Generalized Thermalization in Integrable Lattice Systems

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November 13, 2019

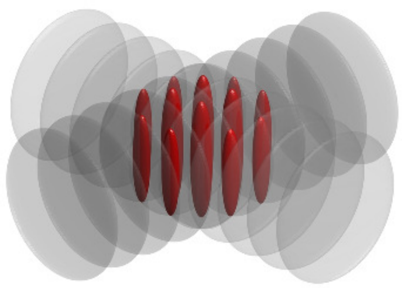
L. Vidmar and MR, *Generalized Gibbs ensemble in integrable lattice models*, J. Stat. Mech. 064007 (2016).

Outline

1. Introduction
   - Experiments with ultracold gases in one dimension
   - Absence of thermalization in 1D?
   - Classical and quantum integrability
   - Hard-core bosons in one-dimensional lattices
2. Generalized Gibbs Ensemble (GGE)
   - Maximal entropy and the GGE
3. Generalized Thermalization
   - GGE vs quantum mechanics
   - Generalized eigenstate thermalization
4. Equilibration: Few-body vs local observables
   - Noninteracting fermions & hard-core anyons
5. Summary

Outline

Experiments in the 1D regime



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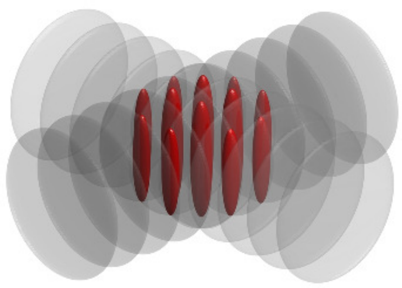

Effective one-dimensional δ potential

M. Olshanii, PRL **81**, 938 (1998).

$$U_{1D}(x) = g_{1D}\delta(x)$$

where

$$g_{1D} = \frac{2\hbar a_s \omega_\perp}{1 - C a_s \sqrt{\frac{m\omega_\perp}{2\hbar}}}$$

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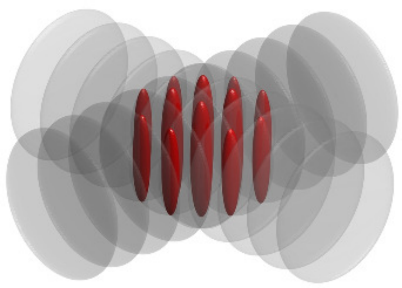

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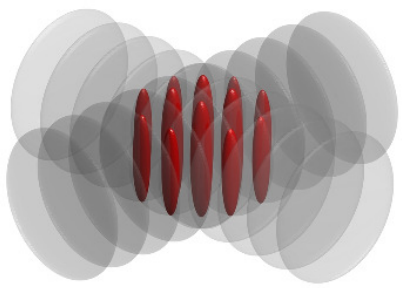

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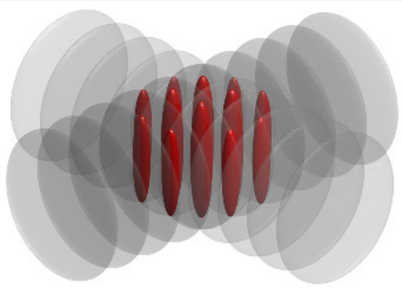

T. Kinoshita, T. Wenger, and D. S. Weiss, Science **305**, 1125 (2004).

T. Kinoshita, T. Wenger, and D. S. Weiss, Phys. Rev. Lett. **95**, 190406 (2005).

$g^{(2)}(x) = \frac{\langle \hat{\Psi}^{\dagger 2}(x)\Psi^2(x)\rangle}{n_{1D}^2(x)}$ and $\gamma = \frac{m g_{1D}}{\hbar^2 n_{1D}}$ $\Leftrightarrow$

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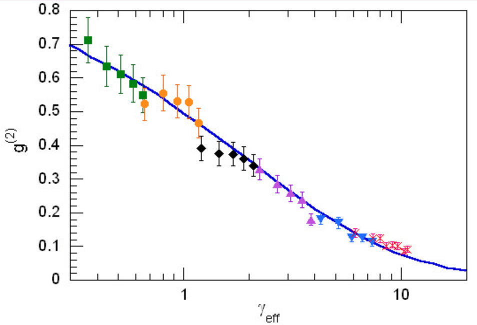

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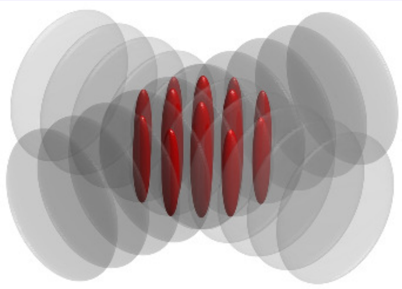

Lieb, Schulz, and Mattis '61 ($U/J=\infty$)

B. Paredes *et al.*,
Nature (London) **429**, 277 (2004).

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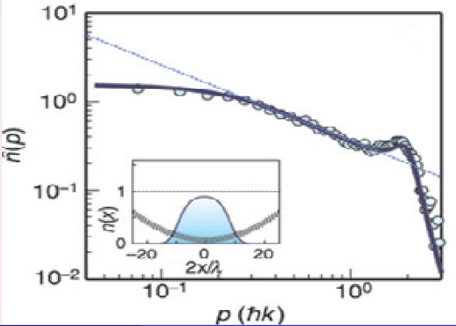
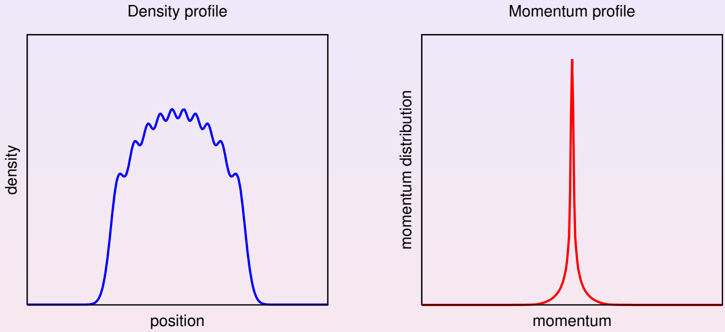


$n(p)$: Momentum distribution $\Leftrightarrow$

$n(x)$: Density distribution $\Leftrightarrow$

Outline

Absence of thermalization in 1D?



