Lieb-Schultz-Mattis type theorem in quantum spin chains without continuous symmetry

a “no non-degenerate scar theorem”

Hal Tasaki

Thermalization, Many body localization and Hydrodynamics
November 11, 2019, ICTS

Yoshiko Ogata and Hal Tasaki, “Lieb–Schultz–Mattis Type Theorems for Quantum Spin Chains Without Continuous Symmetry”
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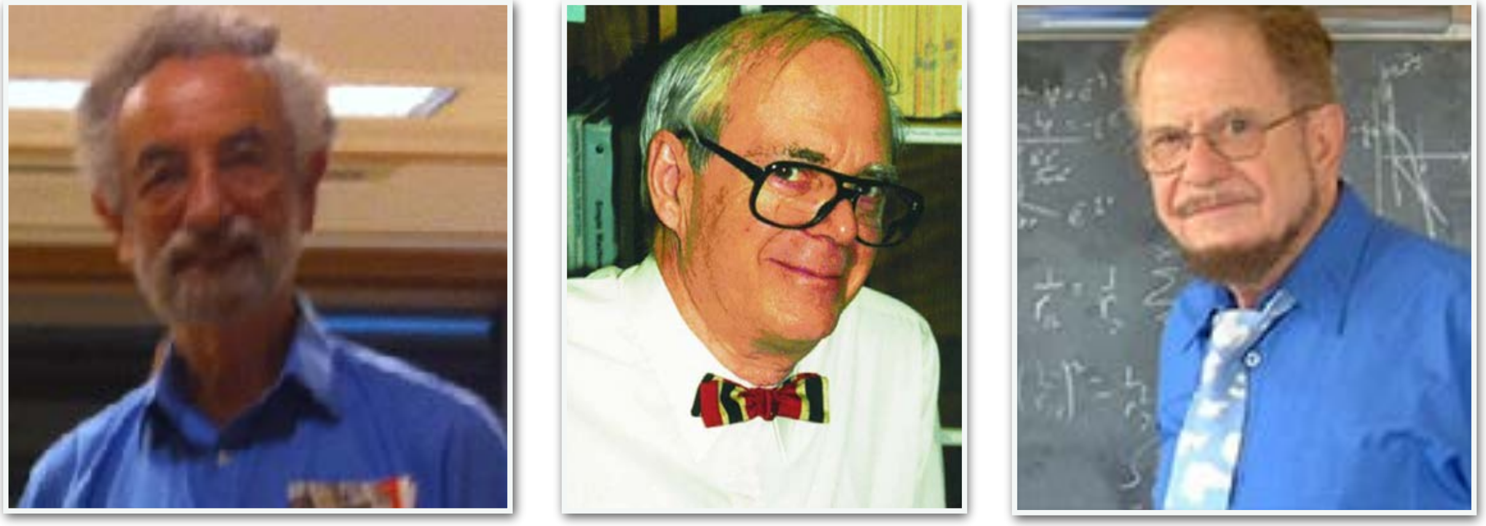

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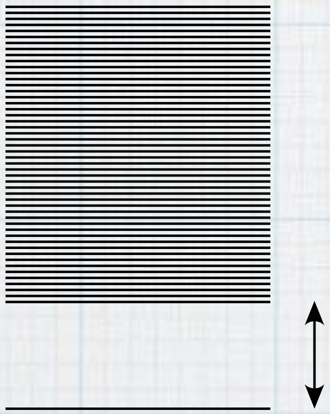
Lieb-Schultz-Mattis Theorem and its Generalizations

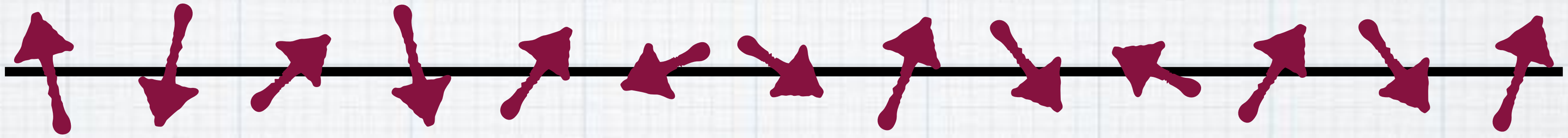
Lieb-Schultz-Mattis (LSM) type theorem

No-go theorem which states that certain quantum many-body systems **CANNNOT** have a gapped unique ground state

(GS may be gapless or exhibit symmetry breaking)



$\Delta E = O(1)$



the original theorem Lieb, Schultz, Mattis 1961, Affleck, Lieb 1986

antiferromagnetic Heisenberg chain

$$\hat{H} = \sum_{j=1}^{L} \hat{\boldsymbol{S}}_j \cdot \hat{\boldsymbol{S}}_{j+1} \quad \text{with} \quad S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$

for any $\ell < L$, there exists an energy eigenvalue E

such that $E_{\mathrm{GS}} < E \leq E_{\mathrm{GS}} + \dfrac{\text{const.}}{\ell}$

there are gapless excitations in the limit $L \uparrow \infty$

Proof of the original theorem

Lieb, Schultz, Mattis 1961, Affleck, Lieb 1986

$\hat{H} = \sum_{j=1}^{L} \hat{\boldsymbol{S}}_j \cdot \hat{\boldsymbol{S}}_{j+1}$ unique ground state $|\mathrm{GS}\rangle$

(1) variational estimate

g.s. is rotation invariant

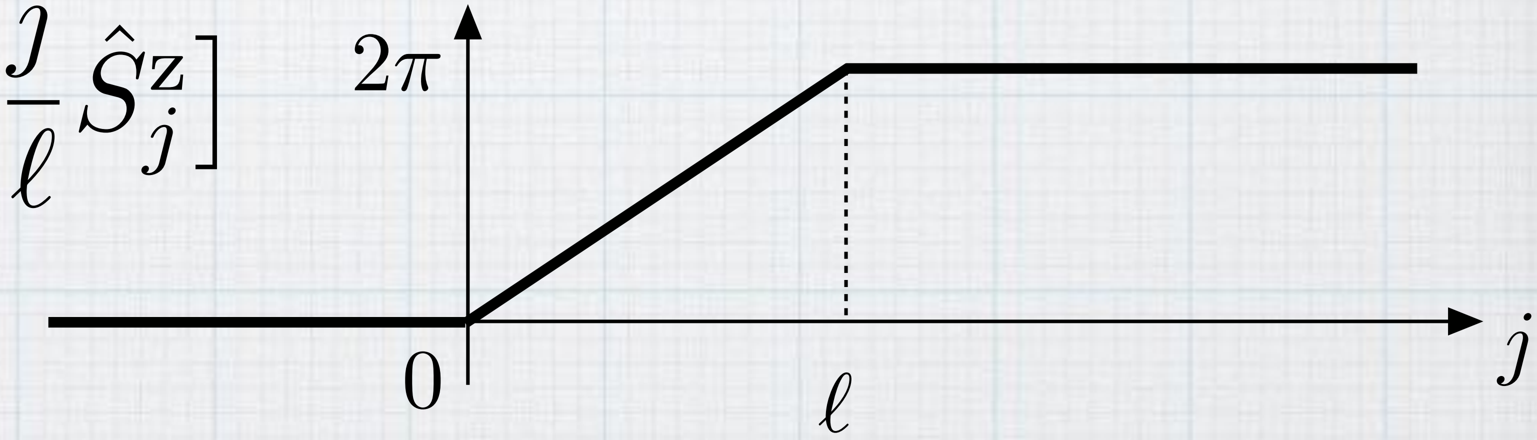
$$\exp\left[i\sum_{j=1}^{L}\theta\, S_j^{\mathrm{z}}\right]|\mathrm{GS}\rangle = |\mathrm{GS}\rangle$$

