1} - t' \left(\hat{b}_i^\dagger \hat{b}_{i+2} + \text{H.c.} \right) + V' \hat{n}_i \hat{n}_{i+2} \right\}$$

Integrable for $t' = V' = 0$

(mappable onto the spin-1/2 XXZ chain)

$t', V' \neq 0$, for $t, V \neq 0$, break integrability

MR, PRL **103**, 100403 (2009); PRA **80**, 053607 (2009).

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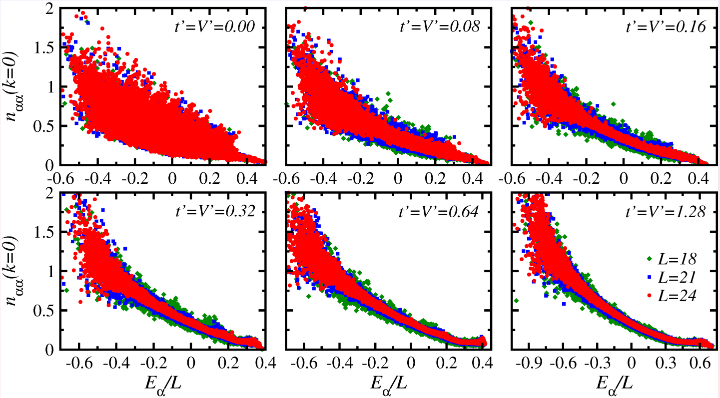
Occupation of momentum $k = 0$ (hard-core bosons, $t = V = 1$, $N_b = L/3$)



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Transverse (+ longitudinal) field Ising model in two dimensions

$$\hat{H} = J \sum_{\langle \mathbf{i},\mathbf{j} \rangle} \hat{\sigma}^z_{\mathbf{i}} \hat{\sigma}^z_{\mathbf{j}} + g \sum_{\mathbf{i}} \hat{\sigma}^x_{\mathbf{i}} + \varepsilon \sum_{\mathbf{i}} \hat{\sigma}^z_{\mathbf{i}}$$

Integrable for $J = 0$ or for $g = 0$
(not mappable onto noninteracting fermions)

$\varepsilon \neq 0$ breaks the Z_2 symmetry of the TFIM

Mondaini *et al.*, PRE **93**, 032104 (2016); Mondaini and MR, PRE **96**, 012157 (2017).

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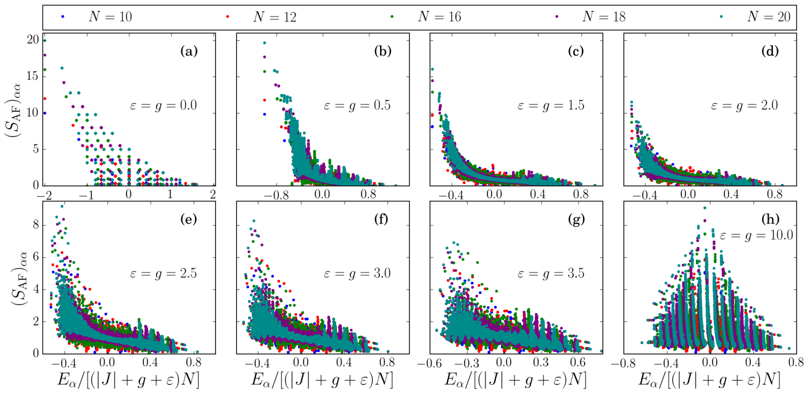
Diagonal matrix elements (2D AF-TFIM)



Mondaini, Fratus, Srednicki, and MR, PRE **93**, 032104 (2016).