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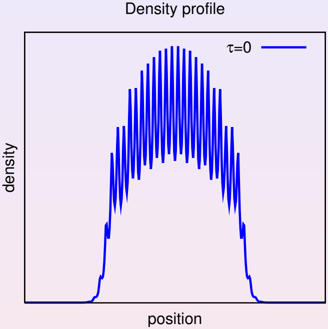

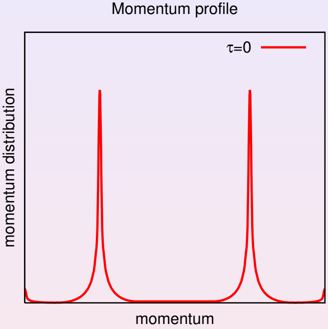

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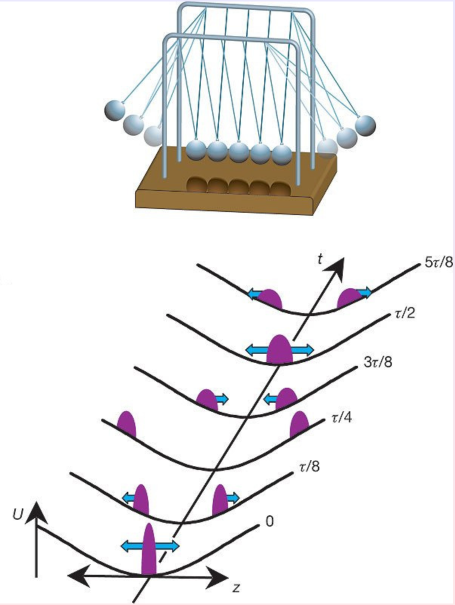

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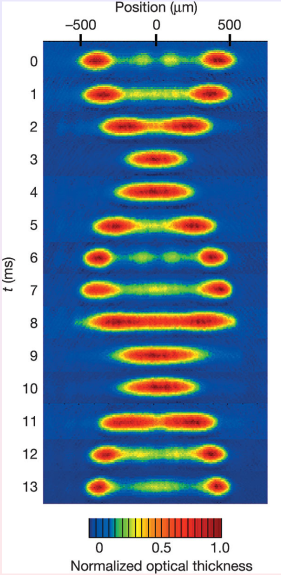
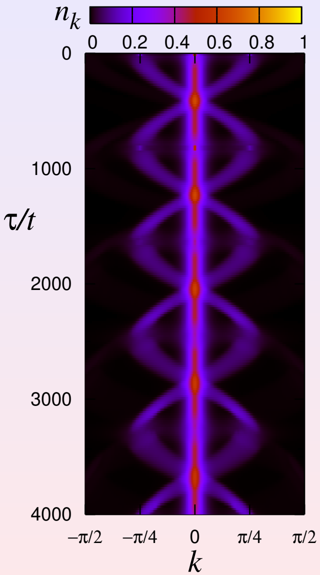
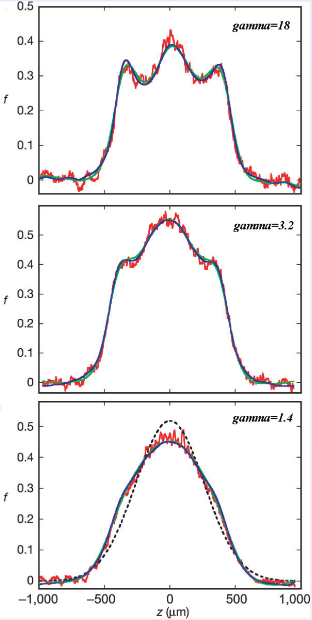




Theory

Absence of thermalization with contact interactions?



T. Kinoshita, T. Wenger, and D. S. Weiss, Nature **440**, 900 (2006).

$$\gamma = \frac{mg_{1D}}{\hbar^2 n_{1D}}$$

g_{1D}: Contact interaction strength
n_{1D}: One-dimensional density

If $\gamma \gg 1$ the system is in the strongly correlated Tonks-Girardeau regime

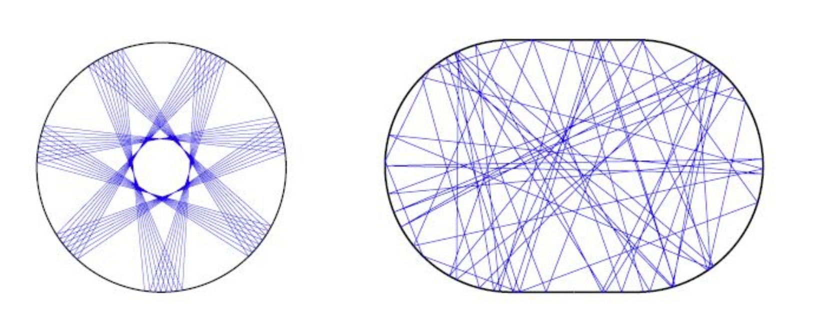
If $\gamma \ll 1$ the system is in the weakly interacting regime

Review of related work in atom chips:
T. Langen, T. Gasenzer, and J. Schmiedmayer, J. Stat. Mech. 064009 (2016).

Outline

Classical chaos and integrability

Particle trajectories in a circular cavity and a Bunimovich stadium (scholarpedia)



- Integrability: A system is said to be integrable if it has as many constants of motion as degrees of freedom
- Chaos: exponential sensitivity of the trajectories to perturbations

Liouville's integrability theorem (Classical)

Hamiltonian

$$H(p,q), \quad \text{coordinates } q = (q_1, \cdots, q_N)$$
$$\text{momenta } p = (p_1, \cdots, p_N)$$

N independent constants of the motion, $I = (I_1, \cdots, I_N)$, in involution

$$\{I_\alpha, H\} = 0, \quad \{I_\alpha, I_\beta\} = 0, \qquad \{f, g\} = \sum_{i=1,N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

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There is a canonical transformation $(p,q) \to (\Theta, I)$ (action-angle variables)

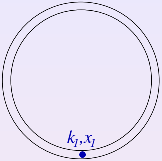
$$H(p,q) = H'(I)$$

Equations of motion

$$\frac{dI_\alpha}{dt} = -\frac{\partial H'}{\partial \Theta_\alpha} = 0 \quad \Rightarrow \quad I_\alpha = \text{constant}$$
$$\frac{d\Theta_\alpha}{dt} = \frac{\partial H'}{\partial I_\alpha} = \Omega_\alpha(I) \quad \Rightarrow \quad \Theta_\alpha = \Omega_\alpha(I)t + \Theta_\alpha^0$$

Scattering without diffraction (Quantum)

One particle



Momentum	Energy	Wavefunction
k_1	$\varepsilon(k_1) = \frac{(k_1)^2}{2}$	$\Psi(x_1) = e^{ik_1 x_1}$

B. Sutherland, *Beautiful Models* (World Scientific, Singapore, 2004).

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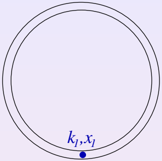

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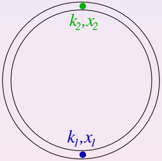
Two particles