uniform rotation about z

gradual non-uniform rotation to the GS

$$\hat{V}_\ell = \exp\left[i\sum_{j=0}^{\ell} 2\pi\frac{j}{\ell}\hat{S}_j^{\mathrm{z}}\right]$$

$$|\Psi_\ell\rangle = \hat{V}_\ell|\mathrm{GS}\rangle$$



from an elementary estimate

$$\langle\Psi_\ell|\hat{H}|\Psi_\ell\rangle - E_{\mathrm{GS}} \le \frac{\mathrm{const.}}{\ell}$$

Proof of the original theorem

Lieb, Shultz, Mattis 1961, Affleck, Lieb 1986

$\hat{H} = \sum_{j=1}^{L} \hat{\boldsymbol{S}}_j \cdot \hat{\boldsymbol{S}}_{j+1}$ unique ground state $|\mathrm{GS}\rangle$

(1) variational estimate

$$\langle \Psi_\ell | \hat{H} | \Psi_\ell \rangle - E_{\mathrm{GS}} \leq \frac{\text{const.}}{\ell} \qquad |\Psi_\ell\rangle = \hat{V}_\ell |\mathrm{GS}\rangle$$

(2) orthogonality

it can be shown (by symmetry) that $\langle \Psi_\ell | \mathrm{GS} \rangle = 0$

for $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

for any $\ell < L$, there exists an energy eigenvalue E such that $E_{\mathrm{GS}} < E \leq E_{\mathrm{GS}} + \frac{\text{const.}}{\ell}$

there cannot be a unique gapped ground state!

Lieb-Schultz-Mattis (LSM) type theorem

No-go theorem which states that certain quantum many-body systems **CANNNOT** have a gapped unique ground state

the original theorem and its extensions

Lieb, Shultz, Mattis 1961, Affleck, Lieb 1986

Oshikawa, Yamanaka, Affleck 1997

Oshikawa 2000, Hastings 2004, Nachtergaele, Sims 2007

U(1) invariance is essential

recent “extensions”

Chen, Gu, Wen 2011   Watanabe, Po, Vishwanath, Zaletel 2013

similar no-go statements for models without continuous symmetry, but with some discrete symmetry

projective representation of the symmetry is inconsistent with the existence of a unique gapped ground state

the argument appears already in Matsui 2001

A Typical Theorem

THEOREM 1: Consider a quantum spin chain with $S=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$ and a short-ranged Hamiltonian that is invariant under translation and $\mathbb{Z}_2\times\mathbb{Z}_2$ transformation. Then it can never be the case that the corresponding infinite volume ground state is unique and accompanied by a nonzero gap.

$\mathbb{Z}_2\times\mathbb{Z}_2$ transformation

π-rotations about the three axes

$$\begin{array}{llll}
\mathcal{R}_{\mathrm{x}} & \hat{S}^{\mathrm{x}}_j\to\hat{S}^{\mathrm{x}}_j & \hat{S}^{\mathrm{y}}_j\to-\hat{S}^{\mathrm{y}}_j & \hat{S}^{\mathrm{z}}_j\to-\hat{S}^{\mathrm{z}}_j \\
\mathcal{R}_{\mathrm{y}} & \hat{S}^{\mathrm{x}}_j\to-\hat{S}^{\mathrm{x}}_j & \hat{S}^{\mathrm{y}}_j\to\hat{S}^{\mathrm{y}}_j & \hat{S}^{\mathrm{z}}_j\to-\hat{S}^{\mathrm{z}}_j \\
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\end{array}$$

invariant Hamiltonian (an example)

$$\hat{H}=\sum_j\{J_{\mathrm{x}}\hat{S}^{\mathrm{x}}_j\hat{S}^{\mathrm{x}}_{j+1}+J_{\mathrm{y}}\hat{S}^{\mathrm{y}}_j\hat{S}^{\mathrm{y}}_{j+1}+J_{\mathrm{z}}\hat{S}^{\mathrm{z}}_j\hat{S}^{\mathrm{z}}_{j+1}+K\hat{S}^{\mathrm{x}}_j\hat{S}^{\mathrm{y}}_j\hat{S}^{\mathrm{z}}_j\}$$

Remarks on the Theorem (1)

THEOREM 1: Consider a quantum spin chain with $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and a short-ranged Hamiltonian that is invariant under translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation. Then it can never be the case that the corresponding infinite volume ground state is unique and accompanied by a nonzero gap.

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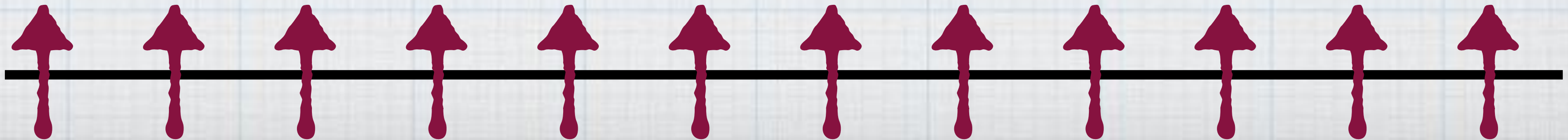
$\mathbb{Z}_2\times\mathbb{Z}_2$ invariance is of course essential

not invariant under $\mathcal{R}_{\rm x}$ or $\mathcal{R}_{\rm y}$

$$\hat{H}=\sum_{j=1}^{L}\{\hat{\boldsymbol{S}}_j\cdot\hat{\boldsymbol{S}}_{j+1}-H\,\hat{S}^{\rm z}_j\}$$

when $H\gg 1$ the ground state (all up!) is unique and accompanied by a gap



Remarks on the Theorem (2)

THEOREM 1: Consider a quantum spin chain with $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and a short-ranged Hamiltonian that is invariant under translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation. Then it can never be the case that the corresponding infinite volume ground state is unique and accompanied by a nonzero gap.

