Discrete and continuous spectrum for free, standing viscous capillary-gravity waves

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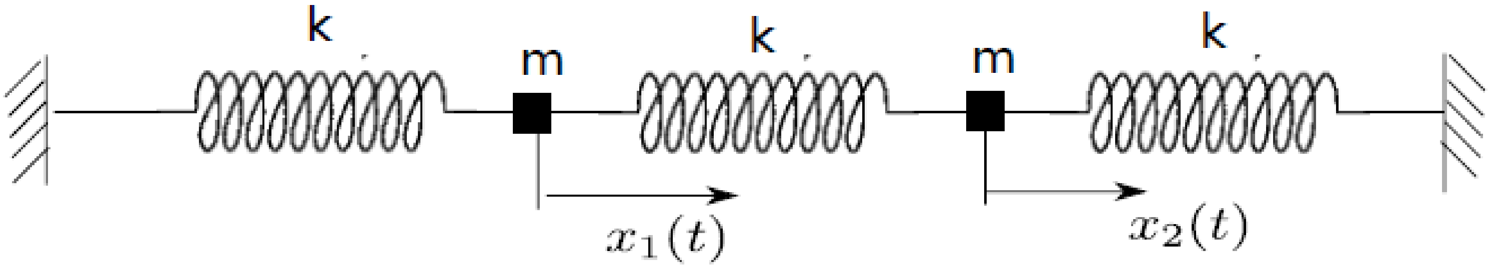
Two degree of freedom spring-mass system



$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2 = -k(x_2 - x_1) - kx_2$$

$$\ddot{\boldsymbol{X}} + \boldsymbol{A} \cdot \boldsymbol{X}(\boldsymbol{t}) = \boldsymbol{0}$$

$$\boldsymbol{X}(t) \equiv \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \mathbf{A} \equiv \begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ \frac{-k}{m} & \frac{2k}{m} \end{bmatrix}$$

Normal mode analysis:
Exponential dependence on time

$$\boldsymbol{X}(t) \equiv \begin{bmatrix} a \\ b \end{bmatrix} \exp(i\omega t)$$

Two degree of freedom spring-mass system

Solve the IVP with: $x_1(0), x_2(0), \dot{x}_1(0)$ and $\dot{x}_2(0)$ given

$$\boldsymbol{X}(t) = (c_1 \exp[i\omega_1 t] + c_1^* \exp[-i\omega_1 t])\, \boldsymbol{e}_1 + (c_2 \exp[i\omega_2 t] + c_2^* \exp[-i\omega_2 t])\, \boldsymbol{e}_2$$

$$\omega_1^2 = \lambda_1 = \frac{k}{m}, \quad \boldsymbol{e}_1 \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \omega_2^2 = \lambda_2 = \frac{3k}{m} \quad \boldsymbol{e}_2 \equiv \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Only if both the eigenvectors are linearly independent and span the 2 (for N coupled oscillators, the N) dimensional space.

What happens if the eigenvectors do not span the space ?

i.e in Linear Algebra terminology, we do not have N Linearly independent eigenvectors?

Another example

$$\ddot{\mathbf{X}} + \boldsymbol{A} \cdot \boldsymbol{X}(\boldsymbol{t}) = \mathbf{0} \qquad \mathbf{A} \equiv \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\ddot{x}_1 + x_1(t) - x_2(t) = 0$$

$$\ddot{x}_2 + x_2(t) = 0$$

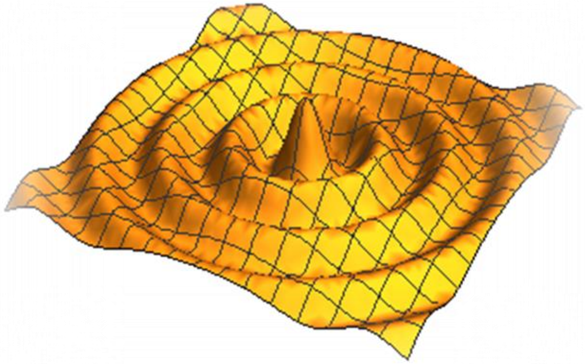
$$\lambda = 1, 1$$

$$\boldsymbol{e} \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

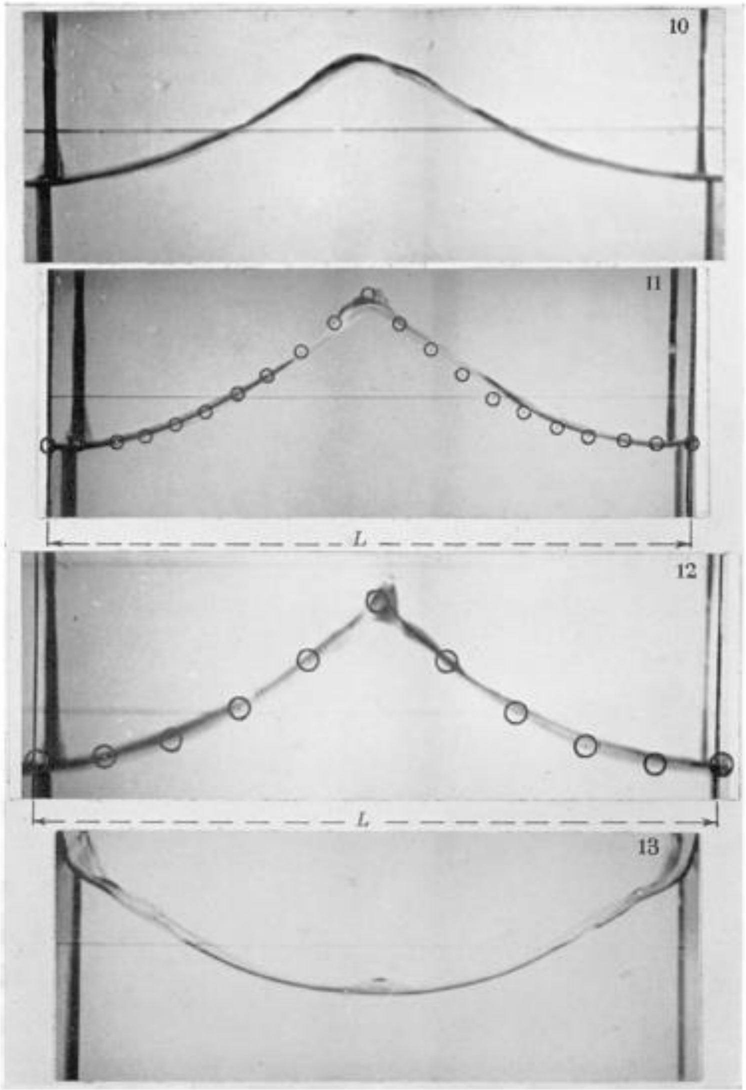
$$x_2(t) = c_2 \cos(t + \phi_2)$$

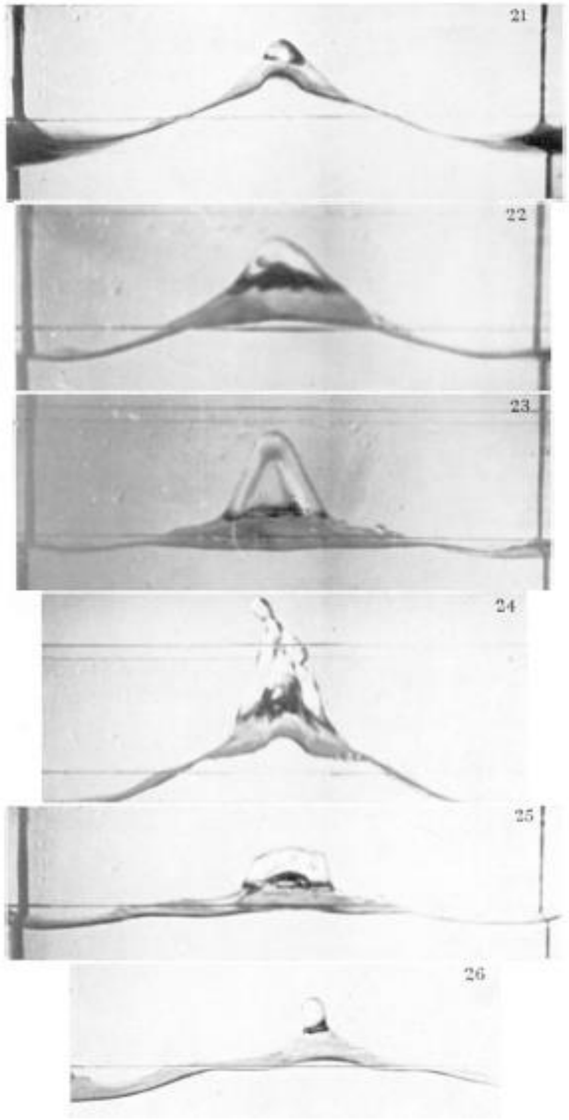
$$x_1(t) = c_1 \cos(t + \phi_1) + \frac{c_2}{2}[t \sin(t + \phi_2) + \cos(t + \phi_2)]$$

