The Rule 54: Exactly solvable deterministic interacting model of transport

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TMH2019, ICTS, 21 November 2019

Questions and Motivation

- Can matrix product ansatz be useful for encoding (time-dependent, or steady) states of deterministic reversible interacting systems?
- Find minimal interacting deterministic $(1+1)$d model about which we can 'know everything' (without approximations and assumptions)
- Check if the model has generic physical (say transport) properties!

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   see also: `A. Inoue, S. Takesue, arXiv:1806.07099`
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6. **Extra:** New integrable SO(3) symmetric classical-spin dynamics on space-time discrete lattice from baxterized set-theretic solution of YB and RLL relations. `Ž. Krajnik, TP, arXiv:1909.03799`

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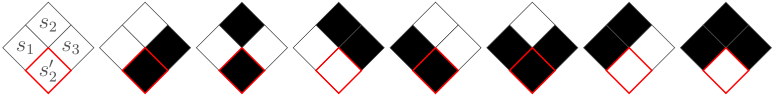

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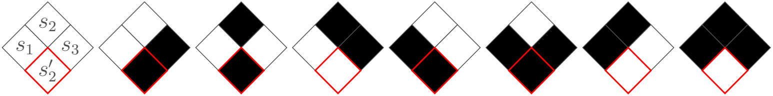

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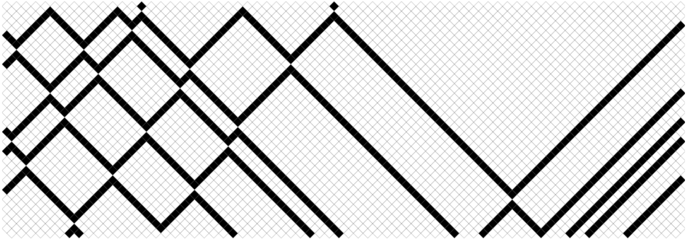
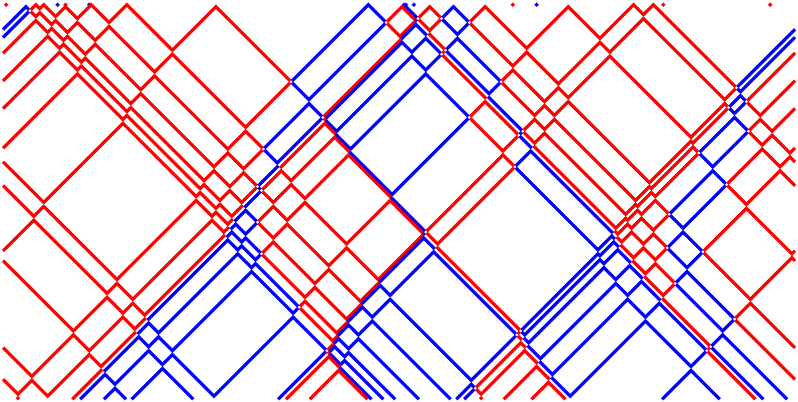


Two color version (low density):



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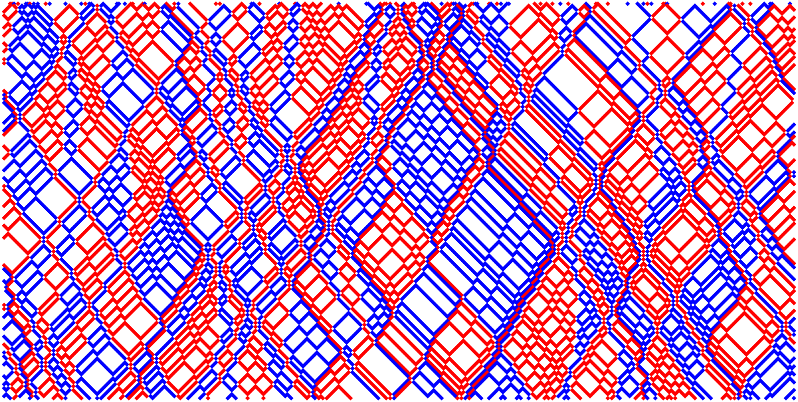

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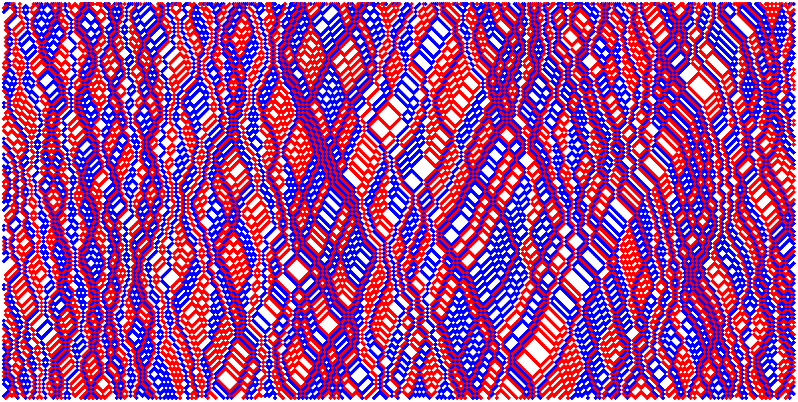
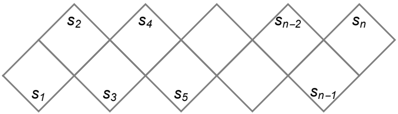


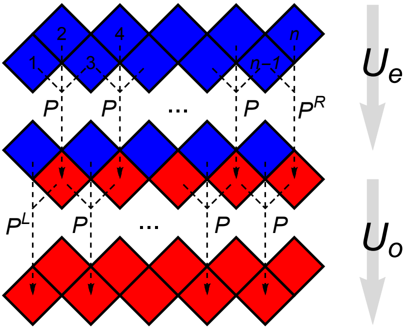
TP and C.Mejia-Monasterio, JPA **49**, 185003 (2016)