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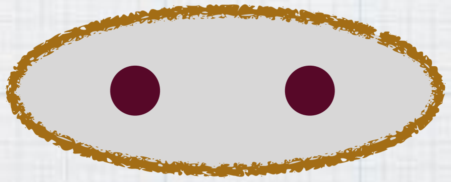
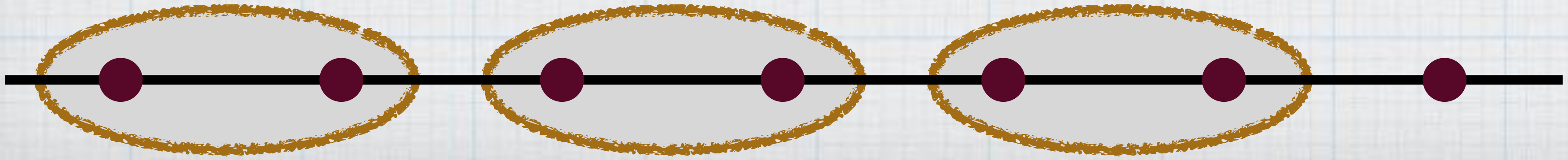
we do need translation invariance

$$\hat{H}=\sum_{k=1}^{L/2}\{\hat{\boldsymbol{S}}_{2k-1}\cdot\hat{\boldsymbol{S}}_{2k}+\zeta\,\hat{\boldsymbol{S}}_{2k}\cdot\hat{\boldsymbol{S}}_{2k+1}\}$$

when $\zeta\gg1$ the ground state is dimerzied; it's unique and accompanied by a gap



$$=\frac{|{\uparrow}\rangle|{\downarrow}\rangle-|{\downarrow}\rangle|{\uparrow}\rangle}{\sqrt{2}}$$



Remarks on the Theorem (3)

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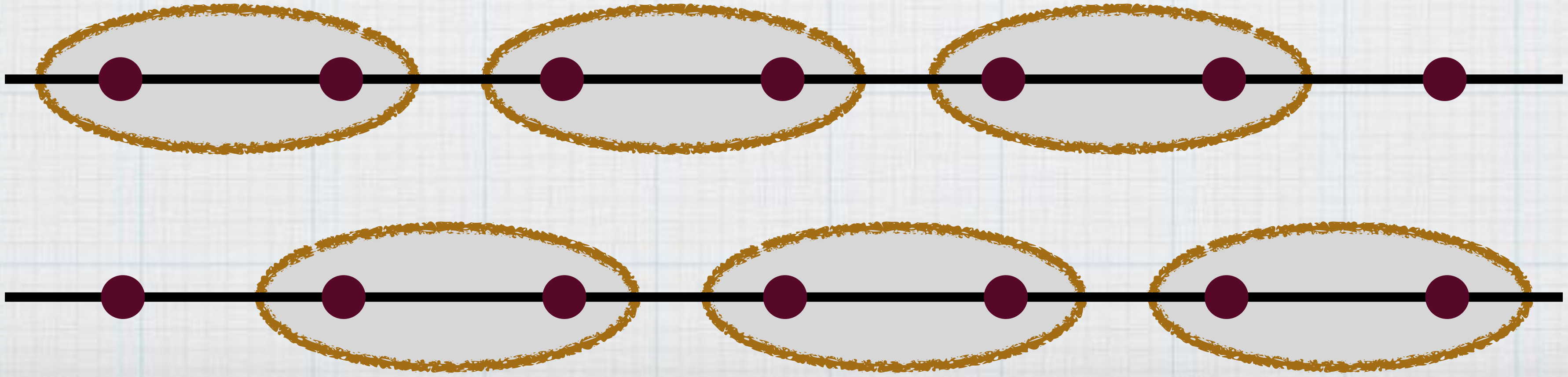
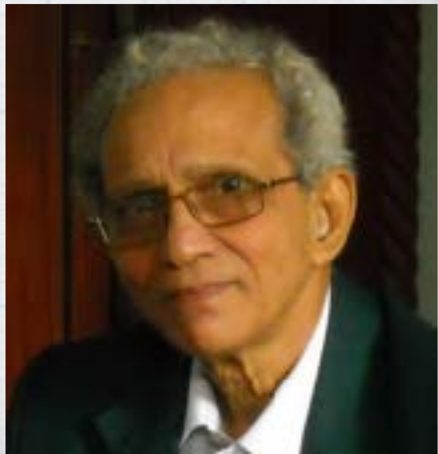
Majumdar-Ghosh model $S = \frac{1}{2}$

$$\hat{H} = \sum_{j=1}^{L} \{ \hat{\boldsymbol{S}}_j \cdot \hat{\boldsymbol{S}}_{j+1} + \hat{\boldsymbol{S}}_j \cdot \hat{\boldsymbol{S}}_{j+2}/2 \}$$

two dimerzed gapped ground states





Remarks on the Theorem (4)

THEOREM 1: Consider a quantum spin chain with $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and a short-ranged Hamiltonian that is invariant under translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation. Then it can never be the case that the corresponding infinite volume ground state is unique and accompanied by a nonzero gap.

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Haldane phenomena

For an integer S, the Heisenberg AF chain

$$\hat{H}=\sum_{j=1}^{L}\hat{\mathbf{S}}_j\cdot\hat{\mathbf{S}}_{j+1}$$

has a unique gapped ground state

AKLT model: rigorous example

Remarks on the Theorem (4)

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π-rotations about the three axes

$$\begin{array}{llll}
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\mathcal{R}_\mathrm{z} & \hat{S}^\mathrm{x}_j\to-\hat{S}^\mathrm{x}_j & \hat{S}^\mathrm{y}_j\to-\hat{S}^\mathrm{y}_j & \hat{S}^\mathrm{z}_j\to\hat{S}^\mathrm{z}_j
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invariant Hamiltonian (an example)

$$\hat{H}=\sum_j\{J_\mathrm{x}\hat{S}^\mathrm{x}_j\hat{S}^\mathrm{x}_{j+1}+J_\mathrm{y}\hat{S}^\mathrm{y}_j\hat{S}^\mathrm{y}_{j+1}+J_\mathrm{z}\hat{S}^\mathrm{z}_j\hat{S}^\mathrm{z}_{j+1}+K\hat{S}^\mathrm{x}_j\hat{S}^\mathrm{y}_j\hat{S}^\mathrm{z}_j\}$$