Introduction to capillary-gravity standing waves



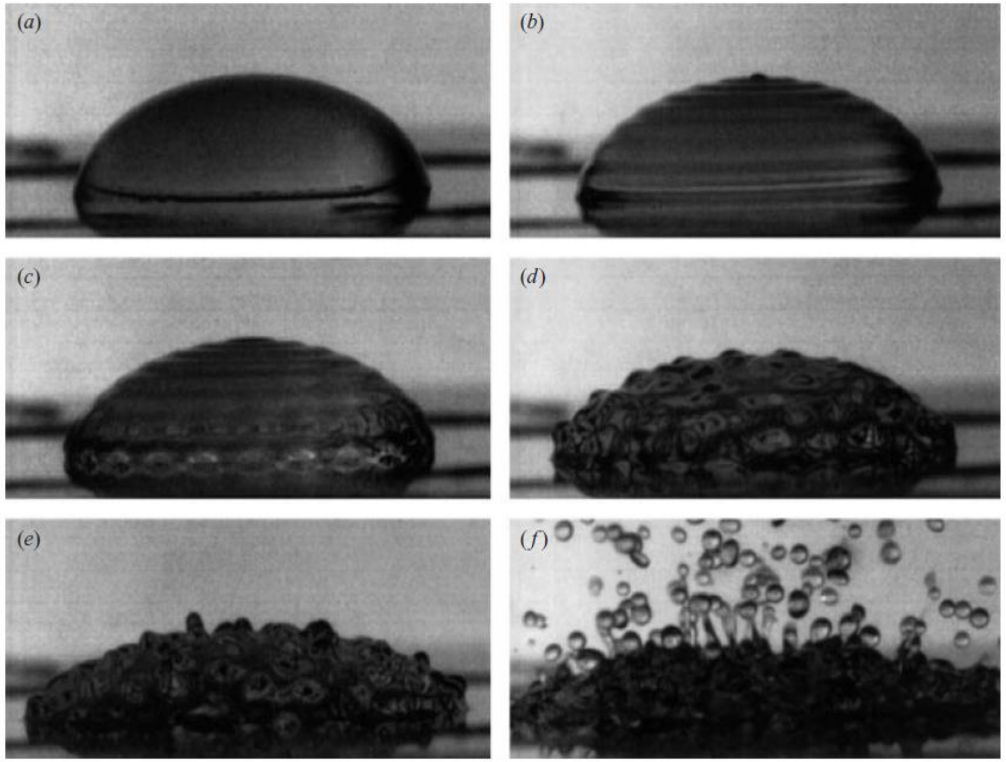
G. I Taylor, An experimental study of standing waves, 1953





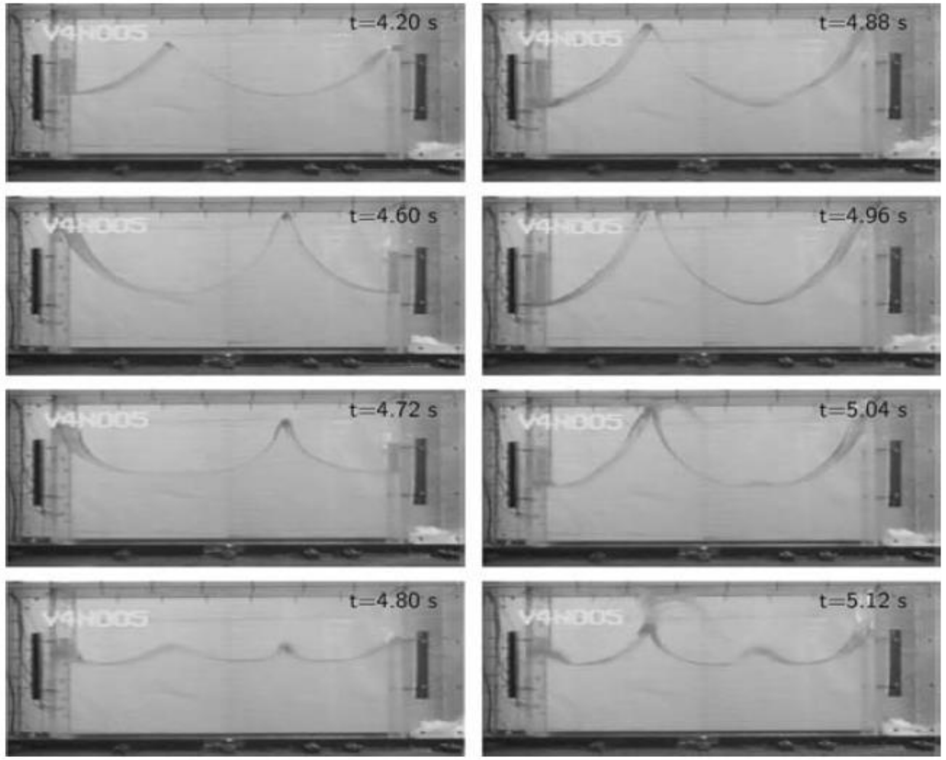
Applications

VIDA



Source:
Vibration induced drop atomization and bursting
J. Fluid Mech., vol. 476, pp. 1-28, 2003