$K = k_1 + k_2$ $\quad$ $E = \varepsilon(k_1) + \varepsilon(k_2)$

$$\Psi(x_1, x_2) \quad \rightarrow \quad \sum_P A(P)\, e^{i(k_{P1}x_1 + k_{P2}x_2)}$$

$$= A(12)\, e^{i(k_1x_1 + k_2x_2)} + A(21)\, e^{i(k_2x_1 + k_1x_2)}$$

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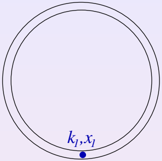

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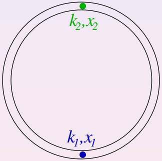

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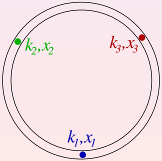
Three particles



$$K = k_1 + k_2 + k_3 \qquad E = \varepsilon(k_1) + \varepsilon(k_2) + \varepsilon(k_3)$$

$$\Psi(x_1, x_2, x_3) \;\rightarrow\; \sum_P A(P)\; e^{i(k_{P1}x_1 + k_{P2}x_2 + k_{P3}x_3)}$$

$+$ ~~diffractive scattering~~

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Semi-classical limit: Statistics of energy levels

- Berry-Tabor conjecture (1977): The statistics of level spacings of quantum systems whose classical counterpart is integrable is described by a Poisson distribution. (Energy eigenvalues behave like a sequence of independent random variables.)

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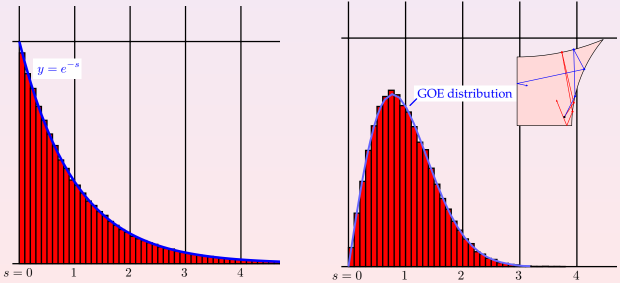
Distribution of level spacings: rectangular and chaotic cavities



Z. Rudnik, Notices AMS **55**, 32 (2008).

Integrability to quantum chaos transition

Spinless fermions (hard-core bosons, spin-1/2) in one dimension

$$\hat{H} = \sum_{i=1}^{L} \left\{ -t \left(\hat{f}_i^\dagger \hat{f}_{i+1} + \text{H.c.} \right) + V \hat{n}_i \hat{n}_{i+1} - t' \left(\hat{f}_i^\dagger \hat{f}_{i+2} + \text{H.c.} \right) + V' \hat{n}_i \hat{n}_{i+2} \right\}$$

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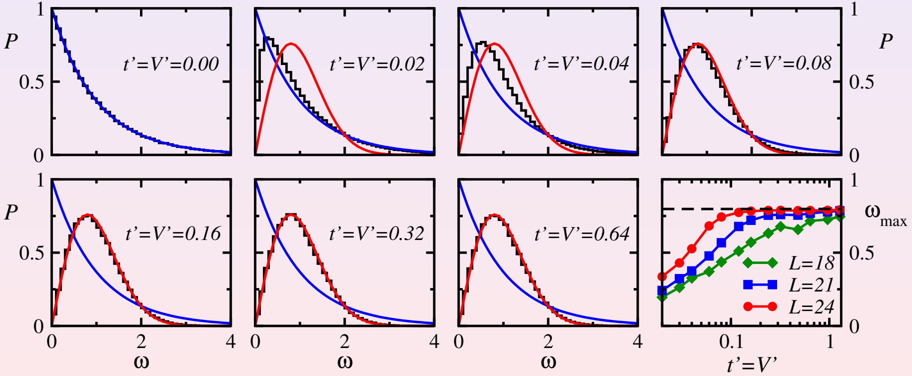
Level spacing distribution $(N_f = L/3)$



L. Santos and MR, PRE **81**, 036206 (2010); PRE **82**, 031130 (2010).

Outline

Bose-Fermi mapping in a 1D lattice

Hard-core boson Hamiltonian in an external potential

$$\hat{H} = -J \sum_i \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + \sum_i v_i \ \hat{n}_i$$

Constraints on the bosonic operators

$$\hat{b}_i^{\dagger 2} = \hat{b}_i^2 = 0$$

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$$\Downarrow$$

Map to spins and then to fermions (Jordan-Wigner transformation)

$$\hat{\sigma}_i^+ = \hat{f}_i^\dagger \prod_{\beta=1}^{i-1} e^{-i\pi \hat{f}_\beta^\dagger \hat{f}_\beta}, \quad \hat{\sigma}_i^- = \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i$$

$$\Downarrow$$

Non-interacting fermion Hamiltonian

$$\hat{H}_F = -J \sum_i \left(\hat{f}_i^\dagger \hat{f}_{i+1} + \text{H.c.} \right) + \sum_i v_i \ \hat{n}_i^f$$

Bose-Fermi mapping in a 1D lattice

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Constraints on the bosonic operators

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$$\Downarrow$$

Set of conserved quantitites

(Occupations of the single-particle energy eigenstates of the noninteracting fermions)

$$\hat{H}_F \hat{\gamma}_m^{f\dagger} |0\rangle = E_m \hat{\gamma}_m^{f\dagger} |0\rangle$$

$$\left\{ \hat{I}_m^f \right\} = \{ \hat{\gamma}_m^{f\dagger} \hat{\gamma}_m^f \}$$

One-body density matrix

One-body Green's function

$$G_{ij} = \langle \Psi_{HCB} | \hat{\sigma}_i^- \hat{\sigma}_j^+ | \Psi_{HCB} \rangle = \langle \Psi_F | \prod_{\beta=1}^{i-1} e^{i\pi \hat{f}_\beta^\dagger \hat{f}_\beta} \hat{f}_i \hat{f}_j^\dagger \prod_{\gamma=1}^{j-1} e^{-i\pi \hat{f}_\gamma^\dagger \hat{f}_\gamma} | \Psi_F \rangle$$

Time evolution

$$|\Psi_F(t)\rangle = e^{-i\hat{H}_F t} |\Psi_F^I\rangle = \prod_{\delta=1}^{N} \sum_{\sigma=1}^{L} P_{\sigma\delta}(t) \hat{f}_\sigma^\dagger \, |0\rangle$$

MR and A. Muramatsu, PRA **70**, 031603(R) (2004); PRL **93**, 230404 (2004).