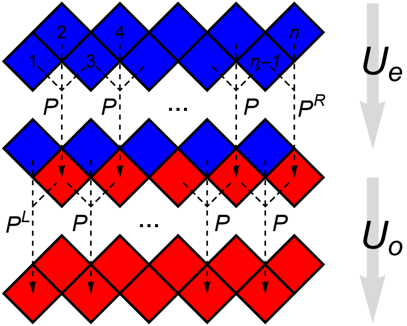
Describe an evolution of *probability state vector* for $n-$cell automaton

$$\mathbf{p}(t) = U^t \mathbf{p}(0)$$

$$\mathbf{p} = (p_0, p_1, \ldots, p_{2^n-1}) \equiv (p_{s_1,s_2,\ldots,s_n}; s_j \in \{0,1\})$$



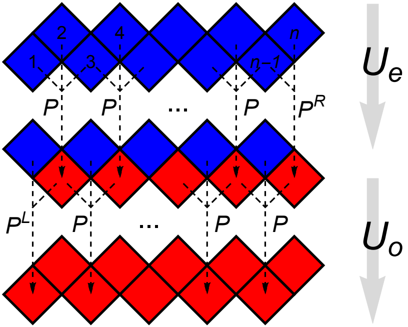

1
2
3
4
n−1
n
P
P
...
P
P^R
P^L
P
...
P
P
U_e
U_o



$$U = U_\mathrm{o} U_\mathrm{e},$$

$$U_\mathrm{e} = P_{123} P_{345} \cdots P_{n-3,n-2,n-1} P^\mathrm{R}_{n-1,n},$$

$$U_\mathrm{o} = P_{n-2,n-1,n} \cdots P_{456} P_{234} P^\mathrm{L}_{12}.$$



$$P = \begin{pmatrix} 1 & & & & & & & \\ & & & 1 & & & & \\ & & 1 & & & & & \\ & 1 & & & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & 1 & & & \\ & & & & & 1 & & \end{pmatrix}$$

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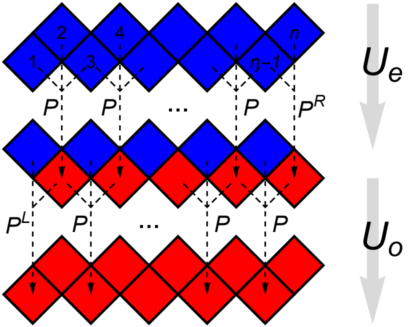

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$$P^{\mathrm{L}} = \begin{pmatrix} \alpha & 0 & \alpha & 0 \\ 0 & \beta & 0 & \beta \\ 1-\alpha & 0 & 1-\alpha & 0 \\ 0 & 1-\beta & 0 & 1-\beta \end{pmatrix}$$

$$P^{\mathrm{R}} = \begin{pmatrix} \gamma & \gamma & 0 & 0 \\ 1-\gamma & 1-\gamma & 0 & 0 \\ 0 & 0 & \delta & \delta \\ 0 & 0 & 1-\delta & 1-\delta \end{pmatrix}$$

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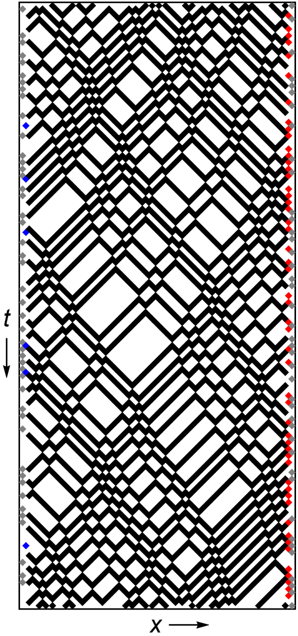
Some Monte-Carlo to warm up...



Theorem

The $2^n \times 2^n$ *matrix* U *is irreducible and aperiodic for generic values of driving parameters, more precisely, for an open set* $0 < \alpha, \beta, \gamma, \delta < 1$.

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Consequence (via *Perron-Frobenius* theorem): Nonequilibrium steady state (NESS), i.e. fixed point of U

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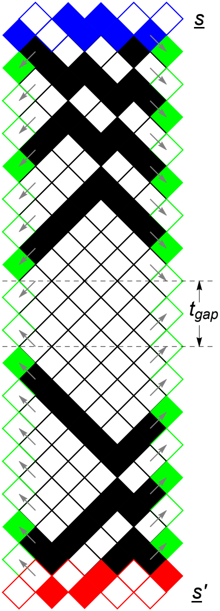

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Idea of the proof:
Show that for any pair of configurations $\mathbf{s}, \mathbf{s}'$, such t_0 exists that

$$(U^t)_{\mathbf{s},\mathbf{s}'} > 0, \quad \forall t \geq t_0.$$

[TP and B. Buča, JPA **50**, 395002 (2017)]

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Consider a pair of matrices:

$$W_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi & \xi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \xi & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \omega \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} W_0 \\ W_1 \end{pmatrix}$$

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and $W'_s(\xi,\omega) := W_s(\omega,\xi)$. These satisfy a remarkable **bulk relation**:

$$P_{123}\mathbf{W}_1 S \mathbf{W}_2 \mathbf{W}'_3 = \mathbf{W}_1 \mathbf{W}'_2 \mathbf{W}_3 S$$

or component-wise

$$W_s S W_{\chi(ss's'')} W'_{s''} = W_s W'_{s'} W_{s''} S.$$

where S is a "delimiter" matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Suppose there exists pairs and quadruples of vectors $\langle l_s|$, $\langle l'_{ss'}|$, $|r_{ss'}\rangle$, $|r'_s\rangle$, and a scalar parameter λ, satisfying the following *boundary equations*