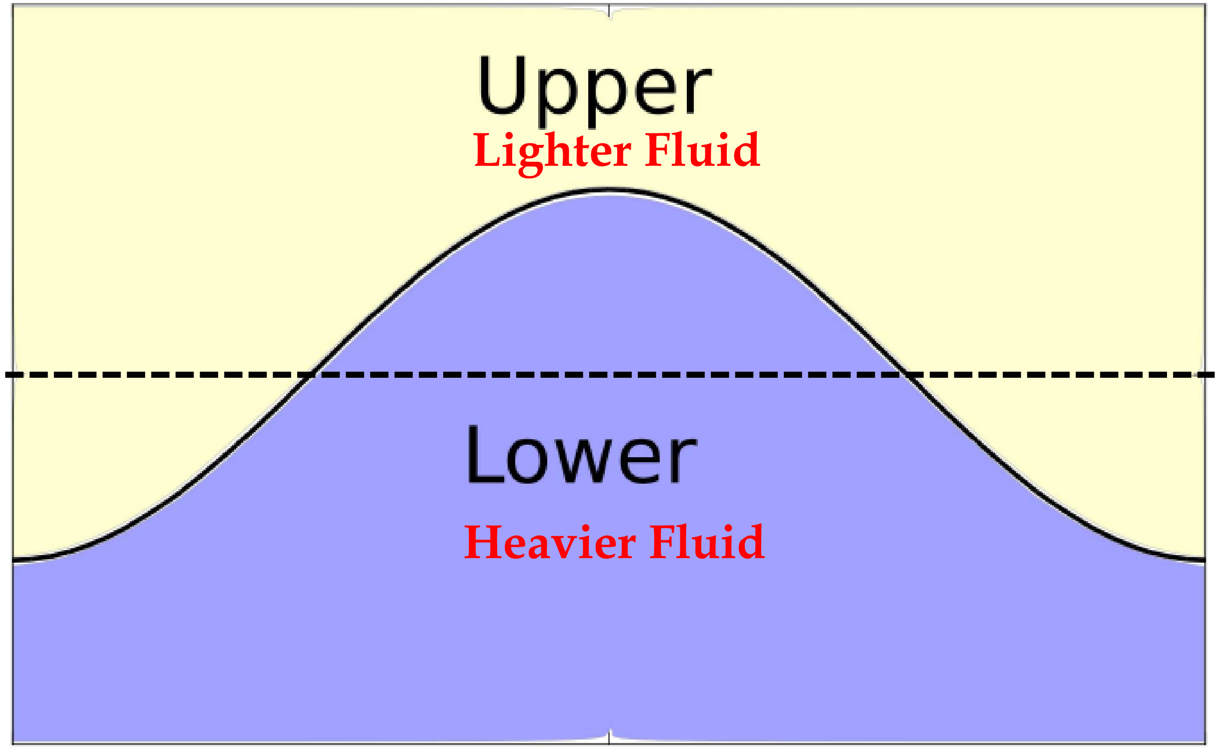
Sloshing



Source:
Experimental investigation and numerical modelling of steep forced water waves
J. Fluid Mech., vol. 490, pp. 217-249, 2003

Initial value problem (IVP) for axi standing capillary-gravity waves

Inviscid, irrotational approximation



2D View

Unbounded horizontally and vertically (Deep water approx.)

$$\eta(x,t) = a(t)\cos(kx)$$

Derive an equation for this, subject to a(0) << 1/k

$$\nabla^2\phi^{\mathcal{U}} = \nabla^2\phi^{\mathcal{L}} = 0$$

Initial Value problem (IVP)….

$$\left.\frac{\partial \phi^{\mathcal{U}}}{\partial y}\right|_{y=0} = \left.\frac{\partial \phi^{\mathcal{L}}}{\partial y}\right|_{y=0} = \eta_t$$

Linearised kinematic b.c.

$$\phi^{\mathcal{U}}(x, \infty, t) \to 0, \quad \phi^{\mathcal{L}}(x, -\infty, t) \to 0$$

$$\phi^{\mathcal{U}} = F(y)\cos(kx)\dot{a}(t), \quad \phi^{\mathcal{L}} = G(y)\cos(kx)\dot{a}(t)$$

$$\frac{d^2F}{dy^2} + k^2F = \frac{d^2G}{dy^2} + k^2G = 0$$

$$\phi^{\mathcal{U}} = -k^{-1}\exp(-ky)\cos(kx)\dot{a}(t) \qquad \phi^{\mathcal{L}} = k^{-1}\exp(ky)\cos(kx)\dot{a}(t)$$

Initial Value problem (IVP)….

$$p^{\mathcal{L}} = -\rho^{\mathcal{L}}\frac{\partial \phi^{\mathcal{L}}}{\partial t} - \rho^{\mathcal{L}} g y$$

$$p^{\mathcal{U}} = -\rho^{\mathcal{U}}\frac{\partial \phi^{\mathcal{U}}}{\partial t} - \rho^{\mathcal{U}} g y$$

Linearized Bernoulli equation

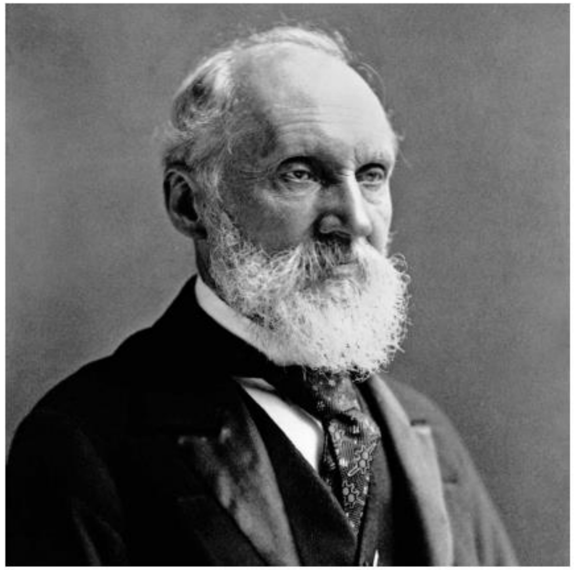
$$p^{\mathcal{L}}(y=\eta) - p^{\mathcal{U}}(y=\eta) = T\left(\boldsymbol{\nabla}\cdot\boldsymbol{n}\right)_{y=\eta}$$

$$\ddot{a} + \underbrace{\left[\left(\frac{\rho^{\mathcal{L}}-\rho^{\mathcal{U}}}{\rho^{\mathcal{U}}+\rho^{\mathcal{L}}}\right)gk + \frac{Tk^3}{\rho^{\mathcal{U}}+\rho^{\mathcal{L}}}\right]}_{\omega^2} a(t) = 0$$

Inviscid dispersion relation

Normal mode analysis will also lead to identical conclusions

$$\omega^2 = \left(\frac{\rho^{\mathcal{L}} - \rho^{\mathcal{U}}}{\rho^{\mathcal{U}} + \rho^{\mathcal{L}}}\right) gk + \frac{Tk^3}{\rho^{\mathcal{U}} + \rho^{\mathcal{L}}}$$



William Thomson,
1st Baron Kelvin

Deep-water dispersion relation for capillary-gravity waves on horizontally unbounded interface

Lord Kelvin, 1871, Waves under motive power of gravity and cohesion jointly without wind, *Phil. Mag.* XLII:370-77

Important for understanding weakly nonlinear resonant interactions for energy transfer between wavetrains: Triadic and quartic resonances

O. M Phillips 1960, McGoldrick 1965, Craik 1985, Chapter 5

Inclusion of viscosity (damping)

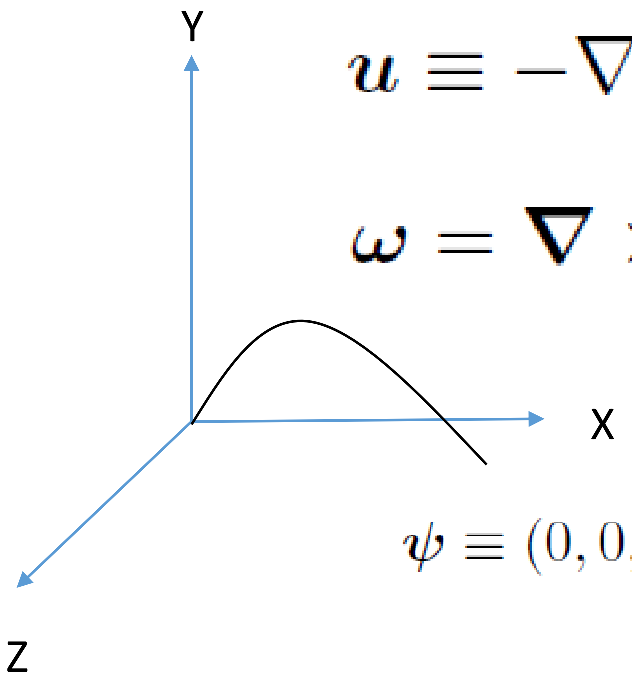
The normal mode approach – Discrete spectrum

Harrison 1908, Lamb 1932, Chandrashekhar 1961

$$\boldsymbol{u}_t = -\frac{1}{\rho}\nabla p + \boldsymbol{g} + \nu\nabla^2\boldsymbol{u}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

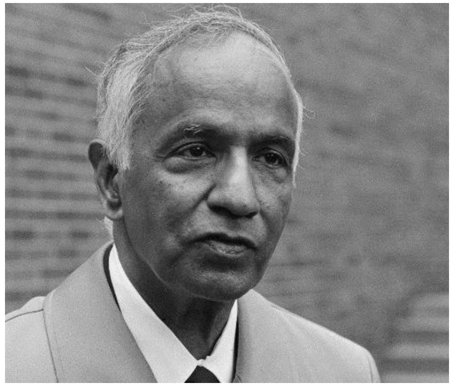
$$\boldsymbol{u} \equiv -\nabla \times \psi$$