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Exact Green's function

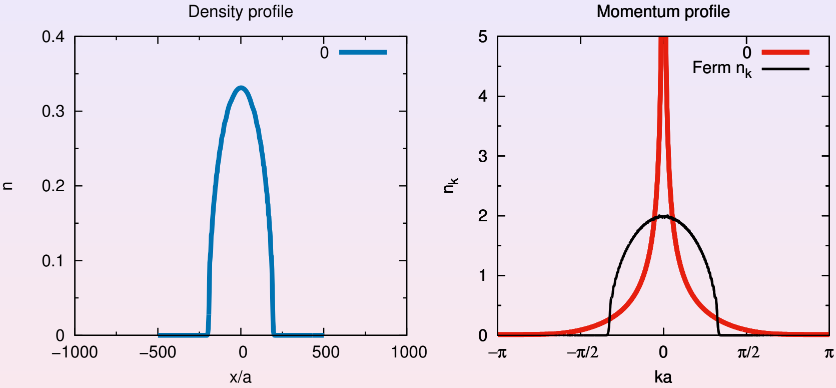
$$G_{ij}(t) = \det \left[\left(\mathbf{P}^l(t) \right)^\dagger \mathbf{P}^r(t) \right]$$

Computation time $\propto L^2 N^3 \rightarrow$ study very large systems

$$\sim 10000 \text{ lattice sites}, \qquad \sim 1000 \text{ particles}$$

MR and A. Muramatsu, PRA **70**, 031603(R) (2004); PRL **93**, 230404 (2004).

Dynamical fermionization



M. Rigol and A. Muramatsu, Phys. Rev. Lett. **94**, 240403 (2005).

"Problem" with TOF: B. Sutherland, Phys. Rev. Lett. **80**, 3678 (1998).

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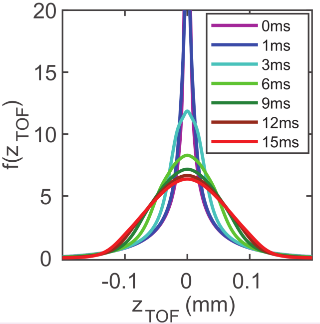

J. M. Wilson, N. Malvania, Y. Le, Y. Zhang, MR, and D. S. Weiss, arXiv:1908.05364.

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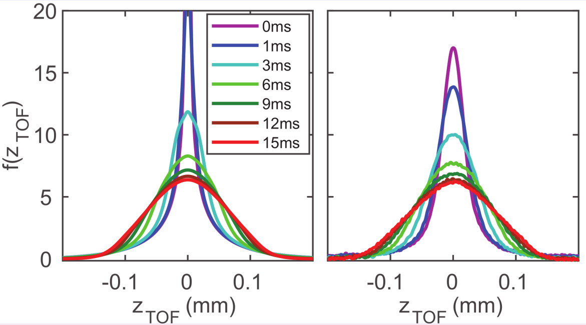

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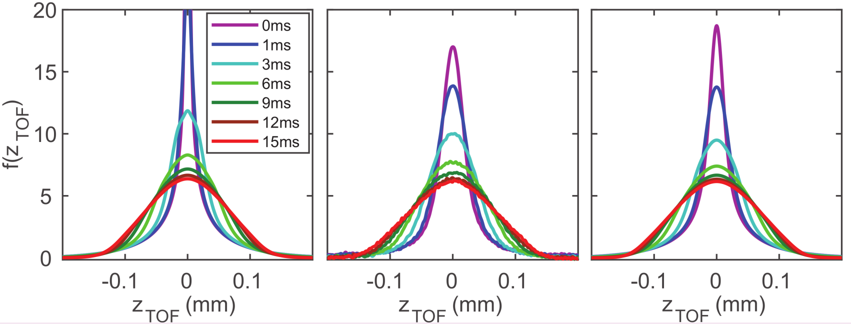

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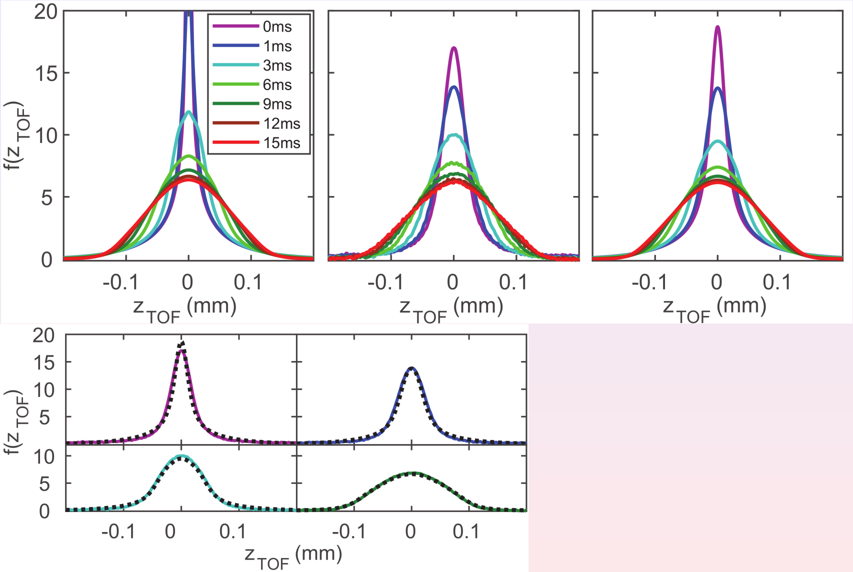

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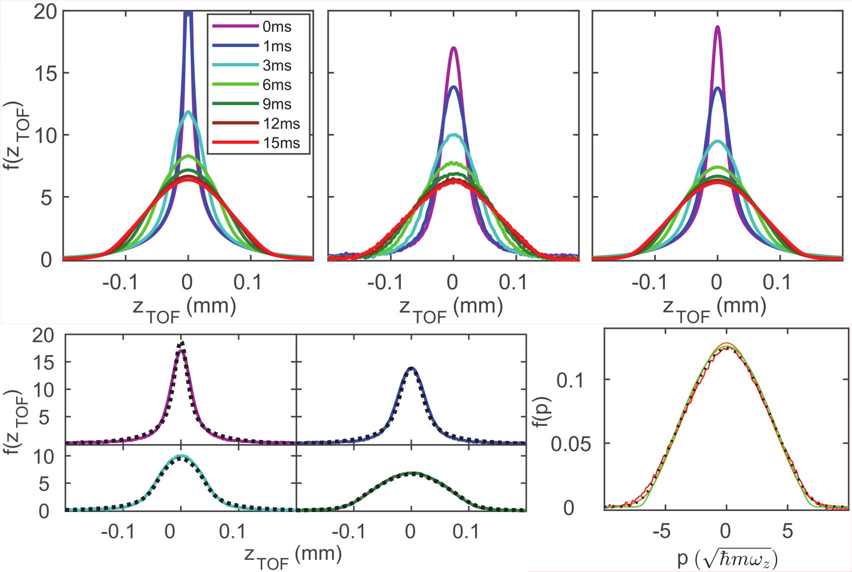

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Finite temperature

One-particle density matrix (grand-canonical ensemble)

$$\rho_{ij} \equiv \frac{1}{Z} \mathrm{Tr} \left\{ \hat{b}_i^\dagger \hat{b}_j e^{-\frac{\hat{H}_{HCB} - \mu \sum_m \hat{b}_m^\dagger \hat{b}_m}{k_B T}} \right\}, \quad Z = \mathrm{Tr} \left\{ e^{-\frac{\hat{H}_{HCB} - \mu \sum_m \hat{b}_m^\dagger \hat{b}_m}{k_B T}} \right\}$$