$$
\begin{aligned}
P_{123}\langle \mathbf{l}_1|\mathbf{W}_2\mathbf{W}'_3 &= \langle \mathbf{l}'_{12}|\mathbf{W}_3 S,\\
P^{\mathrm{R}}_{12}|\mathbf{r}_{12}\rangle &= \mathbf{W}'_1 S|\mathbf{r}'_2\rangle,\\
P_{123}\mathbf{W}'_1\mathbf{W}_2|\mathbf{r}'_3\rangle &= \lambda\mathbf{W}'_1 S|\mathbf{r}_{23}\rangle,\\
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Then, the following probability vectors

$$
\begin{aligned}
\mathbf{p} &\equiv \mathbf{p}_{12\ldots n} = \langle \mathbf{l}_1|\mathbf{W}_2\mathbf{W}'_3\mathbf{W}_4\cdots\mathbf{W}'_{n-3}\mathbf{W}_{n-2}|\mathbf{r}_{n-1,n}\rangle,\\
\mathbf{p}' &\equiv \mathbf{p}'_{12\ldots n} = \langle \mathbf{l}'_{12}|\mathbf{W}_3\mathbf{W}'_4\cdots\mathbf{W}_{n-3}\mathbf{W}'_{n-2}\mathbf{W}_{n-1}|\mathbf{r}'_n\rangle,
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satisfy the NESS fixed point condition

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U_{\mathrm{e}}\mathbf{p} = \mathbf{p}', \quad U_{\mathrm{o}}\mathbf{p}' = \mathbf{p}.
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Proof: Observe the bulk relations

$$
\begin{aligned}
P_{123}\mathbf{W}_1 S\mathbf{W}_2\mathbf{W}'_3 \quad &= \quad \mathbf{W}_1\mathbf{W}'_2\mathbf{W}_3\mathbf{S},\\
P_{123}\mathbf{W}'_1\mathbf{W}_2\mathbf{W}'_3\mathbf{S} \quad &= \quad \mathbf{W}'_1\mathbf{S}\mathbf{W}'_2\mathbf{W}_3
\end{aligned}
$$

to move the delimiter S around, when it 'hits' the boundary observe one of the boundary equations. After the full cycle, you obtain $U_{\mathrm{o}}U_{\mathrm{e}}\mathbf{p} = \lambda\lambda^{-1}\mathbf{p}$.

This yields a consistent system of equations which uniquely determine the unknown parameters, namely for the left boundary:

$$\xi = \frac{(\alpha+\beta-1) - \lambda^{-1}\beta}{\lambda^{-2}(\beta-1)}, \qquad \omega = \frac{\lambda^{-1}(\alpha-\lambda^{-1})}{\beta-1},$$

and for the right boundary:

$$\xi = \frac{\lambda(\gamma-\lambda)}{\delta-1}, \qquad \omega = \frac{\gamma+\delta-1-\lambda\delta}{\lambda^2(\delta-1)},$$

yielding

$$\begin{aligned} \xi &= \frac{(\gamma(\alpha+\beta-1)-\beta)(\beta(\gamma+\delta-1)-\gamma)}{(\alpha-\delta(\alpha+\beta-1))^2}, \\ \omega &= \frac{(\delta(\alpha+\beta-1)-\alpha)(\alpha(\gamma+\delta-1)-\delta)}{(\gamma-\beta(\gamma+\delta-1))^2}, \end{aligned}$$

and explicit expressions for the boundary vectors..

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Yes, a good deal of decay modes can be written as a compact MPA with explicitly positionally dependent matrices

$$\mathbf{W}^{(x)}, \quad \mathbf{W}'^{(x)}$$

depending on $x \in \{2, 3, \ldots, n-1\}$ via multiplicative momentum variable z, containing linear combinations of

$$\{1, z^x, z^{-x}\}$$

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For example:

$$\mathbf{W}^{(x)} = (e_{11} \otimes \mathbf{W}(\xi z, \omega/z) + e_{22} \otimes \mathbf{W}(\xi/z, \omega z)) \left(\mathbb{1}_8 + e_{12} \otimes \frac{c_+ z^x F_+ + c_- z^{-x} F_-}{\xi\omega - 1} \right)$$

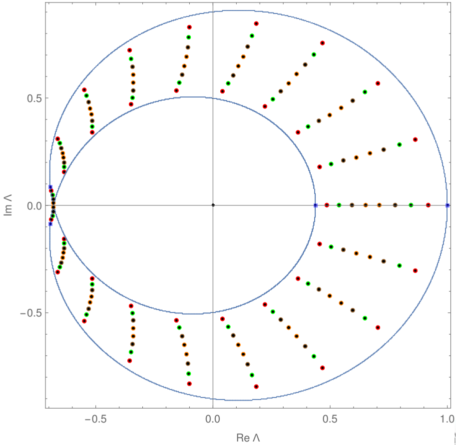
$$F_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & \frac{\xi\omega-1}{\omega z^2} & 0 \\ 0 & 0 & 0 & \xi z^2 \end{pmatrix}, \quad F_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\xi^2 z^3} \\ 0 & 0 & \frac{\xi\omega-1}{\xi z^2} & 0 \\ 0 & 0 & 0 & \omega + \frac{1}{\xi}\left(\frac{1}{z^2} - 1\right) \end{pmatrix}.$$

$$U_{\mathrm{e}}\mathbf{p}(z) = \Lambda_{\mathrm{L}}\mathbf{p}'(z), \quad U_{\mathrm{o}}\mathbf{p}'(z) = \Lambda_{\mathrm{R}}\mathbf{p}(z).$$

$$\frac{z(\alpha+\beta-1)-\beta\Lambda_{\mathrm{L}}}{(\beta-1)\Lambda_{\mathrm{L}}^2} = \frac{\Lambda_{\mathrm{R}}(\gamma z-\Lambda_{\mathrm{R}})}{(\delta-1)z},$$

$$\frac{z(\gamma+\delta-1)-\delta\Lambda_{\mathrm{R}}}{(\delta-1)\Lambda_{\mathrm{R}}^2} = \frac{\Lambda_{\mathrm{L}}(\alpha z-\Lambda_{\mathrm{L}})}{(\beta-1)z},$$

$$z^{2n-6-4p} = \frac{(\alpha+\beta-1)^p(\gamma+\delta-1)^p}{\Lambda_{\mathrm{L}}^{4p}\Lambda_{\mathrm{R}}^{4p}}.$$



Consider a (commutative C^*) **algebra of observables** on infinite lattice $x \in \mathbb{Z}$.