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u} = -\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\psi\right) + \nabla^2\psi$$



$$\psi \equiv (0, 0, \psi(x, y, t)), \quad \boldsymbol{\omega} \equiv (0, 0, \omega(x, y, t))$$ 2D flow

Horace Lamb, 1849-1934



S. Chandrashekhar, 1910-1995

Inclusion of viscosity (damping)

The normal mode approach – Discrete spectrum

$$\omega_t = \nu\nabla^2\omega, \quad \nabla^2\psi = \omega$$

$$\eta_t = \psi_x\big|_{y=0}$$ → **Kin. bc**

$$(\psi_{xx} - \psi_{yy})_{y=0} = 0,$$ → **Shear stress bc**

$$-p(x,0,t) + 2\mu\psi_{xy}\big|_{y=0} = T\eta_{xx}$$ → **Normal stress bc**

Inclusion of viscosity (damping)

The normal mode approach – Discrete spectrum

$$\eta(x,t) = a_0 \exp(\sigma t)\cos(kx+\phi)$$

$$\omega(x,y,t) = \exp(\sigma t)\sin(kx+\phi)\left(A\exp(ly) + B\exp(-ly)\right)$$

$$\frac{\partial^2}{\partial x^2}\sin(kx+\phi) = -k^2\sin(kx+\phi)$$

$$\frac{\partial^2}{\partial y^2}\left[A\exp(ly) + B\exp(-ly)\right] = l^2\left[A\exp(ly) + B\exp(-ly)\right]$$

As long as $Re(l) > 0$ boundedness ⇒ $B = 0$

Inclusion of viscosity (damping)

The normal mode approach – Discrete spectrum

$$\eta(x,t) = a_0 \exp(\sigma t)\cos(kx + \phi)$$

$$\omega(x,y,t) = A \exp(\sigma t)\sin(kx + \phi)\exp(ly)$$

$$l^2 = k^2 + \sigma/\nu$$

$$\psi(x,y,t) = \exp(\sigma t)\sin(kx + \phi)Y(y)$$

Inclusion of viscosity (damping)

The normal mode approach – Discrete spectrum

$$\eta(x,t) = a_0 \exp(\sigma t) \cos(kx + \phi)$$

$$\omega(x,y,t) = A \exp(\sigma t) \sin(kx + \phi) \exp(ly)$$

$$\psi(x,y,t) = \exp(\sigma t) \sin(kx + \phi) \left(C \exp(ky) + \frac{A\nu}{\sigma} \exp(ly) \right)$$

Inclusion of viscosity (damping)

The normal mode approach – Discrete spectrum

$$\sigma a_0 - kC - \left(\frac{k\nu}{\sigma}\right) A = 0$$

$$2k^2C + \left(\frac{2k^2\nu}{\sigma} + 1\right) A = 0$$

$$T' k^2 a_0 + \left(\sigma + 2\nu k^2\right) C + \frac{2\nu^2 l k}{\sigma} A = 0$$

Inclusion of viscosity (damping)

The normal mode approach – Discrete spectrum

$$(\sigma + 2\nu k^2)^2 + gk + \frac{T}{\rho}k^3 = 4\nu^2 k^3 l$$

Lamb, Hydrodynamics

Extension to two fluids worked out by → S. Chandrasekhar

Hydrodynamic & hydromagnetic stability

The dispersion relation constrains the allowable values of σ for a given value of k. The above equation allows for only two values of σ for every k.

$$l^2 = k^2 + \sigma/\nu$$

Hence only two values of l for every k!

The continuous spectrum

$$Re(l) = 0$$

First discussed by Horace Lamb, 1932

Earlier for discrete spectrum $Re(l) > 0$

$$\eta(x,t) = a_0 \exp(\sigma t)\cos(kx+\phi)$$

$$\omega(x,y,t) = A\exp(\sigma t)\sin(kx+\phi)\exp(ly)$$

$$l^2 = k^2 + \sigma/\nu$$

Now $Re(l) = 0$

$$\eta(x,t) = a_0 \exp(\sigma t)\cos(kx+\phi)$$

$$\omega(x,y,t) = \exp(\sigma t)\sin(kx+\phi)\left[A\sin(my) + B\cos(my)\right]$$

$$m^2 = -k^2 - \frac{\sigma}{\nu}$$

The continuous spectrum

$$\eta(x,t) = a_0 \exp(\sigma t)\cos(kx+\phi)$$

$$\omega(x,y,t) = A\exp(\sigma t)\sin(kx+\phi)\,(A\sin(my) + B\cos(my))$$

$$\psi(x,y,t) = \exp(\sigma t)\sin(kx+\phi)\left(C\exp(ky) + \frac{A\nu}{\sigma}\sin(my) + \frac{B\nu}{\sigma}\cos(my)\right)$$

$$\sigma a_0 - kC - \frac{k\nu}{\sigma}B = 0$$

$$2k^2C + \left(\frac{2k^2\nu}{\sigma} + 1\right)B = 0$$

$$T'k^2a_0 + (\sigma + 2\nu k^2)\,C + \frac{2mk\nu^2}{\sigma}A = 0$$

Three equations in 4 unknowns.
Any real value of m gives a nontrivial solution.

The continuous spectrum

$$m^2 = -k^2 - \frac{\sigma}{\nu}$$

$$\text{for real } k,\ m \text{ will be real only if } \sigma \text{ is real}$$

$$-\infty \leq \sigma \leq -\nu k^2,\ 0 \leq m \leq \infty$$

Thus for a given k, the (vorticity) eigenfunctions have a vertical structure like

Discrete spectrum eigenfunctions

$$\sim \exp(ly) \quad , Re(l) > 0$$
$$= \exp(l^r y)\cos(l^i y) \quad \texttt{or} \quad \exp(l^r y)\sin(l^i y)$$