MR, PRA **72**, 063607 (2005)

Finite temperature

One-particle density matrix (grand-canonical ensemble)

$$\rho_{ij} \equiv \frac{1}{Z}\mathrm{Tr}\left\{\hat{b}_i^\dagger \hat{b}_j e^{-\frac{\hat{H}_{HCB}-\mu\sum_m \hat{b}_m^\dagger \hat{b}_m}{k_B T}}\right\}, \quad Z = \mathrm{Tr}\left\{e^{-\frac{\hat{H}_{HCB}-\mu\sum_m \hat{b}_m^\dagger \hat{b}_m}{k_B T}}\right\}$$

$$\Downarrow$$

Mapping to noninteracting fermions

$$\rho_{ij} = \frac{1}{Z}\mathrm{Tr}\left\{\hat{f}_i^\dagger \hat{f}_j \prod_{k=1}^{j-1} e^{i\pi \hat{f}_k^\dagger \hat{f}_k} e^{-\frac{\hat{H}_F-\mu\sum_m \hat{f}_m^\dagger \hat{f}_m}{k_B T}} \prod_{l=1}^{i-1} e^{-i\pi \hat{f}_l^\dagger \hat{f}_l}\right\}$$

$$\Downarrow$$

Exact one-particle density matrix

$$\begin{aligned}\rho_{ij} &= \frac{1}{Z}\left\{\det\left[\mathbf{I} + (\mathbf{I}+\mathbf{A})\mathbf{O}_1\mathbf{U}e^{-(\mathbf{E}-\mu\mathbf{I})/k_B T}\mathbf{U}^\dagger\mathbf{O}_2\right]\right.\\ &\quad \left. -\det\left[\mathbf{I} + \mathbf{O}_1\mathbf{U}e^{-(\mathbf{E}-\mu\mathbf{I})/k_B T}\mathbf{U}^\dagger\mathbf{O}_2\right]\right\}\end{aligned}$$

Computation time $\sim L^5$: 1000 sites

MR, PRA **72**, 063607 (2005)

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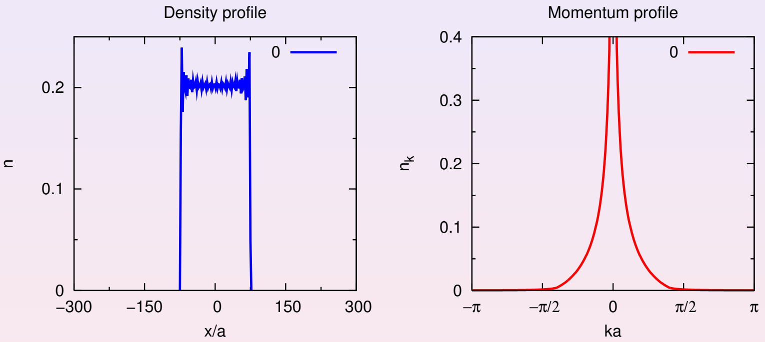
MR, PRA **72**, 063607 (2005); W. Xu and MR, Phys. Rev. A **95**, 033617 (2017).

Outline

Relaxation dynamics in an integrable system



MR, V. Dunjko, V. Yurovsky, and M. Olshanii, PRL **98**, 050405 (2007).

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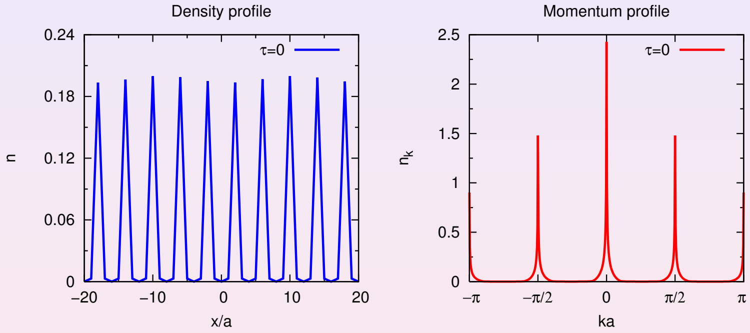

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Generalized Gibbs ensemble (GGE)

Thermal equilibrium

$$\hat{\rho} = Z^{-1} \exp\left[-\left(\hat{H} - \mu\hat{N}\right)/k_B T\right]$$

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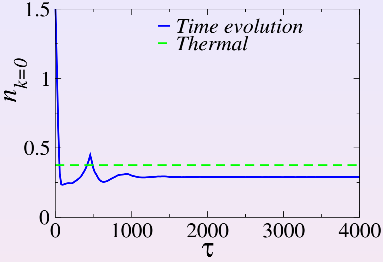
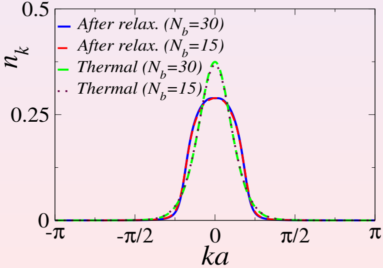
Evolution of $n_{k=0}$



n_k after relaxation



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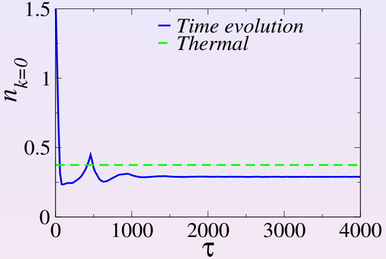

Conserved quantities

(underlying noninteracting fermions)

$$\hat{H}_F \hat{\gamma}_m^{f\dagger}|0\rangle = E_m \hat{\gamma}_m^{f\dagger}|0\rangle$$

$$\left\{\hat{I}_m\right\} = \left\{\hat{\gamma}_m^{f\dagger}\hat{\gamma}_m^{f}\right\}$$

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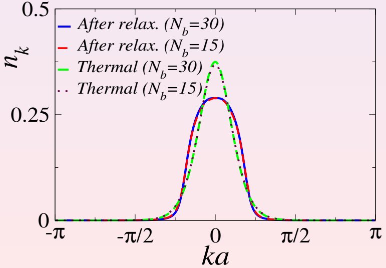

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Generalized Gibbs ensemble

$$\hat{\rho}_{\text{GGE}} = Z_c^{-1} \exp\left[-\sum_m \lambda_m \hat{I}_m\right]$$

$$Z_c = \mathrm{Tr}\left\{\exp\left[-\sum_m \lambda_m \hat{I}_m\right]\right\}$$

$$\mathrm{Tr}\left\{\hat{I}_m \hat{\rho}_{\text{GGE}}\right\} = \langle \hat{I}_m \rangle_{\tau=0}$$