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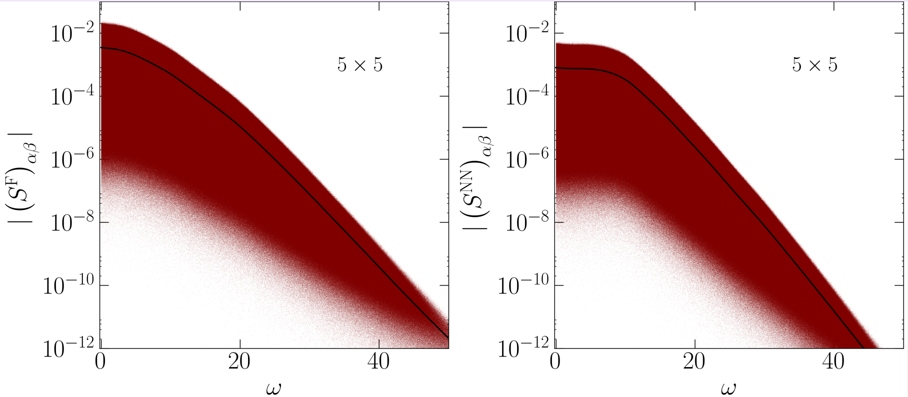
Off-diagonal matrix elements (2D F-TFIM): Results vs ω at constant E



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Outline

Random matrix theory

- Wigner (1955) & Dyson (1962): Statistical properties of the spectra of complex quantum systems (in a narrow energy window) can be predicted from the statistical properties of the spectra of random matrices (with the appropriate symmetries). It was used with great success to understand the spectra of complex nuclei.

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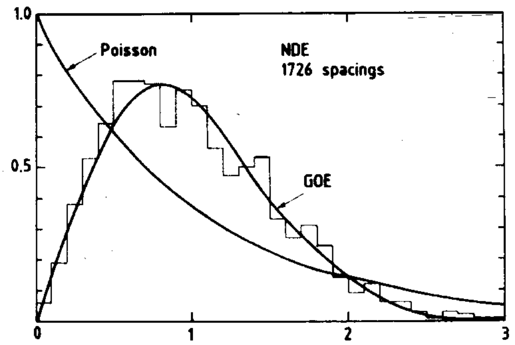
Distribution of level spacings for the “Nuclear Data Ensemble”



T. Guhr *et al.*, Physics Reports **299**, 189 (1998).

Distribution of level spacings $P(\omega)$

$P(\omega)$ can be understood using 2×2 matrices

$$\begin{bmatrix} \varepsilon_1 & \frac{V}{\sqrt{2}} \\ \frac{V^*}{\sqrt{2}} & \varepsilon_2 \end{bmatrix}, \qquad E_{1,2} = \frac{\varepsilon_1 + \varepsilon_2}{2} \pm \frac{1}{2}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + 2|V|^2}.$$

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For systems that are invariant under time reversal, $\hat{H}$ can be written as a real matrix. Draw ε_1, ε_2, and V from a Gaussian distribution with zero mean and variance σ.

$$\begin{aligned} P(\omega \equiv E_1 - E_2) \quad &= \quad \frac{1}{(2\pi)^{3/2}\sigma^3} \int d\varepsilon_1 \int d\varepsilon_2 \int dV \, \delta\left(\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + 2V^2} - \omega\right) \\ &\times \exp\left(-\frac{\varepsilon_1^2 + \varepsilon_2^2 + V^2}{2\sigma^2}\right). \end{aligned}$$

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$$P(\omega) = \frac{\omega}{2\sigma^2} \exp\left[-\frac{\omega^2}{4\sigma^2}\right]$$

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Wigner Surmise (Wigner-Dyson distribution)

$$P(\omega) = A_\beta \, \omega^\beta \, \exp[-B_\beta \omega^2], \ \text{where} \ \beta = 1 \text{ (GOE) and } \beta = 2 \text{ (GUE)}$$

Matrix elements of Hermitian operators within RMT

Let $\hat{O} = \sum_i O_i |i\rangle\langle i|$, where $\hat{O}|i\rangle = O_i|i\rangle$,

$$O_{\alpha\beta} \equiv \langle\alpha|\hat{O}|\beta\rangle = \sum_i O_i \langle\alpha|i\rangle\langle i|\beta\rangle = \sum_i O_i (\psi_i^\alpha)^* \psi_i^\beta$$

$|\alpha\rangle$ and $|\beta\rangle$ are eigenvectors of a random matrix. Averaging over $|\alpha\rangle$ and $|\beta\rangle$ (random orthogonal unit vectors in arbitrary bases): $\overline{(\psi_i^\alpha)^*(\psi_i^\beta)} = \frac{1}{\mathcal{D}}\delta_{\alpha\beta}$.