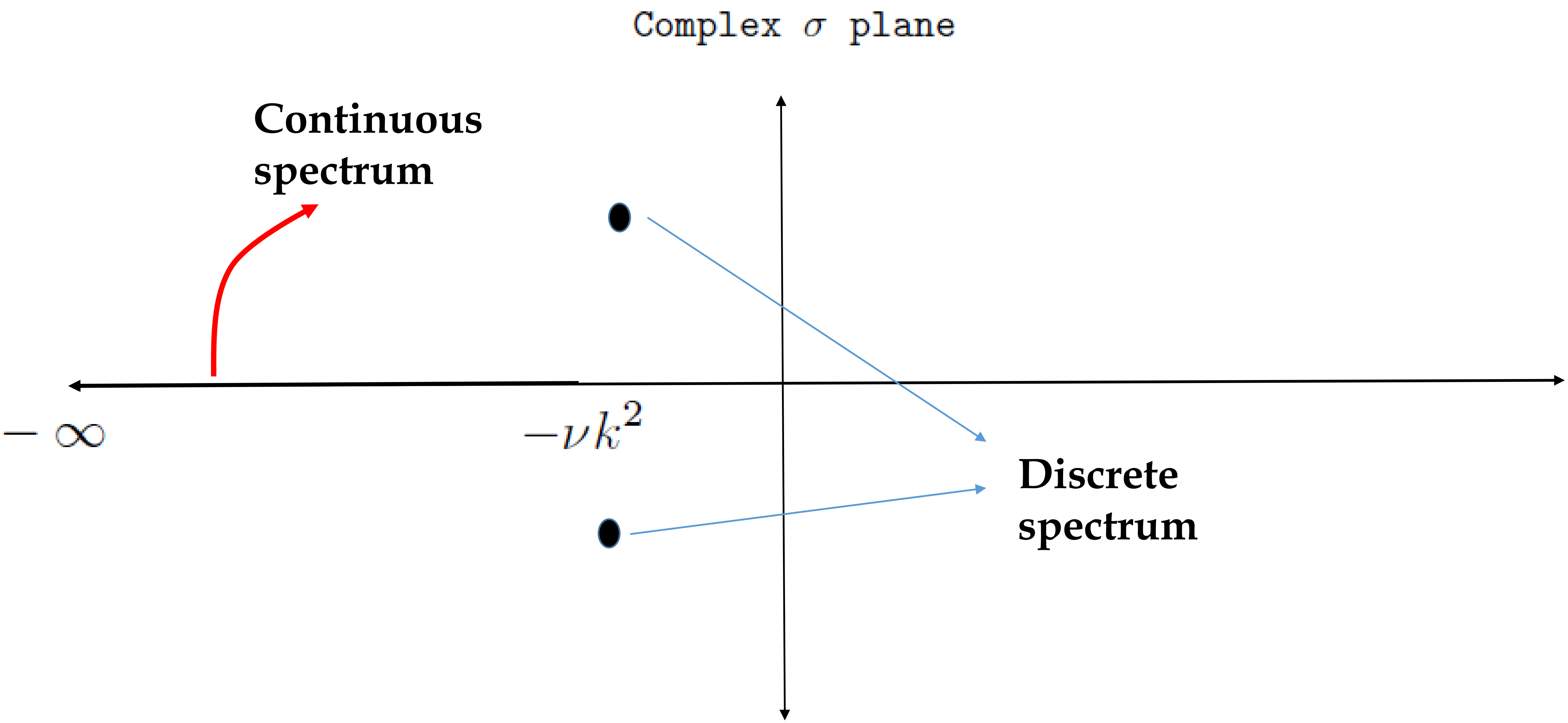
Oscillate and damp out in time

Continuous spectrum eigenfunctions

$$\sim \cos(my) \quad \texttt{or} \quad \sin(my)$$

Only damp out in time

The continuous spectrum



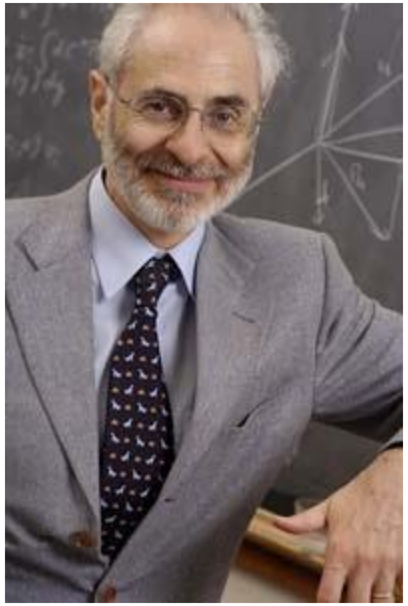
Temporal evolution

$$\omega(x,y,t) = \sin(kx)\Omega(y,t)$$

$$\Omega(y,t) = \sum_l C_l \exp(ly)\exp(\sigma_l t) + \int_{m=0}^{\infty} [A(m)\cos(my) + B(m)\sin(my)] \exp[\sigma(m)t]\, dm$$

$$\Omega(y,0) = \sum_l C_l \exp(ly) + \int_{m=0}^{\infty} [A(m)\cos(my) + B(m)\sin(my)]\, dm$$

$$a(t) = \sum_l p_l \exp(\sigma_l t) + \int_{\sigma=-\infty}^{-\nu k^2} q(\sigma)\exp[\sigma t]\, d\sigma$$



Andrea Prosperetti,
Phys. Fluids 1978, 1981

Actual calculation of a(t) quite tedious using Laplace transforms

Integro-differential equation

Solution to IVP

$$\frac{a_k(t)}{a_k(0)} = \frac{4(\nu k^2)^2(1-4\beta)}{8(\nu k^2)^2(1-4\beta)+\omega_0^2}\mathrm{Erfc}\left(\sqrt{\nu k^2 t}\right) + \sum_{i=1}^{4} \frac{\hat{A}_i \hat{h}_i \omega_0^2 \exp[(\hat{h}_i^2 - \nu k^2)t]\mathrm{Erfc}\,(\hat{h}_i\sqrt{t})}{\nu k^2 - \hat{h}_i^2},$$

Andrea Prosperetti,

Phys. Fluids 1978, Single Fluid

Phys. Fluids 1981, Two Fluids

Cauchy-Poisson problem

See
“The origins of
water-wave theory”

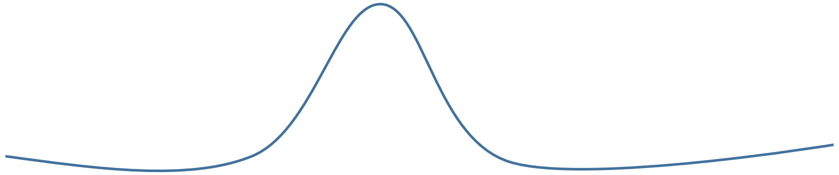
Alex D. D. Craik
Ann. Rev. Fluid
Mech., 2004

Cauchy : 1789-1857

Poisson : 1781-1840

Prize problem by
French Academy of
Science: 1813

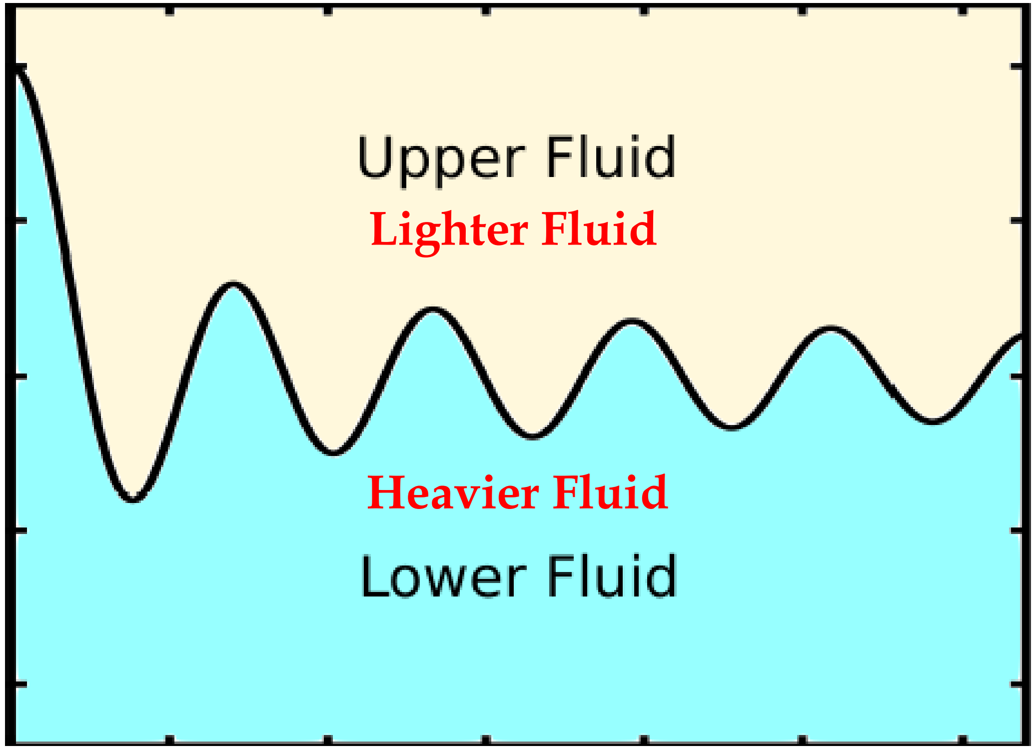
Jointly won by
Cauchy and Poisson
for inviscid,
irrotational