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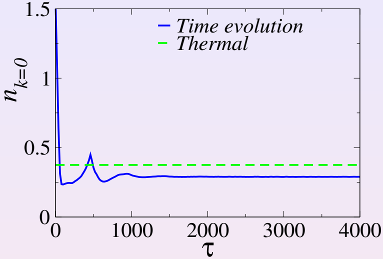

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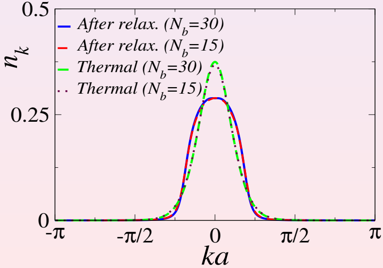

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$$\lambda_m = \ln\left[\frac{1 - \langle \hat{I}_m \rangle_{\tau=0}}{\langle \hat{I}_m \rangle_{\tau=0}}\right]$$

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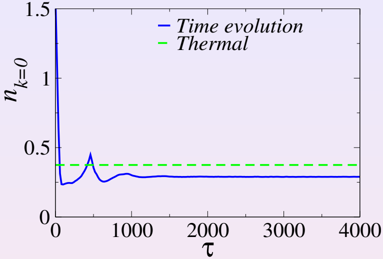

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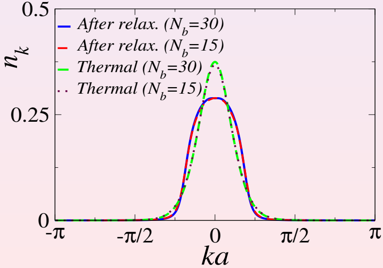

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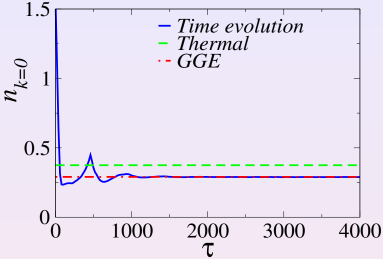

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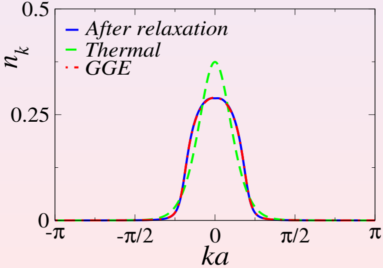

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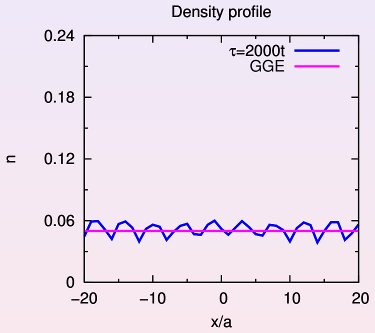

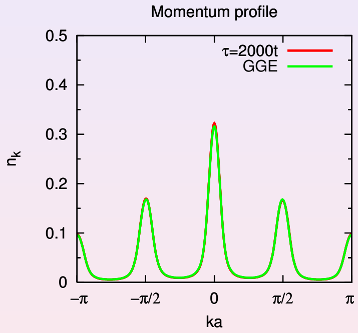

Outline

Exact results from quantum mechanics

If the initial state is not an eigenstate of $\widehat{H}$

$$|\psi_{\text{ini}}\rangle \neq |\alpha\rangle \quad \text{where} \quad \widehat{H}|\alpha\rangle = E_\alpha|\alpha\rangle \quad \text{and} \quad E = \langle\psi_{\text{ini}}|\widehat{H}|\psi_{\text{ini}}\rangle,$$

then observables $\hat{O}$ evolve in time:

$$O(\tau) \equiv \langle\psi(\tau)|\widehat{O}|\psi(\tau)\rangle \quad \text{where} \quad |\psi(\tau)\rangle = e^{-i\widehat{H}\tau}|\psi_{\text{ini}}\rangle.$$

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One can rewrite

$$O(\tau) = \sum_{\alpha,\beta} C_\alpha^\star C_\beta e^{i(E_\alpha - E_\beta)\tau} O_{\alpha\beta} \quad \text{using} \quad |\psi_{\text{ini}}\rangle = \sum_\alpha C_\alpha|\alpha\rangle.$$

Taking the infinite time average (diagonal ensemble $\hat{\rho}_{\text{DE}} \equiv \sum_\alpha |C_\alpha|^2|\alpha\rangle\langle\alpha|$)

$$\overline{O(\tau)} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau d\tau' \langle\Psi(\tau')|\hat{O}|\Psi(\tau')\rangle \overset{?}{=} \sum_\alpha |C_\alpha|^2 O_{\alpha\alpha} \equiv \langle\hat{O}\rangle_{\text{DE}},$$

which depends on the initial conditions through $C_\alpha = \langle\alpha|\psi_{\text{ini}}\rangle$.

Outline

Entropy of the GGE vs the diagonal entropy

$$S_{\mathsf{DE}} = -\mathsf{Tr}[\hat{\rho}_{\mathsf{DE}} \ln \hat{\rho}_{\mathsf{DE}}], \qquad S_{\mathsf{GGE}} = -\mathsf{Tr}[\hat{\rho}_{\mathsf{GGE}} \ln \hat{\rho}_{\mathsf{GGE}}]$$

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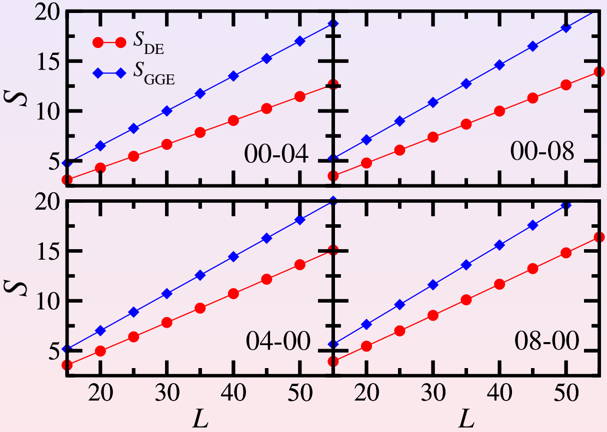
Entropies after quenches in a superlattice potential



S_{DE} & S_{GGE} are extensive but different!

Santos, Polkovnikov, and MR, Phys. Rev. Lett. **107**, 040601 (2011).