The group $\mathbb{Z}_2 \times \mathbb{Z}_2$ its representation and projective representation

the group $\mathbb{Z}_2 \times \mathbb{Z}_2$

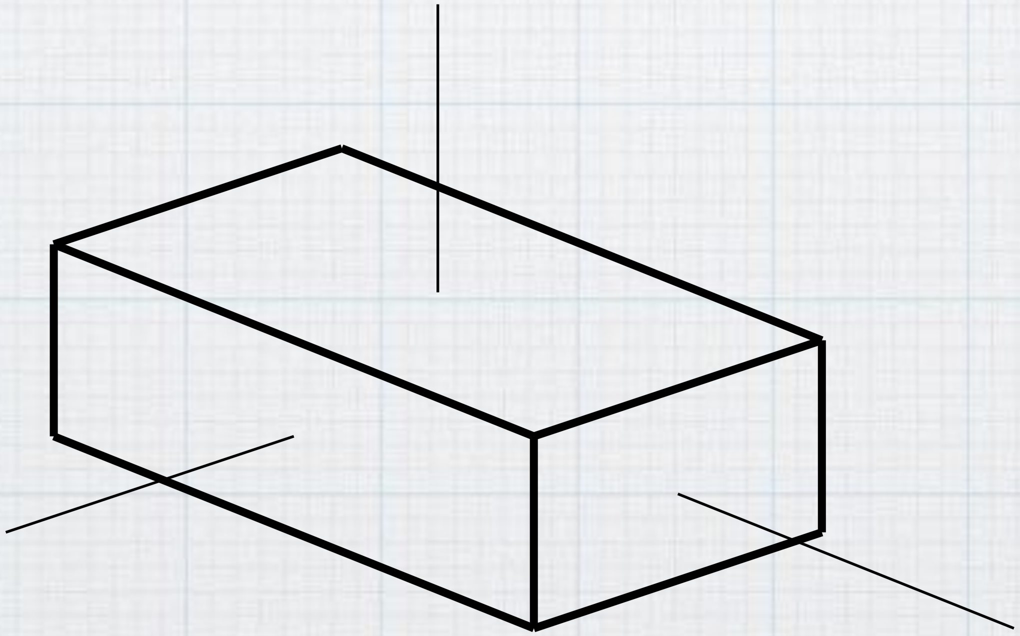
abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, a, b, c\}$

$$eg = ge = g$$

$$a^2 = e \qquad b^2 = e \qquad c^2 = e$$

$$ab = ba = c \qquad bc = cb = a \qquad ca = ac = b$$

example: π-rotations about the three orthogonal axes



$\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation of a single spin

spin operator $\hat{\boldsymbol{S}} = (\hat{S}^{\mathrm{x}}, \hat{S}^{\mathrm{y}}, \hat{S}^{\mathrm{z}})$ $\quad \hat{\boldsymbol{S}}^2 = S(S+1) \quad S = \frac{1}{2}, 1, \frac{3}{2}, \ldots$

π-rotation about the α-axis $\hat{u}_\alpha = \exp[-i\pi\hat{S}^\alpha] \quad \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$

$$\hat{u}_{\mathrm{x}}\hat{u}_{\mathrm{y}} = \hat{u}_{\mathrm{z}} \qquad \hat{u}_{\mathrm{y}}\hat{u}_{\mathrm{z}} = \hat{u}_{\mathrm{x}} \qquad \hat{u}_{\mathrm{z}}\hat{u}_{\mathrm{x}} = \hat{u}_{\mathrm{y}} \qquad \hat{u}_{\mathrm{z}}\hat{u}_{\mathrm{y}}\hat{u}_{\mathrm{x}} = \hat{1}$$

integer S $\quad (S = 1, 2, \ldots)$

$$(\hat{u}_\alpha)^2 = \hat{1} \qquad \hat{u}_\alpha\hat{u}_\beta = \hat{u}_\beta\hat{u}_\alpha$$

$\hat{1}, \hat{u}_{\mathrm{x}}, \hat{u}_{\mathrm{y}}, \hat{u}_{\mathrm{z}}$ **give a genuine representation of** $\mathbb{Z}_2 \times \mathbb{Z}_2$

half-odd-integer S $\quad (S = \frac{1}{2}, \frac{3}{2}, \ldots)$

$$(\hat{u}_\alpha)^2 = -\hat{1} \qquad \hat{u}_\alpha\hat{u}_\beta = -\hat{u}_\beta\hat{u}_\alpha \quad \alpha \neq \beta$$

$\hat{1}, \hat{u}_{\mathrm{x}}, \hat{u}_{\mathrm{y}}, \hat{u}_{\mathrm{z}}$ **give a projective representation of** $\mathbb{Z}_2 \times \mathbb{Z}_2$

for $S = 1/2$ **we have** $\quad \hat{u}^{\mathrm{x}} = -i\hat{\sigma}^{\mathrm{x}} \quad \hat{u}^{\mathrm{y}} = -i\hat{\sigma}^{\mathrm{y}} \quad \hat{u}^{\mathrm{z}} = -i\hat{\sigma}^{\mathrm{z}}$

Theorem for Matrix Product States (MPS)

Watanabe, Po, Vishwanath, Zaletel 2013

Matrix Product States (MPS)

Fannes, Nachtergaele, Werner 1989, 1992

quantum spin system with spin S on $\{1,2,\ldots,L\}$

standard basis states $|\sigma_1,\ldots,\sigma_L\rangle = \bigotimes_{j=1}^{L}|\sigma_j\rangle_j$

$\hat{S}^{\mathrm{z}}|\sigma\rangle = \sigma|\sigma\rangle$

$D \times D$ **matrices** M^{σ} **with** $\sigma = -S,\ldots,S$

translation invariant state (MPS)

$$|\Phi\rangle = \sum_{\sigma_1,\ldots,\sigma_L=-S}^{S} \mathrm{Tr}[\mathsf{M}^{\sigma_1}\ldots\mathsf{M}^{\sigma_L}]\,|\sigma_1,\ldots,\sigma_L\rangle$$

it is known that states with small entanglement (area-law states) can be approximated by MPS

$|\Phi\rangle$ **is said to be injective if** $\sum_{\sigma=-S}^{S}\mathsf{M}^{\sigma}(\mathsf{M}^{\sigma})^{\dagger} = \mathsf{I}$**, and there is** ℓ **such that** $\mathsf{M}^{\sigma_1}\mathsf{M}^{\sigma_2}\cdots\mathsf{M}^{\sigma_\ell}$ **with all possible** $\sigma_1,\ldots,\sigma_\ell$ **span the whole space of** $D \times D$ **matrices**

heuristic

$|\Phi\rangle$ **is injective if it has small entanglement, and not a "cat"**

Theorem for MPS

translation invariant state (MPS)

$$|\Phi\rangle = \sum_{\sigma_1,\ldots,\sigma_L=-S}^{S} \mathrm{Tr}[\mathsf{M}^{\sigma_1}\ldots\mathsf{M}^{\sigma_L}]\,|\sigma_1,\ldots,\sigma_L\rangle$$

$|\Phi\rangle$ is said to be injective if $\sum_{\sigma=-S}^{S} \mathsf{M}^{\sigma}(\mathsf{M}^{\sigma})^{\dagger} = \mathsf{I}$, and there is ℓ such that $\mathsf{M}^{\sigma_1}\mathsf{M}^{\sigma_2}\cdots\mathsf{M}^{\sigma_\ell}$ with all possible $\sigma_1,\ldots,\sigma_\ell$ span the whole space of $D \times D$ matrices

heuristic

$|\Phi\rangle$ is injective if it has small entanglement, and not a “cat”

THEOREM 1’: There cannot be a translation invariant and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant injective MPS for $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

Proof of Theorem 1'

Watanabe, Po, Vishwanath, Zaletel 2013 (arranged by H.T.)