$$\eta(x,0) = f(x) \qquad \eta_t(x,0) = g(x)$$

Research in my group

IVP for axi standing capillary-gravity waves



2D View

$$\eta(r,t) = a(t) J_0(kr)$$

Derive an equation for this

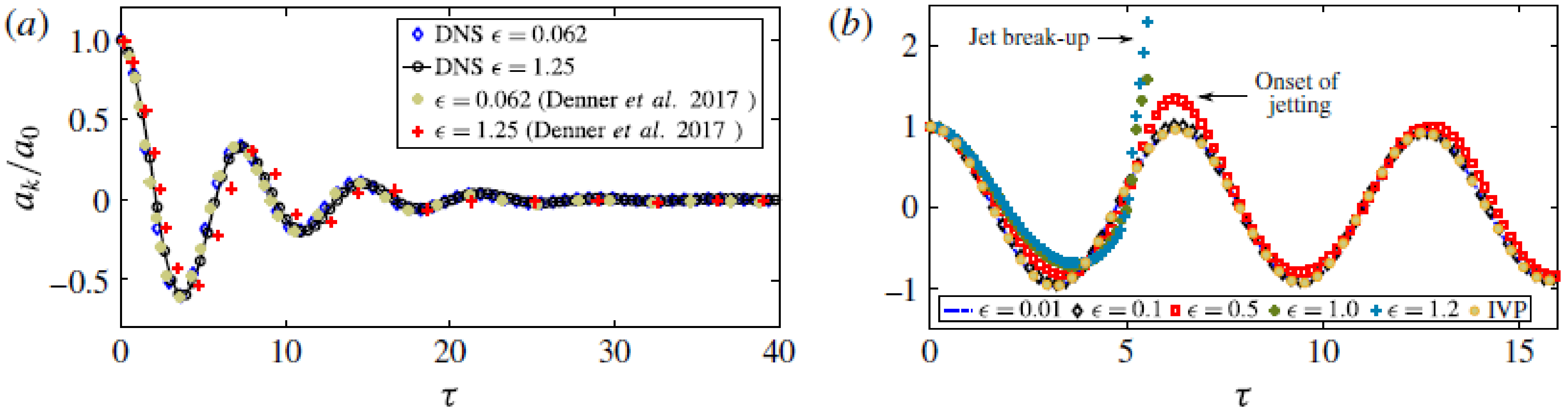
Axisymmetric viscous interfacial oscillations – Theory and simulations, Farsoiya, Mayya and Dasgupta, *J. Fluid Mech.* Vol. 896, pp. 796-818, 2017.

$$\frac{a(t)}{a(0)} = \frac{4(\nu k^2)^2(1-4\beta)}{8(\nu k^2)^2(1-4\beta)+\omega_0^2}\mathrm{Erfc}\left(\sqrt{\nu k^2 t}\right) + \sum_{i=1}^{4} \frac{\hat{A}_i \hat{h}_i \omega_0^2 \exp[(\hat{h}_i^2 - \nu k^2)t]\mathrm{Erfc}\,(\hat{h}_i\sqrt{t})}{\nu k^2 - \hat{h}_i^2},$$

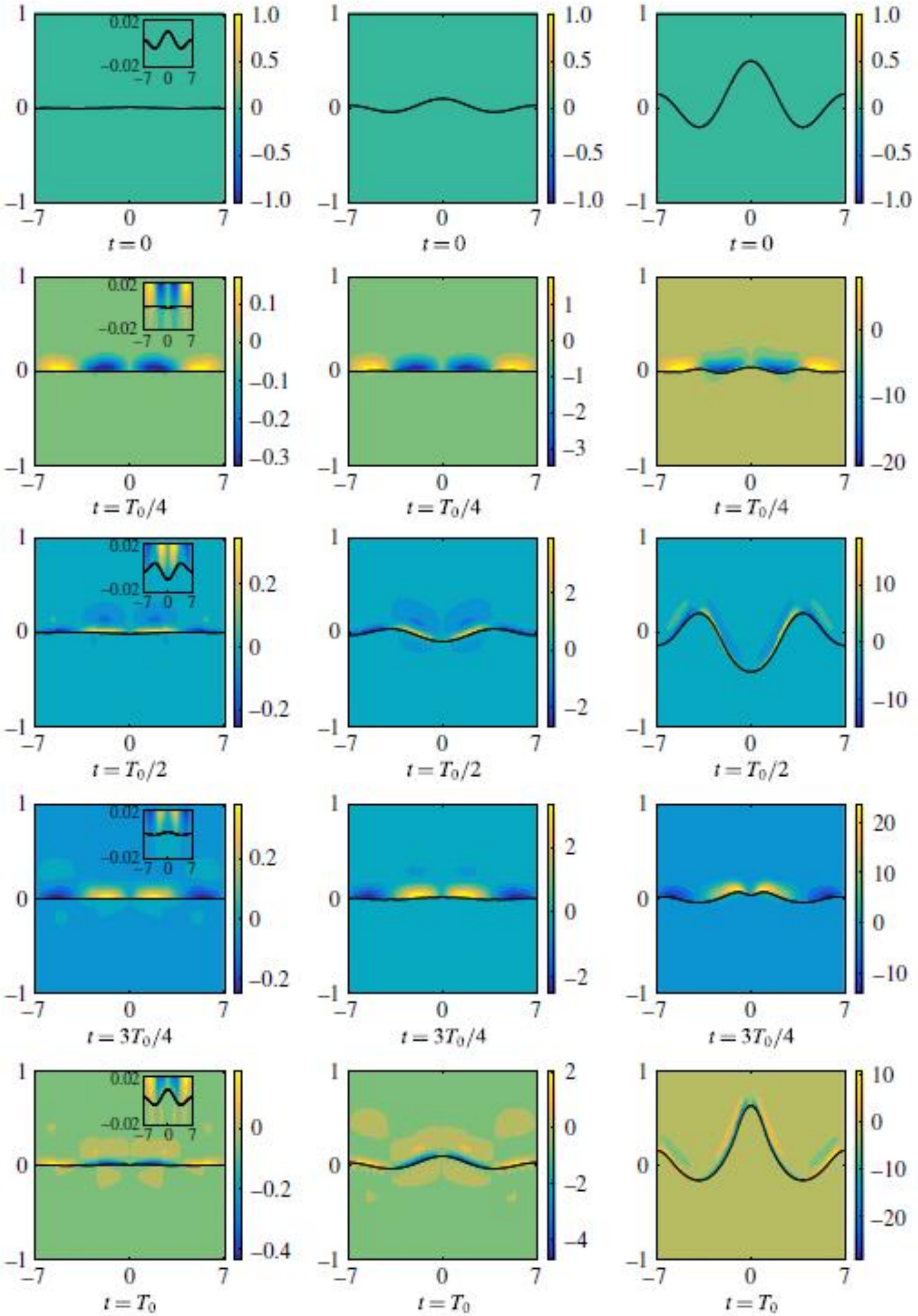
$$\hat{P}(\hat{h}) = \hat{h}^4 - 4(\nu k^2)^{1/2}\beta\hat{h}^3 + 2(\nu k^2)(1-6\beta)\hat{h}^2 + 4(\nu k^2)^{3/2}(1-3\beta)\hat{h} + (\nu k^2)^2(1-4\beta) + \omega_0^2,$$

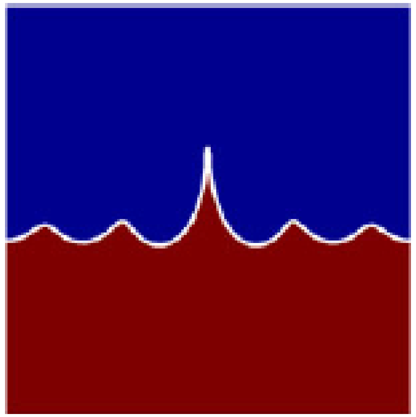
Velocity and pressure field analytically available for equal kinematic viscosity ratios

Axisymmetric viscous interfacial oscillations – Theory and simulations, Farsoiya, Mayya and Dasgupta, *J. Fluid Mech.* Vol. 896, pp. 796-818, 2017.