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The transverse-field Ising model

$$\hat{H}_{\rm TFIM} = -\sum_j \hat{S}^x_j \hat{S}^x_{j+1} - h \sum_j \hat{S}^z_j$$

Entropies after quenches in the translationally invariant case

$$S_{\mathsf{GGE}} = 2 S_{\mathsf{DE}}$$

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Spin-1/2 XXZ chain: Piroli, Vernier, Calabrese, and MR, PRB **95**, 054308 (2017);
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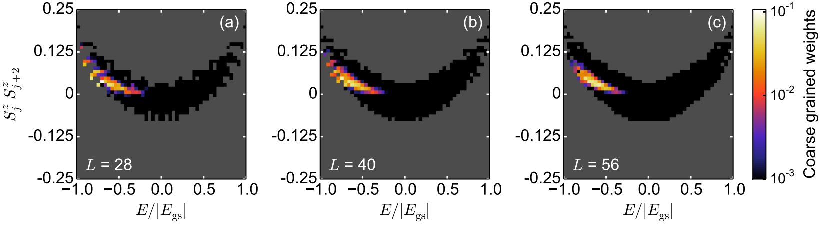
Generalized eigenstate thermalization:

A. C. Cassidy, C. W. Clark, and MR, PRL **106**, 140405 (2011).
L. Vidmar and MR, J. Stat. Mech. 064007 (2016).

Behind generalized thermodynamic Bethe ansatz approaches:
J.-S. Caux and F. H. L. Essler, PRL **110**, 257203 (2013).
B Pozsgay, J. Stat. Mech. P09026 (2014).

Generalized eigenstate thermalization (1D-TFIM)

Weight of eigenstate expectation values after equilibration



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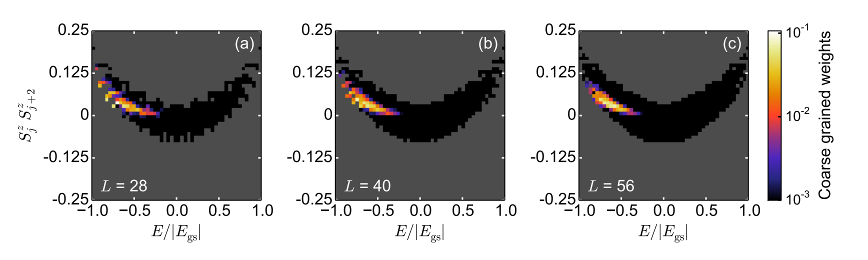
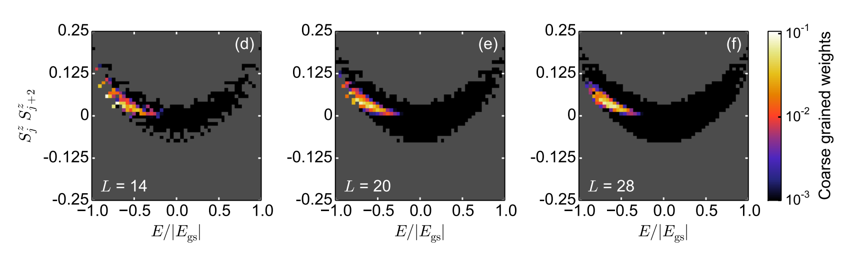


Weight of eigenstate expectation values in the GGE



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Generalized eigenstate thermalization (1D-TFIM)

Variance of observable $\hat{\mathcal{O}}$ after quenches $h_0 \to h$ (ENS $= \mathrm{DE}, \mathrm{GGE}$)

$$\Sigma^2_{\hat{\mathcal{O}},\mathrm{ENS}} = \sum_n \rho_n^{\mathrm{ENS}} \langle n|\hat{\mathcal{O}}|n\rangle^2 - \left(\sum_n \rho_n^{\mathrm{ENS}} \langle n|\hat{\mathcal{O}}|n\rangle \right)^2$$

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For all local observables for which $\Sigma^2_{\hat{\mathcal{O}},\text{ENS}}$ was calculated analytically

$$\frac{\Sigma^2_{\hat{\mathcal{O}},\mathrm{DE}}}{\Sigma^2_{\hat{\mathcal{O}},\mathrm{GGE}}} = 2, \qquad \text{and} \qquad \Sigma_{\hat{\mathcal{O}},\text{ENS}} \sim \frac{1}{\sqrt{L}}$$

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For example, for quenches across the critical field:

$$\Sigma^2_{\hat{S}^x_j \hat{S}^x_{j+1},\mathrm{DE}} = \begin{cases} \frac{1}{64L}\left[1 + h_0^2 - \frac{2h_0}{h}\right] & \text{if} \quad h_0 < 1,\ h > 1 \\ \frac{1}{64L}\left[4 - 3h^2 - \left(\frac{h}{h_0}\right)(4 - 2h^2) + \left(\frac{h}{h_0}\right)^2\right] & \text{if} \quad h_0 > 1,\ h < 1 \end{cases}$$

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Outline

Time averages vs instantaneous values

Noninteracting fermions

- Time average of one-body observables is always GGE:

$$\overline{\sigma(\tau)} = \lim_{\tau' \to \infty} \frac{1}{\tau'} \int_0^{\tau'} d\tau \sum_{s,s'} c_{ss'} e^{-i(e_s - e_{s'})\tau} |s\rangle\langle s'| = \sum_s c_{ss} |s\rangle\langle s|.$$

By construction, ρ_{GGE} is diagonal with $\text{tr}[\hat{\rho}_{\text{GGE}}\, \hat{I}_s] = c_{ss}$.

K. He, L. F. Santos, T. M. Wright, and MR, PRA **87**, 063637 (2013).

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Time independence of the trace distance

- Trace distance in the one-body sector

$$\mathcal{D}\left[\sigma(\tau), \sigma_{\text{GGE}}\right] = \frac{1}{2N} \text{Tr}\left\{\sqrt{(\sigma(\tau) - \sigma_{\text{GGE}})^2}\right\}$$

  where $\text{Tr}\{\sigma(\tau)\} = \text{Tr}\{\sigma_{\text{GGE}}\} = N$ (hence the $1/N$ factor in $\mathcal{D}$).

- $\sigma(\tau) = U(\tau)\sigma(0)U^\dagger(\tau)$ and σ_{GGE} is diagonal in the one-particle eigenbasis

$$\mathcal{D}[\sigma(\tau), \sigma_{\text{GGE}}] = \mathcal{D}[U(\tau)\sigma(0)U^\dagger(\tau), U(\tau)\sigma_{\text{GGE}}U^\dagger(\tau)] = \mathcal{D}[\sigma(0), \sigma_{\text{GGE}}].$$

- $\sigma(\tau)$ does not equilibrate.

Hamiltonian and quenches

Hard-core anyon Hamiltonian in 1D

$$\hat{H} = -J \sum_{j=1}^{L-1} \left(\hat{a}_j^\dagger \hat{a}_{j+1} + \text{H.c.} \right),$$

where $\hat{a}_j \hat{a}_k^\dagger = \delta_{jk} - e^{-i\theta\,\mathrm{sgn}(j-k)} \hat{a}_k^\dagger \hat{a}_j$ and $\hat{a}_j \hat{a}_k = -e^{i\theta\,\mathrm{sgn}(j-k)} \hat{a}_k \hat{a}_j$.