THEOREM 1': There cannot be a translation invariant and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant injective MPS for $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

assume that $|\Phi\rangle = \sum_{\sigma_1,\ldots,\sigma_L=-S}^{S} \mathrm{Tr}[\mathsf{M}^{\sigma_1} \ldots \mathsf{M}^{\sigma_L}]\,|\sigma_1,\ldots,\sigma_L\rangle$

is injective, and $\mathbb{Z}_2 \times \mathbb{Z}_2$ **invariant, i.e.,**

$\exp[-i\pi \sum_j \hat{S}_j^\alpha]|\Phi\rangle = \mathrm{const}\,|\Phi\rangle$ **for** $\alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$

$$\sum \mathrm{Tr}[\tilde{\mathsf{M}}^{\sigma_1} \ldots \tilde{\mathsf{M}}^{\sigma_L}]\,|\sigma_1,\ldots,\sigma_L\rangle = \mathrm{const} \sum \mathrm{Tr}[\mathsf{M}^{\sigma_1} \ldots \mathsf{M}^{\sigma_L}]\,|\sigma_1,\ldots,\sigma_L\rangle$$

with $\tilde{\mathsf{M}}^\sigma = \sum_{\sigma'} \langle\sigma|\hat{u}_\alpha|\sigma'\rangle \mathsf{M}^{\sigma'}$ $\qquad$ $\hat{u}_\alpha = \exp[-i\pi \hat{S}^\alpha]$

Proof of Theorem 1'

$$\sum \mathrm{Tr}[\tilde{\mathsf{M}}^{\sigma_1} \dots \tilde{\mathsf{M}}^{\sigma_L}]\,|\sigma_1, \dots, \sigma_L\rangle = \mathrm{const} \sum \mathrm{Tr}[\mathsf{M}^{\sigma_1} \dots \mathsf{M}^{\sigma_L}]\,|\sigma_1, \dots, \sigma_L\rangle$$

with $\tilde{\mathsf{M}}^{\sigma} = \sum_{\sigma'} \langle \sigma | \hat{u}_\alpha | \sigma' \rangle \mathsf{M}^{\sigma'}$

$\Updownarrow$ uniqueness of injective MPS

Fannes, Nachtergaele, Werner 1992

Perez-Garcia, Wolf, Sanz, Verstraete, and Cirac 2008
Pollmann, Turner, Berg, Oshikawa 2010

there are $D \times D$ unitary matrices $\mathsf{U}_{\mathrm{x}}, \mathsf{U}_{\mathrm{y}}, \mathsf{U}_{\mathrm{z}}$ which form a projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and constants $\zeta_\alpha \in \mathbb{C}$ with $|\zeta_\alpha| = 1$ for $\alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$, such that

$$\tilde{\mathsf{M}}^{\sigma} = \zeta_\alpha\, \mathsf{U}_\alpha^\dagger \mathsf{M}^\sigma \mathsf{U}_\alpha$$

thus the matrices satisfy nontrivial constraints

$$\mathsf{M}^{\sigma} = \zeta_\alpha \sum_{\sigma'} \langle \sigma | \hat{u}_\alpha^\dagger | \sigma' \rangle \mathsf{U}_\alpha^\dagger \mathsf{M}^{\sigma'} \mathsf{U}_\alpha \quad \text{for } \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$$

$$\hat{u}_\alpha = \exp[-i\pi \hat{S}^\alpha]$$

Proof of Theorem 1'

thus the matrices satisfy nontrivial constraints

$$\mathsf{M}^{\sigma} = \zeta_{\alpha} \sum_{\sigma'} \langle\sigma|\hat{u}_{\alpha}^{\dagger}|\sigma'\rangle \mathsf{U}_{\alpha}^{\dagger}\mathsf{M}^{\sigma'}\mathsf{U}_{\alpha} \quad \text{for } \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$$

$$\hat{u}_{\alpha} = \exp[-i\pi\hat{S}^{\alpha}]$$

we then find

S is a half-odd integer

$$\mathsf{M}^{\sigma} = (\zeta_{\alpha})^2 \sum_{\sigma'} \langle\sigma|\underbrace{(\hat{u}_{\alpha}^{\dagger})^2}_{=-\hat{1}}|\sigma'\rangle \underbrace{(\mathsf{U}_{\alpha}^{\dagger})^2\mathsf{M}^{\sigma'}(\mathsf{U}_{\alpha})^2}_{=\mathsf{M}^{\sigma'}} = -(\zeta_{\alpha})^2\, \mathsf{M}^{\sigma}$$

and

$$\mathsf{M}^{\sigma} = \zeta_{\mathrm{x}} \sum_{\sigma'} \langle\sigma|\hat{u}_{\mathrm{x}}^{\dagger}|\sigma'\rangle \mathsf{U}_{\mathrm{x}}^{\dagger}\mathsf{M}^{\sigma'}\mathsf{U}_{\mathrm{x}}$$

$$= \zeta_{\mathrm{x}}\zeta_{\mathrm{y}} \sum_{\sigma'} \langle\sigma|(\hat{u}_{\mathrm{y}}\hat{u}_{\mathrm{x}})^{\dagger}|\sigma'\rangle (\mathsf{U}_{\mathrm{y}}\mathsf{U}_{\mathrm{x}})^{\dagger}\mathsf{M}^{\sigma'}\mathsf{U}_{\mathrm{y}}\mathsf{U}_{\mathrm{x}}$$

$$= \zeta_{\mathrm{x}}\zeta_{\mathrm{y}}\zeta_{\mathrm{z}} \sum_{\sigma'} \langle\sigma|\underbrace{(\hat{u}_{\mathrm{z}}\hat{u}_{\mathrm{y}}\hat{u}_{\mathrm{x}})^{\dagger}}_{=\hat{1}}|\sigma'\rangle \underbrace{(\mathsf{U}_{\mathrm{z}}\mathsf{U}_{\mathrm{y}}\mathsf{U}_{\mathrm{x}})^{\dagger}\mathsf{M}^{\sigma'}\mathsf{U}_{\mathrm{z}}\mathsf{U}_{\mathrm{y}}\mathsf{U}_{\mathrm{x}}}_{=\mathsf{M}^{\sigma'}}$$

$$= \zeta_{\mathrm{x}}\zeta_{\mathrm{y}}\zeta_{\mathrm{z}}\, \mathsf{M}^{\sigma}$$

$$(\zeta_{\alpha})^2 = -1$$

$$\zeta_{\mathrm{x}}\zeta_{\mathrm{y}}\zeta_{\mathrm{z}} = 1$$

contradiction!

Theorem for MPS

THEOREM 1': There cannot be a translation invariant and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant injective MPS for $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$

nontrivial projective representation of the on-site $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is inconsistent with the existence of an injective MPS

Matsui 2001 Chen, Gu, Wen 2011 Watanabe, Po, Vishwanath, Zaletel 2013

$$\mathsf{M}^{\sigma} = \zeta_{\alpha} \sum_{\sigma'} \langle \sigma | \hat{u}_{\alpha}^{\dagger} | \sigma' \rangle \mathsf{U}_{\alpha}^{\dagger} \mathsf{M}^{\sigma'} \mathsf{U}_{\alpha}$$

projective representation

genuine representation

contradiction!