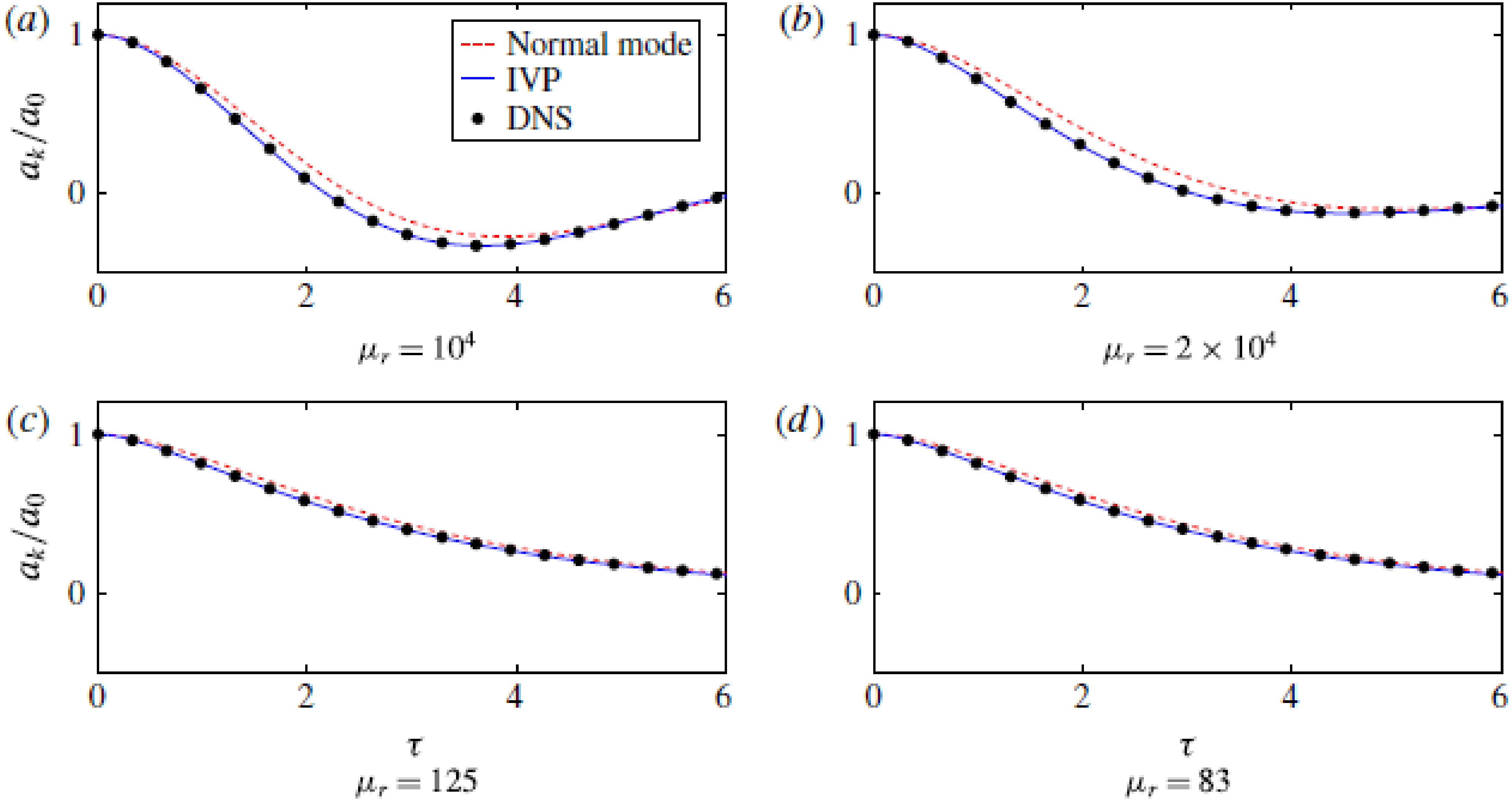
All simulations done using Basilisk http://basilisk.fr/