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This means that (to leading order):

$$\overline{O_{\alpha\alpha}} = \frac{1}{\mathcal{D}} \sum_i O_i \equiv \bar{O}, \quad \text{while} \quad \overline{O_{\alpha\beta}} = 0 \quad \text{for} \quad \alpha \neq \beta.$$

Matrix elements of Hermitian operators within RMT

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$$O_{\alpha\beta} \equiv \langle\alpha|\hat{O}|\beta\rangle = \sum_i O_i \langle\alpha|i\rangle\langle i|\beta\rangle = \sum_i O_i (\psi_i^\alpha)^* \psi_i^\beta$$

$|\alpha\rangle$ and $|\beta\rangle$ are eigenvectors of a random matrix. Averaging over $|\alpha\rangle$ and $|\beta\rangle$ (random orthogonal unit vectors in arbitrary bases): $\overline{(\psi_i^\alpha)^*(\psi_i^\beta)} = \frac{1}{\mathcal{D}}\delta_{\alpha\beta}$.

This means that (to leading order):

$$\overline{O_{\alpha\alpha}} = \frac{1}{\mathcal{D}} \sum_i O_i \equiv \bar{O}, \quad \text{while} \quad \overline{O_{\alpha\beta}} = 0 \quad \text{for} \quad \alpha \neq \beta.$$

One can also show that ($\eta = 2$ for GOE and $\eta = 1$ for GUE):

$$\overline{O_{\alpha\alpha}^2} - \overline{O_{\alpha\alpha}}^2 = \eta\overline{|O_{\alpha\beta}|^2} = \frac{\eta}{\mathcal{D}^2} \sum_i O_i^2 \equiv \frac{\eta}{\mathcal{D}}\overline{O^2}.$$

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Combining these results one can write

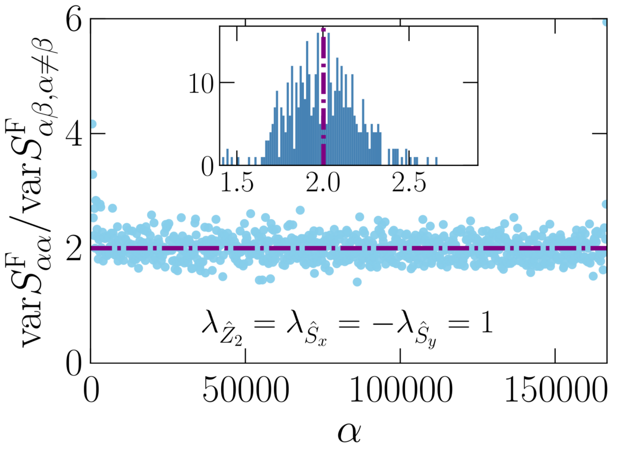
$$O_{\alpha\beta} \approx \bar{O}\delta_{\alpha\beta} + \sqrt{\frac{\overline{O^2}}{\mathcal{D}}} R_{\alpha\beta},$$

where $R_{\alpha\beta}$ is a random variable (real for GOE and complex for GUE).

Ratio of variances in the 2D F-TFIM

Hamiltonian: $\hat{H} = -J \sum_{\langle \mathbf{i},\mathbf{j} \rangle} \hat{\sigma}^z_{\mathbf{i}} \hat{\sigma}^z_{\mathbf{j}} + g \sum_{\mathbf{i}} \hat{\sigma}^x_{\mathbf{i}}$

Ratio of variances for the ferromagnetic structure factor



R. Mondaini and MR, PRE **96**, 012157 (2017).

Outline

S_{vN} in eigenstates of quantum chaotic Hamiltonians

Entanglement entropy in eigenstates of quantum chaotic Hamiltonians

Znidaric'07, Santos *et al.*'12, Deutsch *et al.*'13, Beugeling *et al.*'15, Yang *et al.*'15, Garrison & Grover'15, Vivo *et al.*'16, Dymarsky *et al.*'16, Fujita *et al.*'17, MR & Vidmar'17, Huang'17. . .