Toward the Full Theorem

From Theorem 1' to Theorem 1

we have proved

THEOREM 1': There cannot be a translation invariant and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant injective MPS for $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

this seems to imply the desired

THEOREM 1: Consider a quantum spin chain with $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and a short-ranged Hamiltonian that is invariant under translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation. Then it can never be the case that the corresponding ground state is unique and accompanied by a nonzero gap.

From Theorem 1' to Theorem 1

assume that the GS is unique and gapped

the GS has area-law entanglement

the GS is translationally and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant since Hamiltonian has translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

area-law states can be approximated by MPS

there exists an injective MPS that is translationally and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant

this contradicts Theorem 1'

From Theorem 1' to Theorem 1

assume that the GS is unique and gapped

the GS has area-law entanglement

the GS is translationally and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant since Hamiltonian has translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

area-law states can be approximated by MPS

there exists an injective MPS that is translationally and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant

this contradicts Theorem 1'

this "proof" looks plausible, but does not work!!!
the approximation by MPS is not that precise

From Theorem 1' to Theorem 1

assume that the GS is unique and gapped

the GS has area-law entanglement

the GS is translationally and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant since Hamiltonian has translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

area-law states can be approximated by MPS

there exists an injective MPS that is translationally and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant

this contradicts Theorem 1'

this "proof" looks plausible, but does not work!!!

the approximation by MPS is not that precise

the proof of Theorem 1 makes an essential use of operator algebraic formulation

About operator algebraic approaches in quantum systems with infinite degrees of freedom

famous quote:

the contribution of axiomatic quantum field theory to physics is

About operator algebraic approaches in quantum systems with infinite degrees of freedom

~~famous quote:~~ notorious joke:

the contribution of axiomatic quantum field theory to physics is less than any given positive ε

Opinions of a mathematical physicist on operator algebraic approaches to spin systems

Opinions of a mathematical physicist on operator algebraic approaches to spin systems

student

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Hey! Here's a formulation that allows us to treat infinite systems as they are! Probably we can solve phase transitions, renormalization, and everything!

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old guy

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Hey! Here's a formulation that allows us to treat

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g!

d

It's

inf

results

WOW!

IT'S USEFUL!!!!!

posdoc

Ogata, Tasaki 2018

Ogata 2018, 2019

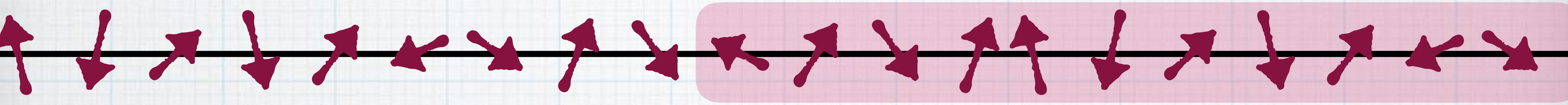
old guy

index theorems for SPT phases

Outline of the Proof

algebras for the half-infinite chain

$\mathfrak{A}_{\mathrm{R}}$ C^*-algebra of local operators on the half-infinite chain



$\omega(\cdot)$ the unique ground state of the whole infinite chain

GNS construction

$\mathcal{H}$ **Hilbert space** $\Omega \in \mathcal{H}$ **representation** $\pi : \mathfrak{A}_{\mathrm{R}} \to B(\mathcal{H})$

$$\omega(\hat{A}) = \langle \Omega, \pi(\hat{A})\Omega \rangle \text{ for any } \hat{A} \in \mathfrak{A}_{\mathrm{R}}$$

the set of all bounded operators

von Neumann algebra

$$\pi(\mathfrak{A}_{\mathrm{R}}) \xrightarrow[\text{bicommutant}]{} \pi(\mathfrak{A}_{\mathrm{R}})''$$

representation of the C* algebra

von Neumann algebra

$$\pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}}) \subset \pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}})'' \subset B(\mathcal{H})$$

illustration by Chisato Naruse

algebras for the half-infinite chain

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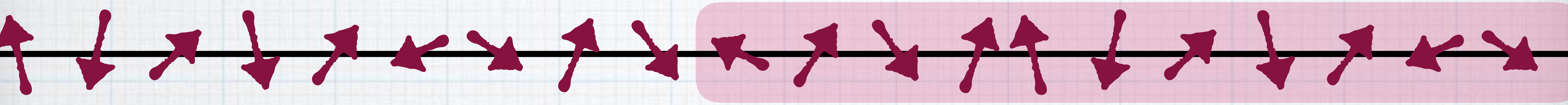

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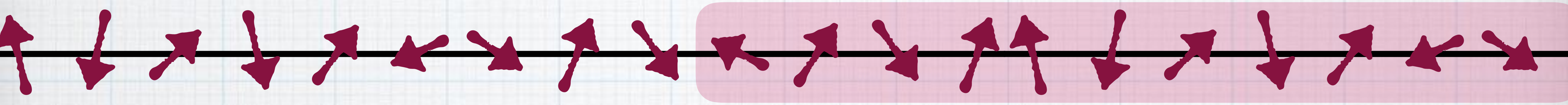

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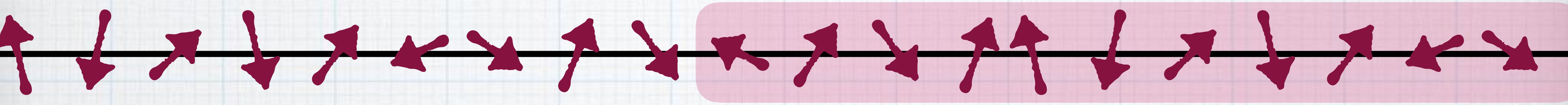
new Hilbert space $\tilde{\mathcal{H}}$

$\mathfrak{A}_{\mathrm{R}}$ C^*-algebra of local operators on the half-infinite chain



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representation of the C* algebra

von Neumann algebra

Hastings 2007
Matsui 2013

unique gapped GS satisfies this

$$\pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}}) \subset \pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}})'' \subset B(\mathcal{H})$$

if the state $\omega(\cdot)$ satisfies the split property, then $\pi(\mathfrak{A}_{\mathrm{R}})''$ is a type-I factor, which is the most well-behaved von Neumann algebra

then $\pi(\mathfrak{A}_{\mathrm{R}})'' \cong B(\tilde{\mathcal{H}})$ for some Hilbert space $\tilde{\mathcal{H}}$

the Cuntz algebra

by using the translation invariance we can construct a representation of the Cuntz algebra $c^\sigma \in B(\tilde{\mathcal{H}})$