Jetting

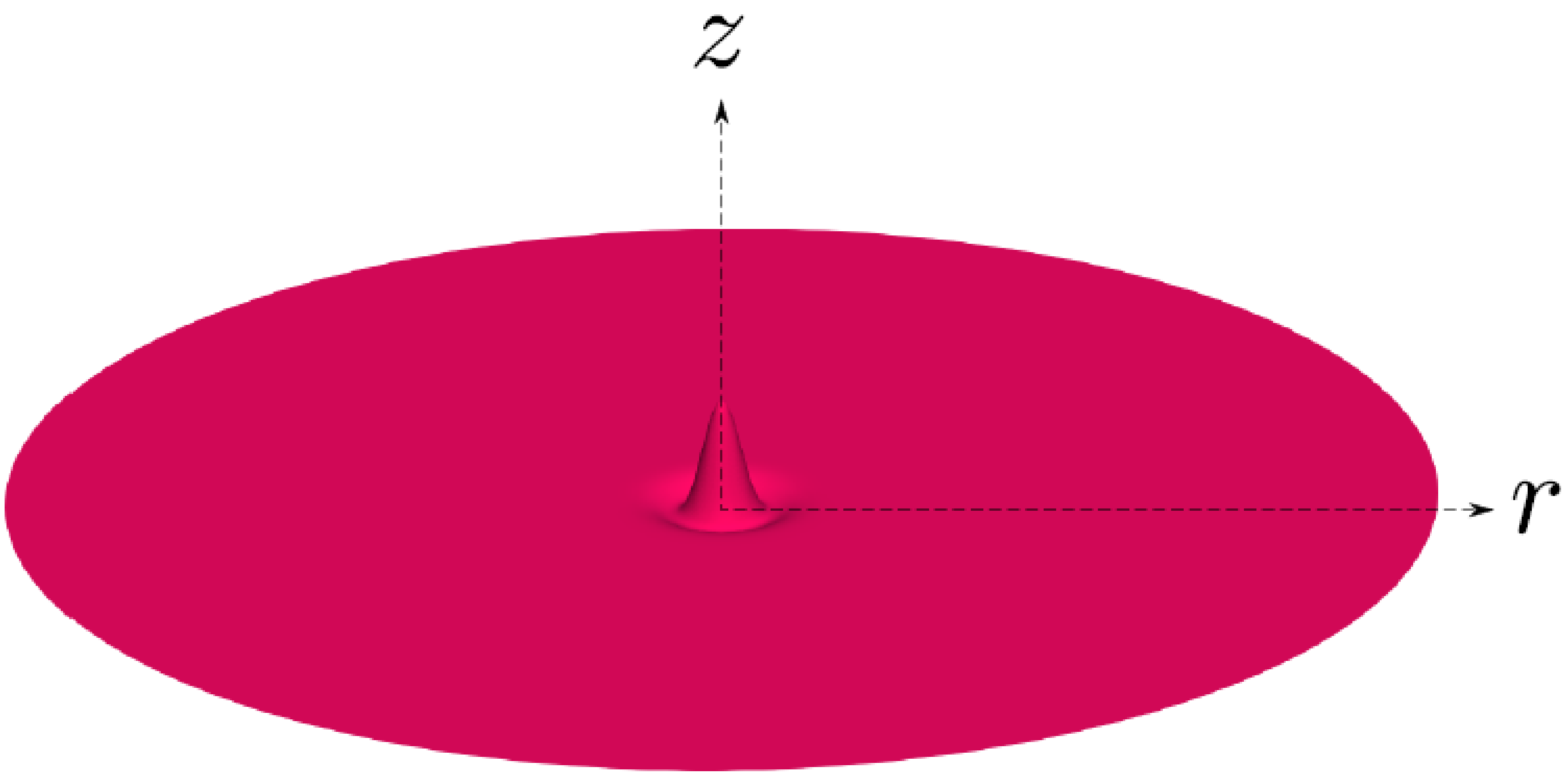
Comparison of IVP, normal mode and DNS



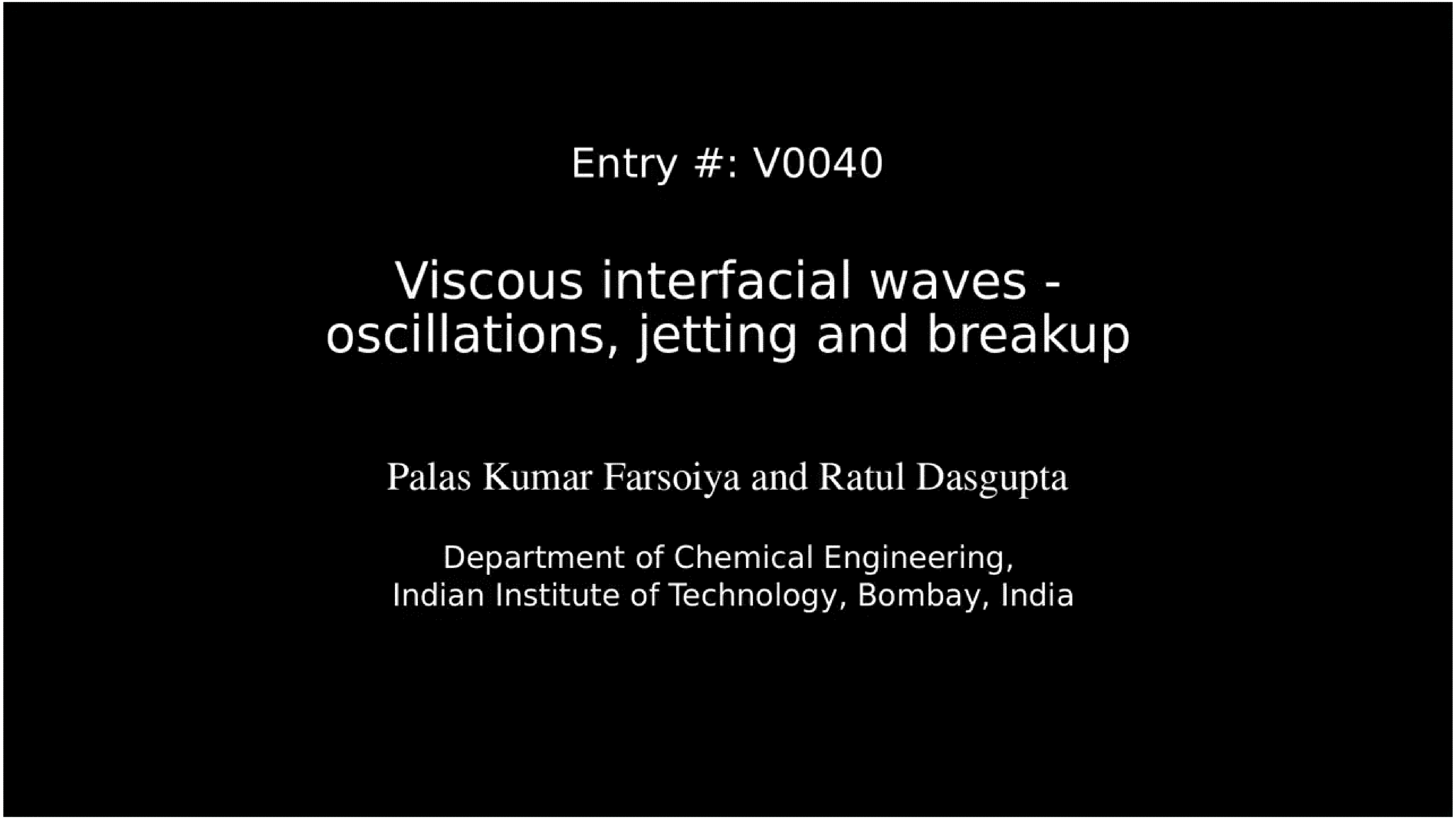
Axisymmetric viscous interfacial oscillations – Theory and simulations, Farsoiya, Mayya and Dasgupta, *J. Fluid Mech.* Vol. 896, pp. 796-818, 2017.

Solution to two fluid Cauchy-Poisson problem

$$\eta(r,t) = \int_{k=0}^{\infty} dk\; k J_0(kr)\tilde{\eta}_0(k) a(k,t),$$



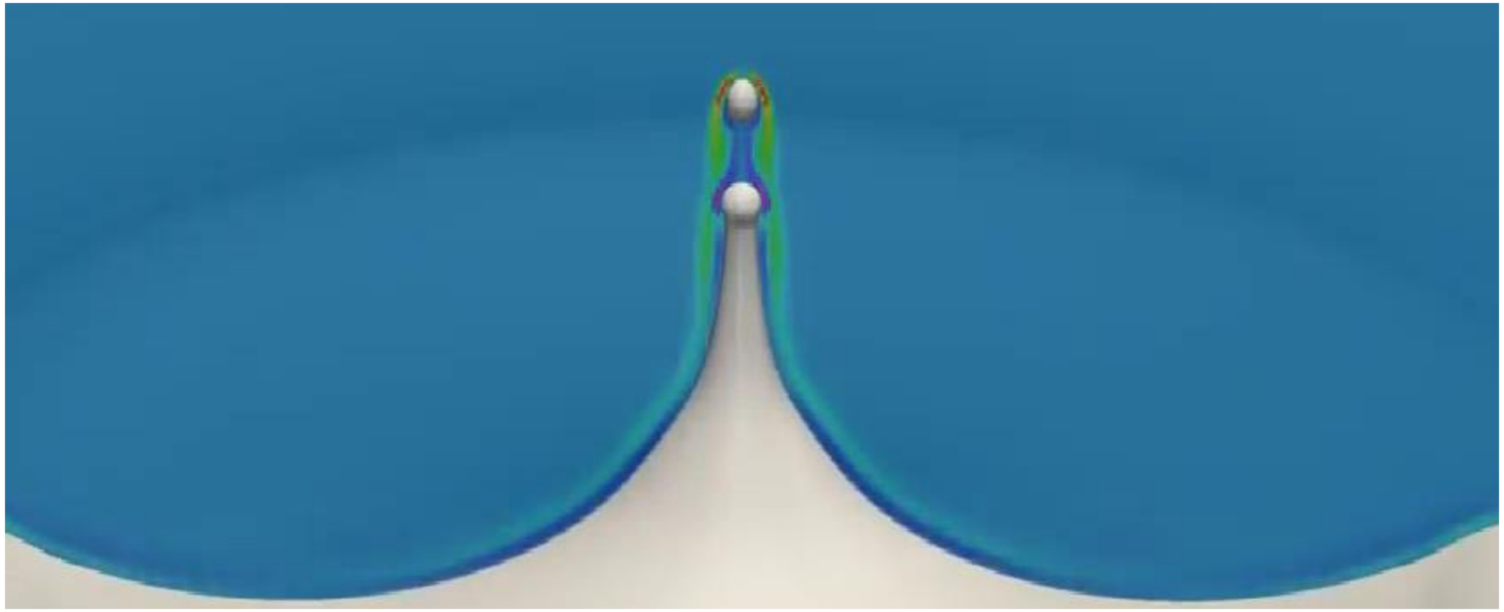
Viscous axisymmetric waves – the interfacial Cauchy-Poisson problem, Farsoiya, Nair and Dasgupta, 2018



Gallery of Fluid Motion, 2017, https://gfm.aps.org/
Video Entry No. V0040, Also on Youtube

Open Questions and ongoing work

- Generalized solution to Initial Value Problem independent of base state geometry.
- Parametric regime where the contribution from the continuous spectrum is sizeable.
- Generalized Faraday waves (forced oscillations on drops, filaments and cylindrical and planar pools).
- Nonlinear theory of jetting and breakup.



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- IRCC IITB, DST-SERB and IITB Chemical Engg. for funding studies on free and forced (Faraday waves) capillary/capillary-gravity oscillations.

- Dr. Anubhab Roy, IIT Madras and Dr. Y. S. Mayya, IITB for many interesting discussions.

Thank You