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What is the (subleading) effect of adding a conservation law?

Focus on the conservation of the number of particles

S_{vN} in eigenstates of quantum chaotic Hamiltonians

Hard-core bosons in one dimension ($t = t' = 1$ and $V = V' = 1.1$)

$$\hat{H} = \sum_{i=1}^{L} \left\{ -t \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + V \hat{n}_i \hat{n}_{i+1} - t' \left(\hat{b}_i^\dagger \hat{b}_{i+2} + \text{H.c.} \right) + V' \hat{n}_i \hat{n}_{i+2} \right\}$$

Random canonical states (z_j: normally distributed real random number)

$$|\psi_N\rangle = \frac{1}{\sqrt{\mathcal{D}_N}} \sum_{j=1}^{\mathcal{D}_N} z_j |j\rangle, \qquad |j\rangle \text{ are base kets for } N \text{ particles in the site basis}$$

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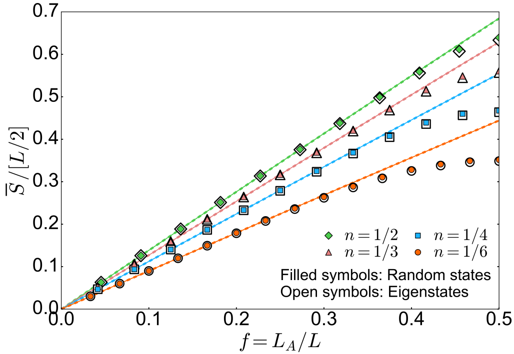
$\bar{S}_{\text{vN}}$ vs subsystem fraction $f = L_A/L$ [$L = 22$ (n=1/2), 24 (n=1/3,1/4) and 30 (n=1/6)]



L. Vidmar and MR,
PRL **119**, 220603 (2017).

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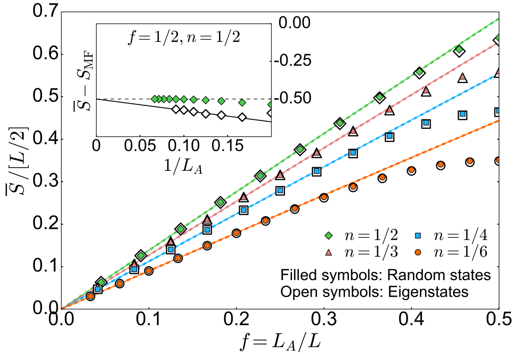

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For large systems, using Stirling's approximation and replacing $\sum_{N_A} \to \int dn$:

$$S^*_{\text{MF}} = -L_A\left[n\ln n + (1-n)\ln(1-n)\right] + \frac{f+\ln(1-f)}{2}$$

Leading order: J. R. Garrison and T. Grover, Phys. Rev. X **8**, 021026 (2018).

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$$S^{\mathrm{bound}*}_{\mathrm{fluct}} = -\sqrt{L_A}\,\ln\left(\frac{1-n}{n}\right)\sqrt{\frac{n(1-n)}{\pi}} + \frac{1}{\sqrt{L_A}}\,\frac{(1-2n)}{3\sqrt{\pi n(1-n)}}$$