$$\sigma = -S, \dots, S$$

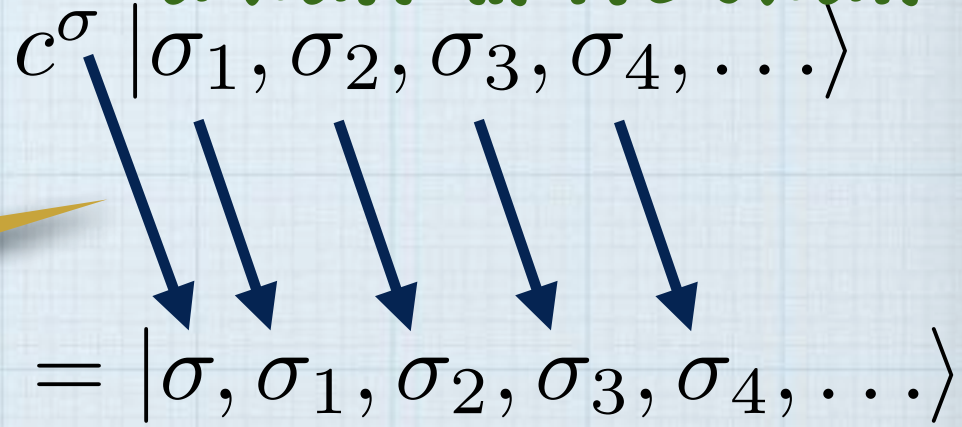
$(c^\sigma)_{\sigma=-S,\dots,S}$ "infinite dimension version" of matrices for MPS

$$(c^\sigma)^* c^{\sigma'} = \delta_{\sigma,\sigma'} \hat{1}$$

$$\pi_{\mathrm{R}}(|\sigma\rangle\langle\sigma'| \otimes \hat{1}_{[1,\infty)}) = c^\sigma (c^{\sigma'})^*$$

$$\sum_\sigma c^\sigma \pi_{\mathrm{R}}(\hat{A})(c^\sigma)^* = \pi_{\mathrm{R}}(\tau(\hat{A}))$$

related to the shift in a half-finite chain



$$c^\sigma |\sigma_1, \sigma_2, \sigma_3, \sigma_4, \dots\rangle$$

$$= |\sigma, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \dots\rangle$$

non-rigorous picture!

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the core of the proof

by using the translation invariance we can construct a representation of the Cuntz algebra $c^\sigma \in B(\tilde{\mathcal{H}})$

$\sigma = -S, \dots, S$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation of c^σ

$$c^\sigma = \zeta_\alpha \sum_{\sigma'} \langle \sigma | \hat{u}_\alpha^\dagger | \sigma' \rangle \tilde{\mathcal{R}}_\alpha (c^{\sigma'}) \quad \zeta_\alpha \in \mathbb{C} \quad |\zeta_\alpha| = 1 \quad \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$$

$$\hat{u}_\alpha = \exp[-i\pi \hat{S}^\alpha]$$

$\tilde{\mathcal{R}}_\mathrm{x}, \tilde{\mathcal{R}}_\mathrm{y}, \tilde{\mathcal{R}}_\mathrm{z}$ *-automorphisms on $B(\tilde{\mathcal{H}})$

give a genuine representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$

exactly the same transformation rule as in MPS!

$$\mathsf{M}^\sigma = \zeta_\alpha \sum_{\sigma'} \langle \sigma | \hat{u}_\alpha^\dagger | \sigma' \rangle \mathsf{U}_\alpha^\dagger \mathsf{M}^{\sigma'} \mathsf{U}_\alpha$$

the same argument leads to contradiction Matsui 2001

$$c^\sigma = \zeta_\alpha \sum_{\sigma'} \langle \sigma | \hat{u}_\alpha^\dagger | \sigma' \rangle \tilde{\mathcal{R}}_\alpha (c^{\sigma'})$$

projective representation

genuine representation

Extensions

symmetry

THEOREM 1: Consider a quantum spin chain with $S=\frac{1}{2},\frac{3}{2},\frac{5}{2},\dots$ and a short-ranged Hamiltonian that is invariant under translation and $\mathbb{Z}_2\times\mathbb{Z}_2$ transformation. Then it can never be the case that the corresponding ground state is unique and accompanied by a nonzero gap.

$\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry

↓

Chen, Gu, Wen 2011
Yuji Tachikawa, private communication

any on-site symmetry whose representation on a single spin is projective

example: time-reversal symmetry for $S=\frac{1}{2},\frac{3}{2},\frac{5}{2},\dots$

$$\hat{S}^\alpha_j \to -\hat{S}^\alpha_j$$

state

THEOREM 1: Consider a quantum spin chain with $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and a short-ranged Hamiltonian that is invariant under translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation. Then it can never be the case that the corresponding ground state is unique and accompanied by a nonzero gap.

it is only essential that the state is pure, translation invariant and satisfies the split property

any translation invariant pure state with area law entanglement is excluded Matsui 2013

general theorem

THEOREM 2: In quantum spin chains, there can be no translation invariant pure states with area law entanglement and on-site symmetry whose representation on a single spin is projective

COROLLARY: In a translation invariant spin chain with $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and time-reversal or $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, any "scar" state must be degenerate and break symmetry

no non-degenerate scar theorem

Summary

☑ LSM-type no-go theorem is proved for quantum spin chains with translation and on-site symmetry whose representation is projective

☑ the proof is based on the inconsistency between the projective symmetry and the transformation property of the Cuntz algebra

☑ it is surprising (to me, at least) that such a mathematically abstract object as the von Neumann algebra is useful in proving physically natural theorems

(cf. Ogata's fully rigorous index theorem for SPT)