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Ultralocal basis $\{[0]_x, [1]_x\}$:

$$[\alpha]_x(\mathbf{s}) = \delta_{\alpha,s_x}, \qquad ([\alpha]_x[\beta]_y)(\mathbf{s}) = [\alpha]_x(\mathbf{s})\,[\beta]_y(\mathbf{s}), \quad \alpha,\beta \in \{0,1\},\ \mathbf{s} \in \{0,1\}^{\mathbb{Z}}$$

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r-local basis centred on site x:

$$[\alpha_1\alpha_2\ldots\alpha_r]_x \equiv [\alpha_1]_{x-\lfloor\frac{r}{2}\rfloor}[\alpha_2]_{x-\lfloor\frac{r}{2}\rfloor+1}\cdots[\alpha_r]_{x+\lfloor\frac{r-1}{2}\rfloor}.$$

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Using unit element $\mathbb{1} = [0]_x + [1]_x$, we can extend the support of each r-local basis element as

$$[\alpha_1\alpha_2\ldots\alpha_r]_x \equiv \mathbb{1}_{x-\lfloor\frac{r+2}{2}\rfloor}\cdot[\alpha_1\alpha_2\ldots\alpha_r]_x\cdot\mathbb{1}_{x+\lfloor\frac{r+1}{2}\rfloor} \equiv$$
$$\equiv [0\alpha_1\alpha_2\ldots\alpha_r0]_x + [0\alpha_1\alpha_2\ldots\alpha_r1]_x + [1\alpha_1\alpha_2\ldots\alpha_r0]_x + [1\alpha_1\alpha_2\ldots\alpha_r1]_x.$$

Separable (strongly clustering) states p defined by expectation values $p(x)$ of ultralocal observables

$$\langle [\alpha_1\alpha_2\ldots\alpha_r]_x\rangle_p = p_{x-\lfloor\frac{r}{2}\rfloor}(\alpha_1)\cdot p_{x-\lfloor\frac{r}{2}\rfloor+1}(\alpha_2)\cdots p_{x+\lfloor\frac{r-1}{2}\rfloor}(\alpha_r).$$

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Two examples of separable states that we consider:

1. A *maximum entropy state*

$$p_x(0) = p_x(1) = 1/2, \quad \forall x \in \mathbb{Z}.$$

2. An *inhomogeneous initial state*

$$\begin{cases} p_x(0) = p_x(1) = 1/2, & \text{for } x \leq 0 \\ p_x(0) = 1, \quad p_x(1) = 0. & \text{for } x > 0 \end{cases}$$

Dynamics: Time automorphism of algebra of observables

$$a^t(\mathbf{s}^0) = a(\mathbf{s}^t)$$

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For 3-site observables, dynamical automorphism is defined as

$$U_x[\alpha\ \beta\ \gamma]_y = \begin{cases} [\alpha\ \chi(\alpha,\beta,\gamma)\ \gamma]_y; & x = y, \\ [\alpha\ \beta\ \gamma]_y; & |x-y| \geq 2, \end{cases}$$

while for any r-local observable it is defined as
a *t-staggered linear homomorphism*

$$a^{t+1} = U(t)a^t$$

$$U(t) = \begin{cases} \prod_{x\in 2\mathbb{Z}} U_x; & t \equiv 0 \pmod 2, \\ \prod_{x\in 2\mathbb{Z}+1} U_x; & t \equiv 1 \pmod 2. \end{cases}$$

Theorem (Klobas *et al.* 18): Time evolution of a local observable $[1]_x$ reads

$$[1]_x^t = \sum_{s_{-t},\ldots,s_t\in\{0,1\}} c_{s_{-t},\ldots,s_t}(t)[s_{-t}s_{-t+1}\cdots s_t]_x,$$

where the amplitudes $c_{s_{-t},\ldots,s_t}(t)\in\{0,1\}$ can be represented as MPA

$$c_{s_{-t},\ldots s_t}(t) = \langle l(t)|V_{s_{-t}}W_{s_{-t+1}}V_{s_{-t+2}}\cdots W_{s_{t-1}}V_{s_t}|r\rangle + \\ + \langle l'|V'_{s_{-t}}W'_{s_{-t+1}}V'_{s_{-t+2}}\cdots W'_{s_{t-1}}V'_{s_t}|r'(t)\rangle.$$

$V_s,\ W_s,\ V'_s,\ W'_s \in \mathrm{End}(\mathcal{V})$, $s\in\{0,1\}$, are linear operators over auxiliary Hilbert space $\mathcal{V}=\mathrm{lsp}\{|c,w,n,a\rangle;\ c,w\in\mathbb{N}_0,\ n\in\{0,1,2\},\ a\in\{0,1\}\}$, and can be explicitly expressed in terms of ladder operators and projectors

$$\mathbf{c}^+ = \sum_{c,w,n,a}|c+1,w,n,a\rangle\langle c,w,n,a|, \qquad \mathbf{c}^- = \left(\mathbf{c}^+\right)^T,$$

$$\mathbf{w}^+ = \sum_{c,w,n,a}|c,w+1,n,a\rangle\langle c,w,n,a|, \qquad \mathbf{w}^- = \left(\mathbf{w}^+\right)^T,$$

$$\mathbf{e}_{c_2w_2n_2a_2,c_1w_1n_1a_1} = |c_2,w_2,n_2,a_2\rangle\langle c_1,w_1,n_1,a_1|,$$