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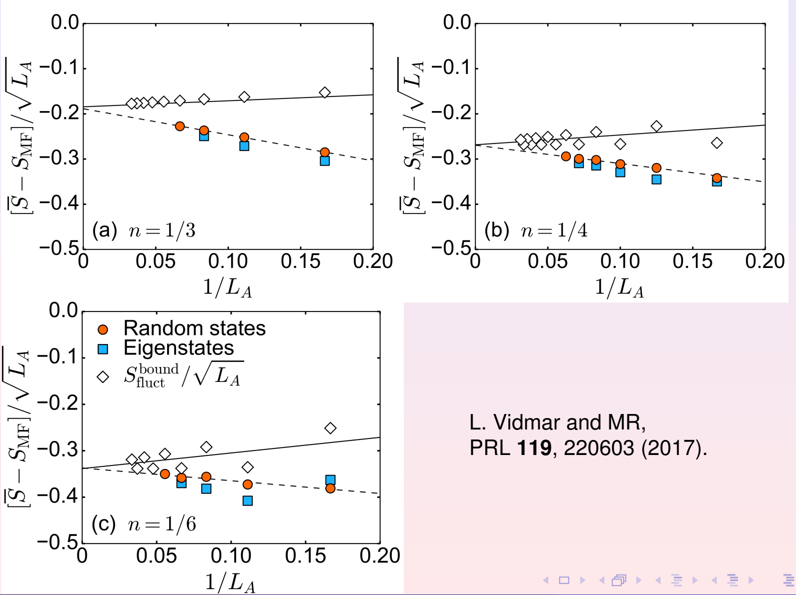
For the canonical ensemble at infinite temperature

$$S_A = \ln\mathcal{D}_N/2 \approx -L_A\left[n\ln n + (1-n)\ln(1-n)\right] - (1/4)\ln(L_A) + C_n$$

S_{vN} in eigenstates of quantum chaotic Hamiltonians

Fluctuation contribution to the average entanglement entropy for $f = 1/2$



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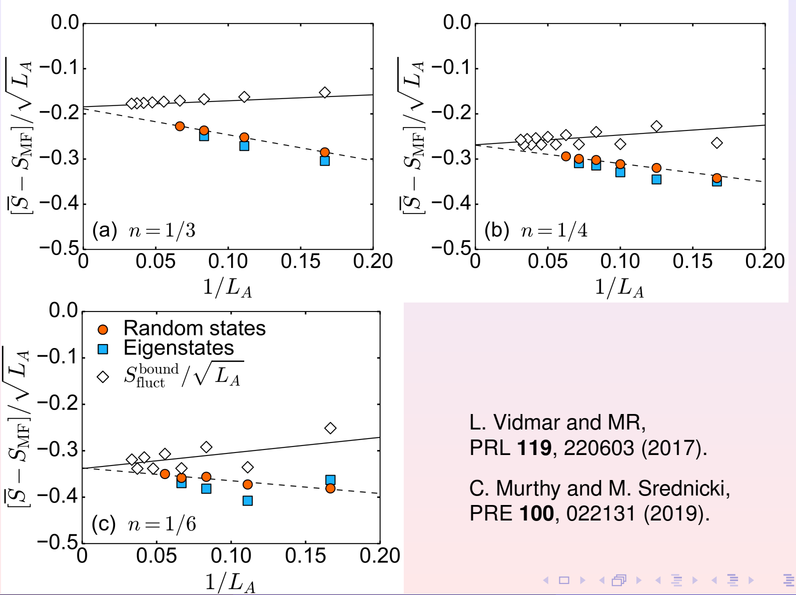

L. Vidmar and MR, PRL **119**, 220603 (2017).

C. Murthy and M. Srednicki, PRE **100**, 022131 (2019).

Summary

- The entanglement properties of typical eigenstates of integrable models are different from those of typical eigenstates of nonintegrable ones
  - Need a nonvanishing subsytem fraction to detect them!
- The leading and first subleading [when $O(1)$ or larger] terms in the von Neumann entanglement entropy of typical eigenstates of quantum chaotic Hamiltonians are universal and given by random matrix theory.

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Collaborators

- *Lev Vidmar (Jožef Stefan Institute)*
- Tyler LeBlond (Penn State)
- Krishna Mallayya (Penn State)
- Eugenio Bianchi (Penn State)
- Lucas Hackl (MPQ)

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