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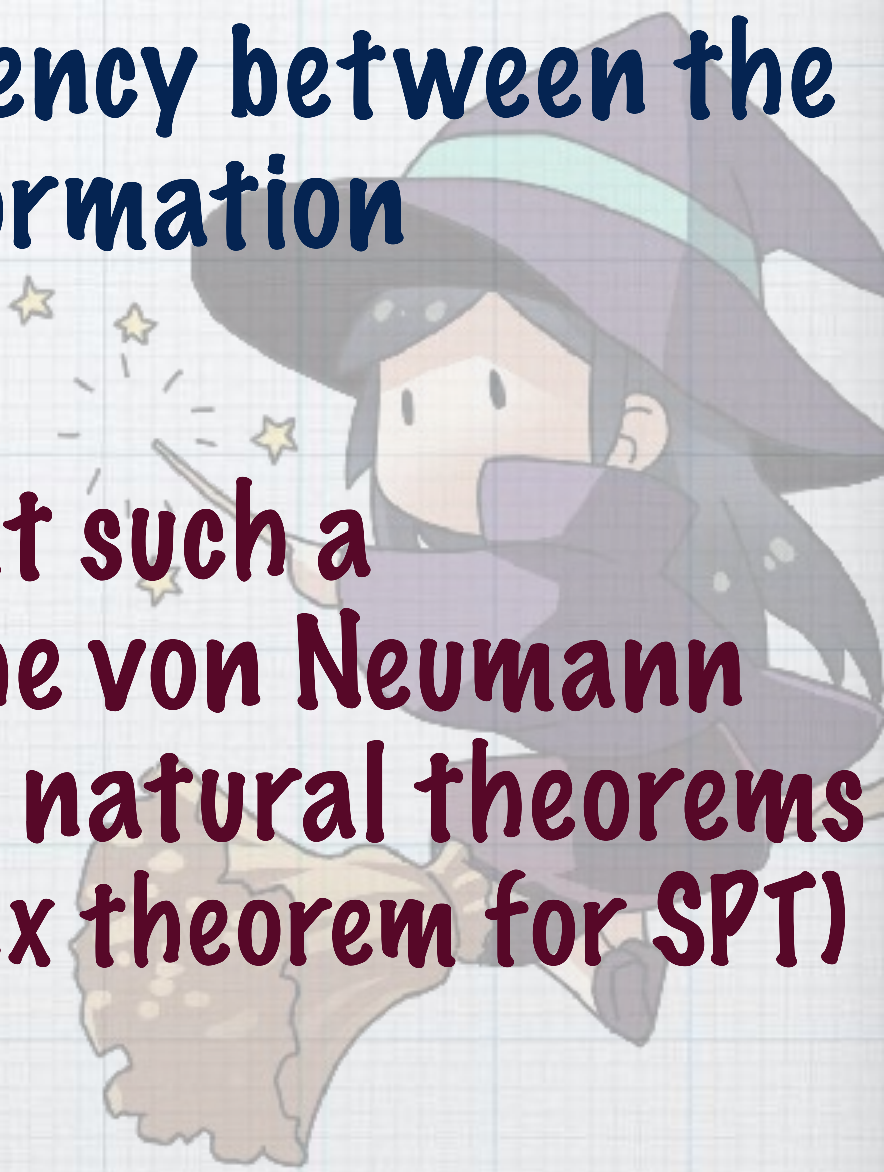

illustration by Chisato Naruse

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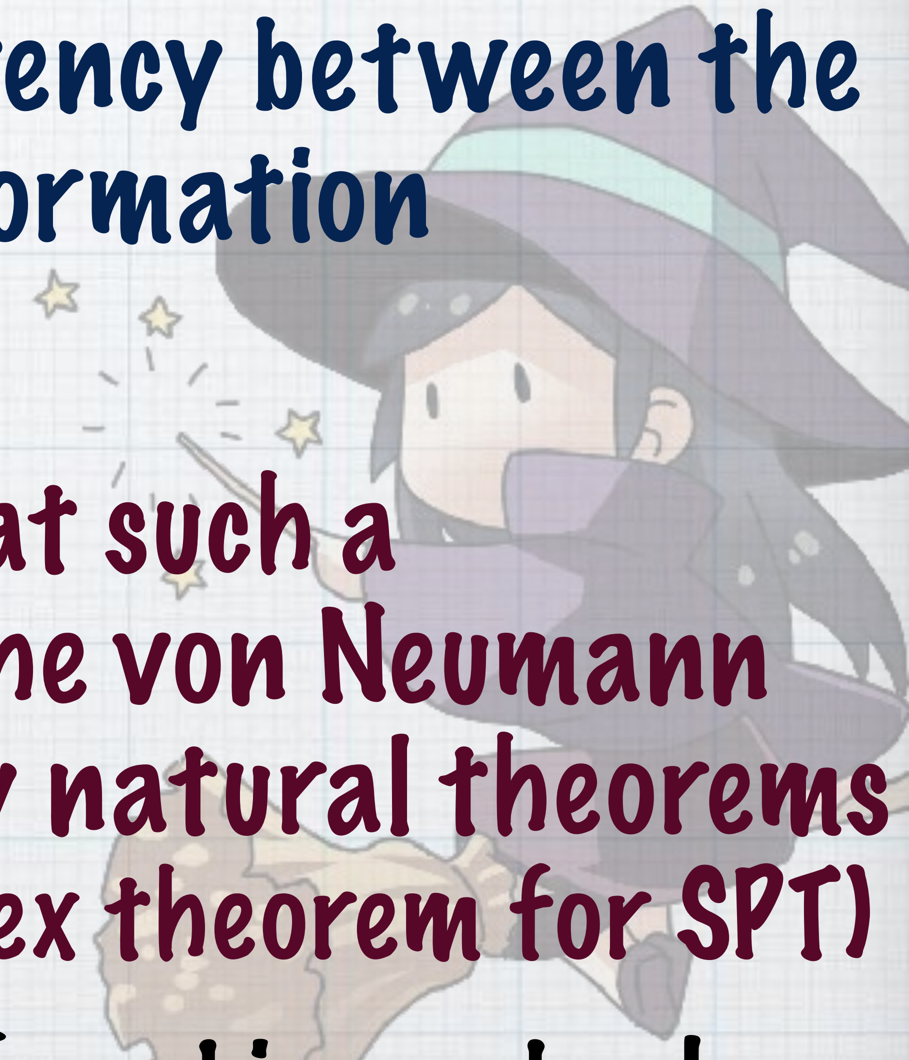

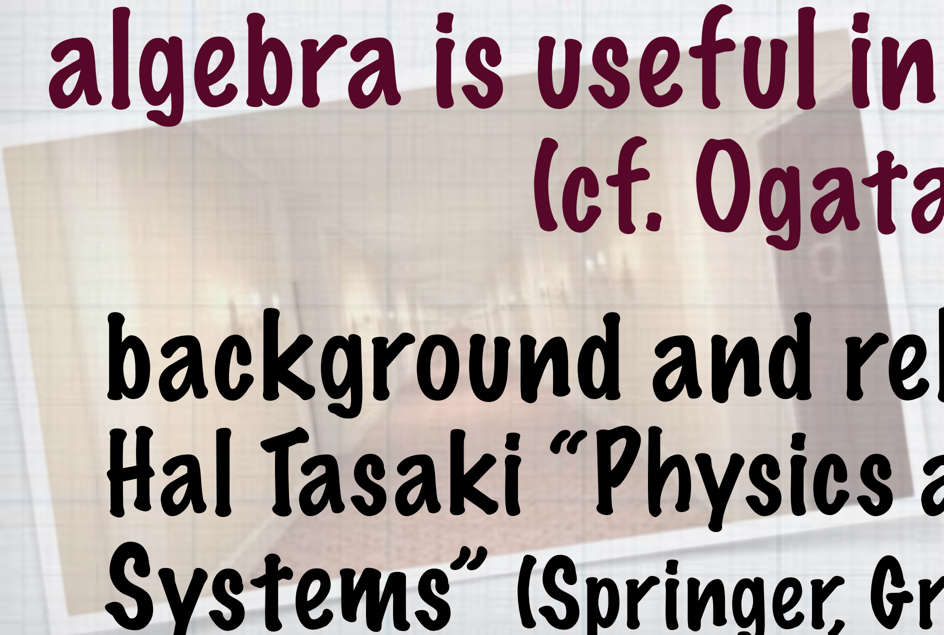

background and related topics can be found in my book: Hal Tasaki "Physics and Mathematics of Quantum Many-Body Systems" (Springer, Graduate Texts in Physics, 2020)

illustration by Chisato Naruse