$$\mathbf{e}_{n_2a_2,n_1a_1} = \sum_{c,w}|c,w,n_2,a_2\rangle\langle c,w,n_1,a_1|,$$

$$
\begin{aligned}
V_0 &= \mathbf{e}_{00,00} + \mathbf{e}_{10,00} + \mathbf{e}_{20,00} + \mathbf{c}^+\mathbf{e}_{10,01} + \mathbf{e}_{01,01} + \mathbf{c}^+\mathbf{w}^+\mathbf{e}_{11,01} + \mathbf{e}_{21,01} + \\
&\quad + \mathbf{e}_{0001,0001} + \mathbf{e}_{0011,0001} + \mathbf{e}_{0021,0001}, \\
V_1 &= \mathbf{e}_{00,10} + \mathbf{e}_{10,20} + \mathbf{e}_{20,20} + \mathbf{e}_{00,11} + \mathbf{e}_{10,21} + \mathbf{e}_{20,21} + \mathbf{e}_{01,11} + \\
&\quad + \mathbf{w}^+\mathbf{e}_{11,21} + \mathbf{w}^+\mathbf{e}_{21,21} + \mathbf{e}_{0001,0011} + \mathbf{e}_{0011,0021} + \mathbf{e}_{0021,0021}, \\
W_0 &= \mathbf{c}^-\mathbf{w}^+\left(\mathbf{e}_{00,00} + \mathbf{e}_{10,00} + \mathbf{e}_{20,00}\right) + \mathbf{w}^+\mathbf{e}_{10,01} + \mathbf{w}^+\mathbf{e}_{01,01} + \\
&\quad + \mathbf{c}^+\left(\mathbf{w}^+\right)^2\mathbf{e}_{11,01} + \mathbf{w}^+\mathbf{e}_{21,01} + \mathbf{e}_{1111,0001} + \mathbf{e}_{0001,0001} + \mathbf{e}_{0011,0001} + \mathbf{e}_{0021,0001}, \\
W_1 &= \mathbf{c}^-\mathbf{w}^+\left(\mathbf{e}_{00,10} + \mathbf{e}_{10,20} + \mathbf{e}_{20,20}\right) + \mathbf{w}^+\mathbf{e}_{01,11} + \mathbf{c}^+\mathbf{w}^+\mathbf{e}_{11,21} + \\
&\quad + \mathbf{c}^+\mathbf{w}^+\mathbf{e}_{21,21} + \mathbf{e}_{0001,0011} + \mathbf{e}_{0011,0021} + \mathbf{e}_{0021,0021}, \\
V_0' &= V_0^T - \left(\mathbf{e}_{0001,1111} + \mathbf{e}_{0101,1211} + \mathbf{e}_{0101,1110}\right), \\
V_1' &= V_1^T, \\
W_0' &= W_0^T - \left(\mathbf{e}_{0001,1111} + \mathbf{e}_{0000,1211}\right), \\
W_1' &= W_1^T - \left(\mathbf{e}_{0021,1111} + \mathbf{e}_{0021,1121} + \mathbf{e}_{0121,1211} + \mathbf{e}_{0121,1221}\right).
\end{aligned}
$$

$$
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V_0 &= \mathbf{e}_{00,00} + \mathbf{e}_{10,00} + \mathbf{e}_{20,00} + \mathbf{c}^+\mathbf{e}_{10,01} + \mathbf{e}_{01,01} + \mathbf{c}^+\mathbf{w}^+\mathbf{e}_{11,01} + \mathbf{e}_{21,01} + \\
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&\quad + \mathbf{c}^+\left(\mathbf{w}^+\right)^2\mathbf{e}_{11,01} + \mathbf{w}^+\mathbf{e}_{21,01} + \mathbf{e}_{1111,0001} + \mathbf{e}_{0001,0001} + \mathbf{e}_{0011,0001} + \mathbf{e}_{0021,0001}, \\
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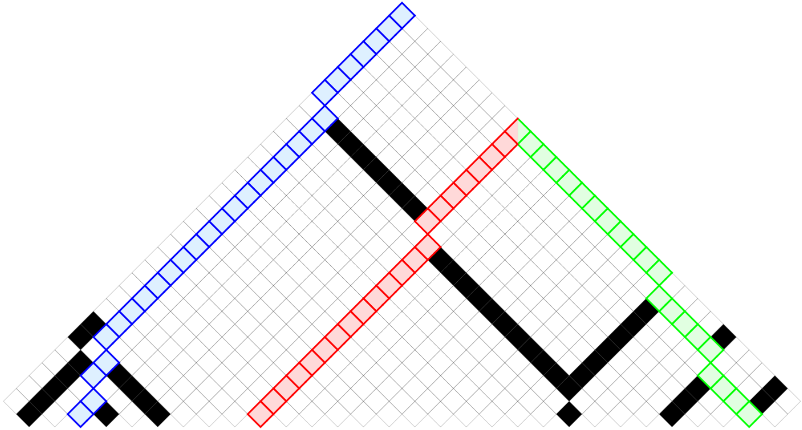
&\quad + \mathbf{c}^+\mathbf{w}^+\mathbf{e}_{21,21} + \mathbf{e}_{0001,0011} + \mathbf{e}_{0011,0021} + \mathbf{e}_{0021,0021}, \\
V'_0 &= V_0^T - \left(\mathbf{e}_{0001,1111} + \mathbf{e}_{0101,1211} + \mathbf{e}_{0101,1110}\right), \\
V'_1 &= V_1^T, \\
W'_0 &= W_0^T - \left(\mathbf{e}_{0001,1111} + \mathbf{e}_{0000,1211}\right), \\
W'_1 &= W_1^T - \left(\mathbf{e}_{0021,1111} + \mathbf{e}_{0021,1121} + \mathbf{e}_{0121,1211} + \mathbf{e}_{0121,1221}\right).
\end{aligned}
$$

The time-dependent auxiliary space boundary vectors take the following form:

$$
\begin{aligned}
\langle l(t)| &= \langle 0, t, 0, 0|, \\
|r\rangle &= |0,0,0,0\rangle + |0,0,0,1\rangle + |0,0,0,2\rangle, \\
\langle l'| &= \langle 0,0,0,1| + \langle 0,0,1,1| + \langle 0,0,2,1| + \langle 0,1,0,1| + \langle 0,1,2,1|, \\
|r'(t)\rangle &= |0, t+1, 0, 0\rangle.
\end{aligned}
\tag{1}
$$

Proof: *'Real space, real time inverse scattering transform'*



The weight of left MPA $\langle l(t)|V_{s_{-t}}W_{s_{-t+1}}V_{s_{-t+2}}\cdots W_{s_{t-1}}V_{s_t}|r\rangle$ is 1 (or 0) if the configuration $(s_{-t}, s_{-t+1}, \ldots, s_t)$ can (cannot) be obtained in a light-cone with the *left-mover at the origin*!

$$C(x,t) = \langle [1]_x [1]_0^t \rangle_p - \langle [1]_x \rangle_p \langle [1]_0^t \rangle_p = \langle [1]_x [1]_0^t \rangle_p - \frac{1}{4}$$

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Using time-dependent MPA:

$$C(x,t) = \frac{1}{2^{2t+1}} \left(\langle l(t) | T^{\frac{x+t}{2}} V_1 \overline{T}^{t-\frac{x+t}{2}} | r \rangle + \langle l' | \overline{T}'^{\frac{x+t}{2}} V_1' T'^{t-\frac{x+t}{2}} | r'(t) \rangle \right) - \frac{1}{4}$$

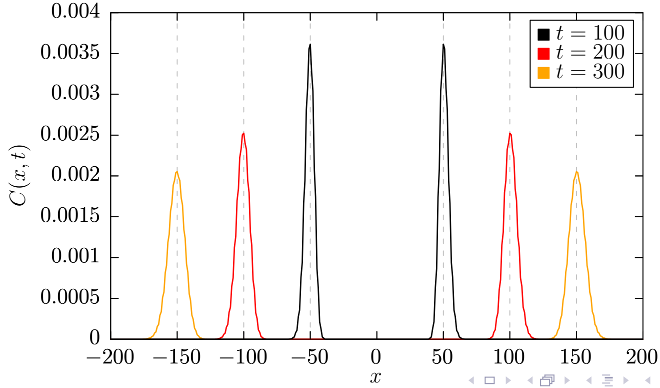
with

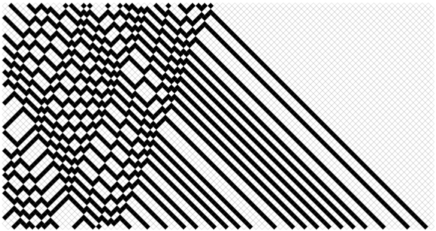
$$\begin{aligned} T &= (V_0 + V_1)(W_0 + W_1), & \overline{T} &= (W_0 + W_1)(V_0 + V_1), \\ T' &= (W_0' + W_1')(V_0' + V_1'), & \overline{T}' &= (V_0' + V_1')(W_0' + W_1'). \end{aligned}$$

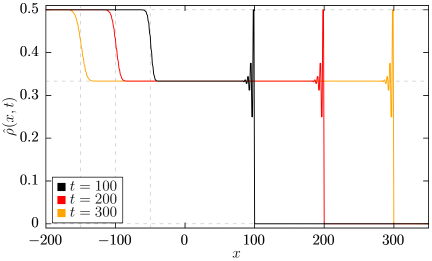
We find normal hydrodynamic scaling!

For maximum entropy ('infinite temperature') state, tMPA yields

$$C(x,t) = 2^{-t-1} \sum_{m=0}^{\frac{t-|x|-2}{2}} 4^m \left(2\binom{t-2m-3}{m} - \binom{t-2m-2}{m} \right)$$
$$\simeq \frac{1}{16\sqrt{t\pi}} \exp\left(-\frac{4}{t} \left(|x| - \frac{t}{2} \right)^2 \right).$$







$$\hat{\rho}(x,t) = \langle [1]_x \rangle_{p^t_{\mathrm{inhom}}} = \langle [1]_x^{-t} \rangle_{p_{\mathrm{inhom}}}$$

Exact solution exhibits the following simple asymptotic behavior:

- Quasi-free regime

$$\hat{\rho}\left(t \geq x \geq -\frac{t}{3} + 1, t\right) = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^{\lfloor \frac{t-x+1}{2} \rfloor}\right)$$

- Thermalizing (diffusive) regime

$$\lim_{t\to\infty} \hat{\rho}\left(-\frac{t}{2} + \zeta\sqrt{t}, t\right) = \frac{1}{12}\left(5 - \mathrm{erf}(2\zeta)\right)$$

[B.Buča, J.P.Garrahan, T.Prosen, M.Vanicat, arXiv:1901.00845]
Large deviation theory for arbitrary observable of the form:

$$\mathcal{O}_T = \sum_{t=0}^{T-1} \sum_{x=1}^{N-1} \left[f_x(s_x^t, s_{x+1}^t) + g_x(s_x^{t+1/2}, s_{x+1}^{t+1/2}) \right]$$

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Tilted Markov generator:

$$\tilde{U}(s) = U_{\rm o}\, G(s)\, U_{\rm e}\, F(s).$$

$$F(s) = F_{12}^{(1)} F_{23}^{(2)} F_{34}^{(3)} \ldots F_{N-1,N}^{(N-1)} \quad \text{and} \quad G(s) = G_{12}^{(1)} G_{23}^{(2)} G_{34}^{(3)} \ldots G_{N-1,N}^{(N-1)}$$

where

$$F^{(x)} = \begin{pmatrix} f_{0,0}^{(x)} & 0 & 0 & 0 \\ 0 & f_{0,1}^{(x)} & 0 & 0 \\ 0 & 0 & f_{1,0}^{(x)} & 0 \\ 0 & 0 & 0 & f_{1,1}^{(x)} \end{pmatrix}, \qquad f_{s,s'}^{(x)} \equiv e^{s f_x(s,s')}$$

and similar for $G^{(x)}$.

There exist 3×3 matrices satisying bulk algebraic conditions:

$$f^{(j-1)}_{ss'} f^{(j)}_{s's''} W^{(j-1)}_{s} W^{(j)}_{s'} X^{(j+1)}_{s''} = X^{(j-1)}_{s} V^{(j)}_{\chi(ss's'')} V^{(j+1)}_{s''},$$
$$g^{(j-2)}_{ss'} g^{(j-1)}_{s's''} X^{(j-2)}_{s} V^{(j-1)}_{s'} V^{(j)}_{s''} = W^{(j-2)}_{s} W^{(j-1)}_{\chi(ss's'')} X^{(j)}_{s''},$$

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and boundary equations

$$\begin{aligned}
f^{(1)}_{ss'} f^{(2)}_{s's''} \langle l_s | W^{(2)}_{s'} X^{(3)}_{s''} &= \langle l'_{s\chi(ss's'')} | V^{(3)}_{s''}, \\
\sum_{m,m'=0,1} R^{mm'}_{ss'} f^{(N-1)}_{mm'} | r_{mm'} \rangle &= \lambda_{\mathrm{R}} X^{(N-1)}_{s} | r'_{s'} \rangle, \\
\sum_{m,m'=0,1} L^{mm'}_{ss'} g^{(1)}_{mm'} \langle l'_{mm'} | &= \lambda_{\mathrm{L}} \langle l_s | X^{(2)}_{s'}, \\
g^{(N-2)}_{ss'} g^{(N-1)}_{s's''} X^{(N-2)}_{s} V^{(N-1)}_{s'} | r'_{s''} \rangle &= W^{(N-2)}_{s} | r_{\chi(ss's'')s''} \rangle,
\end{aligned}$$

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\end{aligned}$$

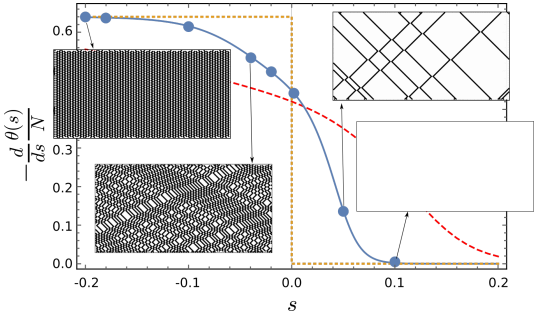
such that MPA:

$$\begin{aligned}
p_{s_1,\ldots,s_N} &= \langle l_{s_1}| W^{(2)}_{s_2} W^{(3)}_{s_3} \cdots W^{(N-3)}_{s_{N-3}} W^{(N-2)}_{s_{N-2}} |r_{s_{N-1}s_N}\rangle \\
p'_{s_1,\ldots,s_N} &= \langle l'_{s_1 s_2}| V^{(3)}_{s_3} V^{(4)}_{s_4} \cdots V^{(N-2)}_{s_{N-2}} V^{(N-1)}_{s_{N-1}} |r'_{s_N}\rangle,
\end{aligned}$$

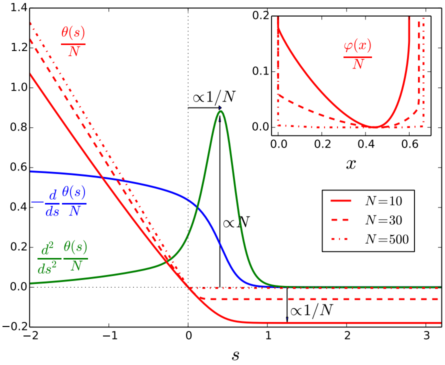
solves the eigenalue equation

$$\tilde{U}(s)\mathbf{p} = \Lambda(s)\mathbf{p}$$

and $\Lambda(s) = e^{\theta(s)}$ is a root of third order polynomial.



$-\frac{d}{ds}\frac{\theta(s)}{N}$
0.0
0.1
0.2
0.3
0.6
-0.2
-0.1
0.0
0.1
0.2
s



$\frac{\theta(s)}{N}$
$-\frac{d}{ds}\frac{\theta(s)}{N}$
$\frac{d^2}{ds^2}\frac{\theta(s)}{N}$
$\propto 1/N$
$\propto N$
$\propto 1/N$
$\frac{\varphi(x)}{N}$
x
$N=10$
$N=30$
$N=500$
s

- Interacting integrable model about which we can compute everything: *quenches, non-equilibrium steady states with baths, relaxation rates, dynamical structure factor, large deviations etc.*
- Generalizations (stochastic/unitary branching)? Link to Yang-Baxter integrability missing?
- Testbed for computing diffusive corrections to generalized hydroduynamics. See e.g.: S. Gopalakrishnan, D. Huse, V. Khemani, R. Vasseur, PRB **98**, 220303 (2018)

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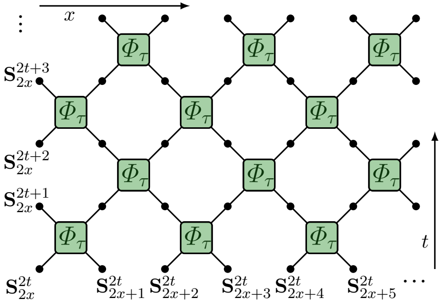
Family of rational-symplectic maps over $\mathcal{S}^2 \times \mathcal{S}^2$ ($\mathbf{S}_1 \cdot \mathbf{S}_1 = \mathbf{S}_2 \cdot \mathbf{S}_2 = 1$):

$$\Phi_\tau(\mathbf{S}_1, \mathbf{S}_2) = \frac{1}{\sigma^2 + \tau^2}\left(\sigma^2 \mathbf{S}_1 + \tau^2 \mathbf{S}_2 + \tau \mathbf{S}_1 \times \mathbf{S}_2, \sigma^2 \mathbf{S}_2 + \tau^2 \mathbf{S}_1 + \tau \mathbf{S}_2 \times \mathbf{S}_1\right),$$

$$\sigma^2 := \frac{1}{2}\left(1 + \mathbf{S}_1 \cdot \mathbf{S}_2\right),$$

defining discrete space-time many-body symplectic dynamics
(classical local Floquet circuit)

$$(S^{2t+1}_{2x}, \mathbf{S}^{2t+1}_{2x+1}) = \Phi_\tau(\mathbf{S}^{2t}_{2x}, \mathbf{S}^{2t}_{2x+1}), \qquad (\mathbf{S}^{2t+2}_{2x-1}, \mathbf{S}^{2t+2}_{2x}) = \Phi_\tau(\mathbf{S}^{2t+1}_{2x-1}, \mathbf{S}^{2t+1}_{2x}),$$



Defining $\mathbb{I}$ a identity map over $\mathcal{S}^2$ we find (note that $\Phi_\infty(\mathbf{S}_1, \mathbf{S}_2) = (\mathbf{S}_2, \mathbf{S}_1)$):

$$\left(\Phi_\lambda \otimes \mathbb{I}\right) \circ \left(\mathbb{I} \otimes \Phi_{\lambda+\mu}\right) \circ \left(\Phi_\mu \otimes \mathbb{I}\right) = \left(\mathbb{I} \otimes \Phi_\mu\right) \circ \left(\Phi_{\lambda+\mu} \otimes \mathbb{I}\right) \circ \left(\mathbb{I} \otimes \Phi_\lambda\right).$$

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Furthermore, defining a matrix valued function $L(\lambda) : \mathcal{S}^2 \to \mathrm{End}(\mathbb{C}^2)$

$$L(\mathbf{S}; \lambda) = \mathbb{1} + \frac{1}{2i\lambda} \mathbf{S} \cdot \boldsymbol{\sigma},$$

we find that RLL relation is obeyed:

$$L(\mathbf{S}_2; \lambda) L(\mathbf{S}_1; \mu) = L(\mathbf{S}_2'; \mu) L(\mathbf{S}_1'; \lambda), \quad (\mathbf{S}_1', \mathbf{S}_2') := \Phi_{\lambda-\mu}(\mathbf{S}_1, \mathbf{S}_2).$$

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$$L(\mathbf{S}_2;\lambda)L(\mathbf{S}_1;\mu)=L(\mathbf{S}'_2;\mu)L(\mathbf{S}'_1;\lambda),\quad(\mathbf{S}'_1,\mathbf{S}'_2):=\Phi_{\lambda-\mu}(\mathbf{S}_1,\mathbf{S}_2).$$

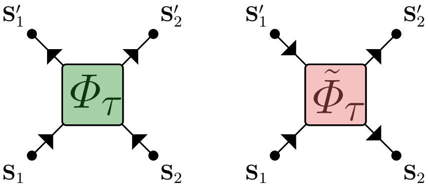
Defining monodromy-matrix

$$T(\mathbf{S}_0,\mathbf{S}_1,\ldots,\mathbf{S}_{N-1};\lambda,\mu)=\mathrm{tr}\left(\overleftarrow{\prod_{x=0}^{N/2-1}}L(\mathbf{S}_{2x+1};\lambda)L(\mathbf{S}_{2x};\mu)\right),$$

subsequent application of RLL relations shows manifest conservation of its trace (for any λ while $\mu=\lambda-\tau$)

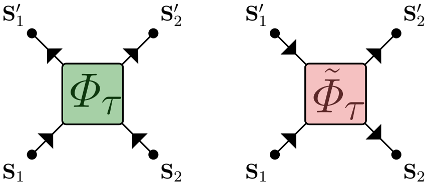
$$T(\mathbf{S}^t_0\ldots\mathbf{S}^t_{N-1};\lambda,\lambda-\tau)=T(\mathbf{S}^{t+1}_0\ldots\mathbf{S}^{t+1}_{N-1};\lambda-\tau,\lambda)=T(\mathbf{S}^{t+2}_0\ldots\mathbf{S}^{t+2}_{N-1};\lambda,\lambda-\tau).$$

$$
\begin{aligned}
Q_k^{\rm even} &= \partial_\lambda^k \log|T(\lambda,\lambda-\tau)|^2|_{\lambda=-\frac{i}{2}}, \\
Q_k^{\rm odd} &= \partial_\lambda^k \log|T(\lambda,\lambda-\tau)|^2|_{\lambda=\tau-\frac{i}{2}}, \qquad k=0,1,2\ldots
\end{aligned}
$$



$$(\mathbf{S}'_1, \mathbf{S}'_2) = \Phi_\tau(\mathbf{S}_1, \mathbf{S}_2), \qquad (\mathbf{S}_2, \mathbf{S}'_2) = \tilde{\Phi}_\tau(\mathbf{S}_1, \mathbf{S}'_1).$$

Space-Time Self-Duality



$$(\mathbf{S}_1', \mathbf{S}_2') = \Phi_\tau(\mathbf{S}_1, \mathbf{S}_2), \qquad (\mathbf{S}_2, \mathbf{S}_2') = \tilde{\Phi}_\tau(\mathbf{S}_1, \mathbf{S}_1').$$

$$\Xi \circ \tilde{\Phi}_\tau = \Phi_\tau \circ (-\Xi),$$

$$\Xi(\mathbf{S}, \mathbf{S}') := (\mathbf{S}, -\mathbf{S}'), \quad (-\Xi)(\mathbf{S}, \mathbf{S}') := (-\mathbf{S}, \mathbf{S}').$$

$$C(x,t) = \langle S_x^t S_0^0 \rangle - \langle S_x^t \rangle \langle S_0^0 \rangle$$