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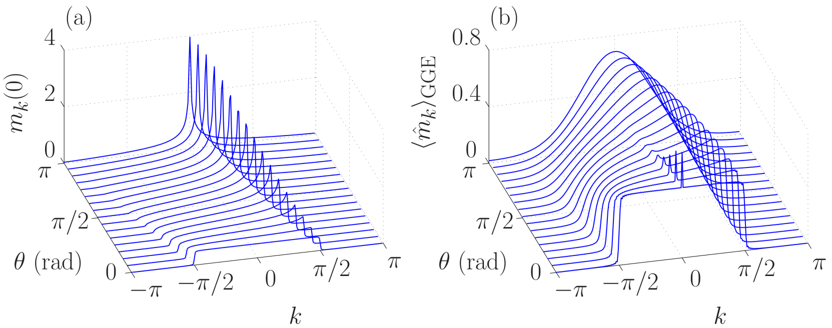
Quench: Open box with $L = 2N \to L = 4N$



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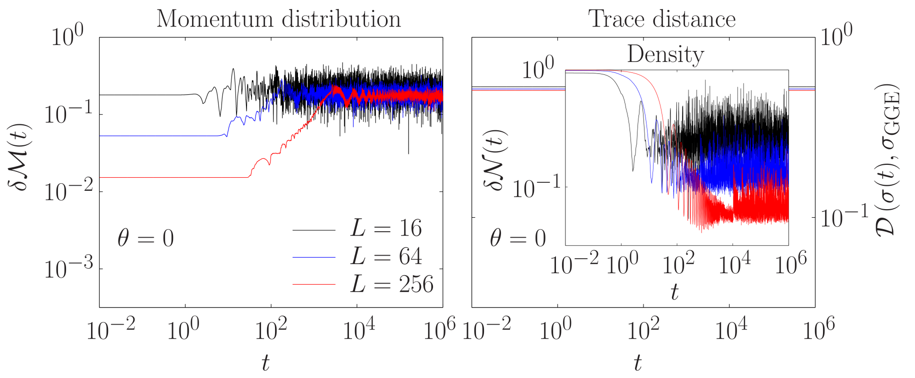
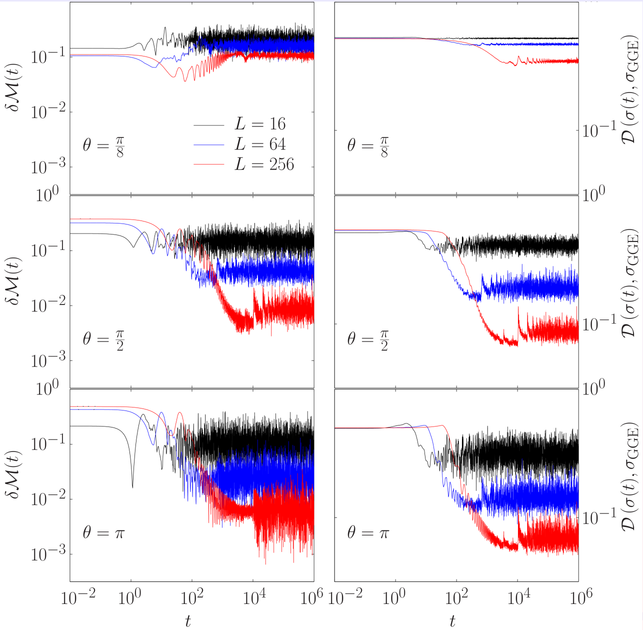
Dynamics after the quench (trace distance vs m_k/n_i)

$$\delta\mathcal{M}(t) = (\sum_k |m_k(t) - \langle\hat{m}_k\rangle_{\rm GGE}|)/\sum_k \langle\hat{m}_k\rangle_{\rm GGE}$$

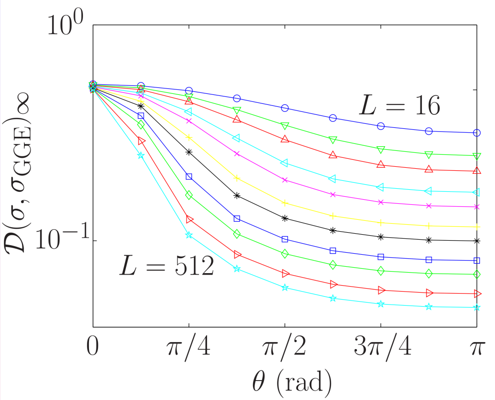
$$\delta\mathcal{N}(t) = (\sum_i |n_i(t) - \langle\hat{n}_i\rangle_{\rm GGE}|)/\sum_i \langle\hat{n}_i\rangle_{\rm GGE}$$



Dynamics after the quench (trace distance vs m_k)

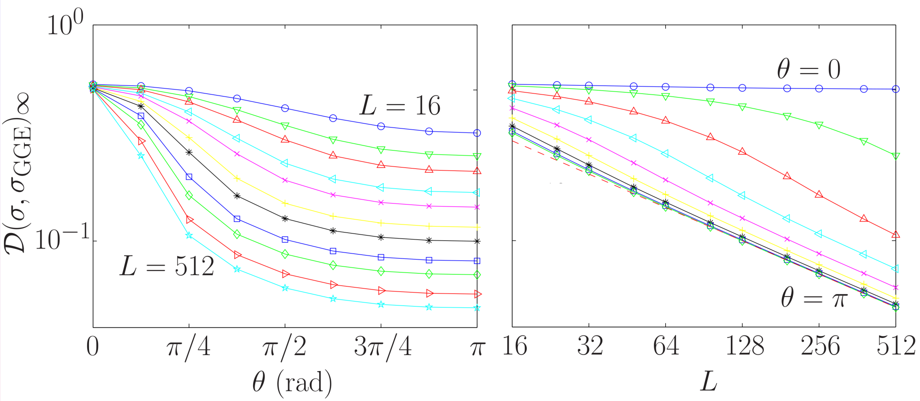


Scaling of the trace distances



- For $\theta \neq 0$: $\mathcal{D}[\sigma, \sigma_{\mathrm{GGE}}]_\infty \propto 1/\sqrt{L}$
- So long as the system is interacting the entire one-body density matrix relaxes to the GGE prediction. All one-body observables, not only $\{n_i\}$ and $\{m_k\}$, are described by the GGE.

Scaling of the trace distances



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  - ★ Microcanonical discussion: Cassidy, Clark & MR, PRL **106**, 140405 (2011).
- In quenches involving hard-core anyons, the entire one-body density matrix equilibrates to the GGE prediction with the singular exception of free fermions
  - ★ The effective bath provided by the other particles (for $\theta \neq 0$) makes relaxation to the GGE possible. No need of tracing out a spatial domain

Collaborators

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Supported by: