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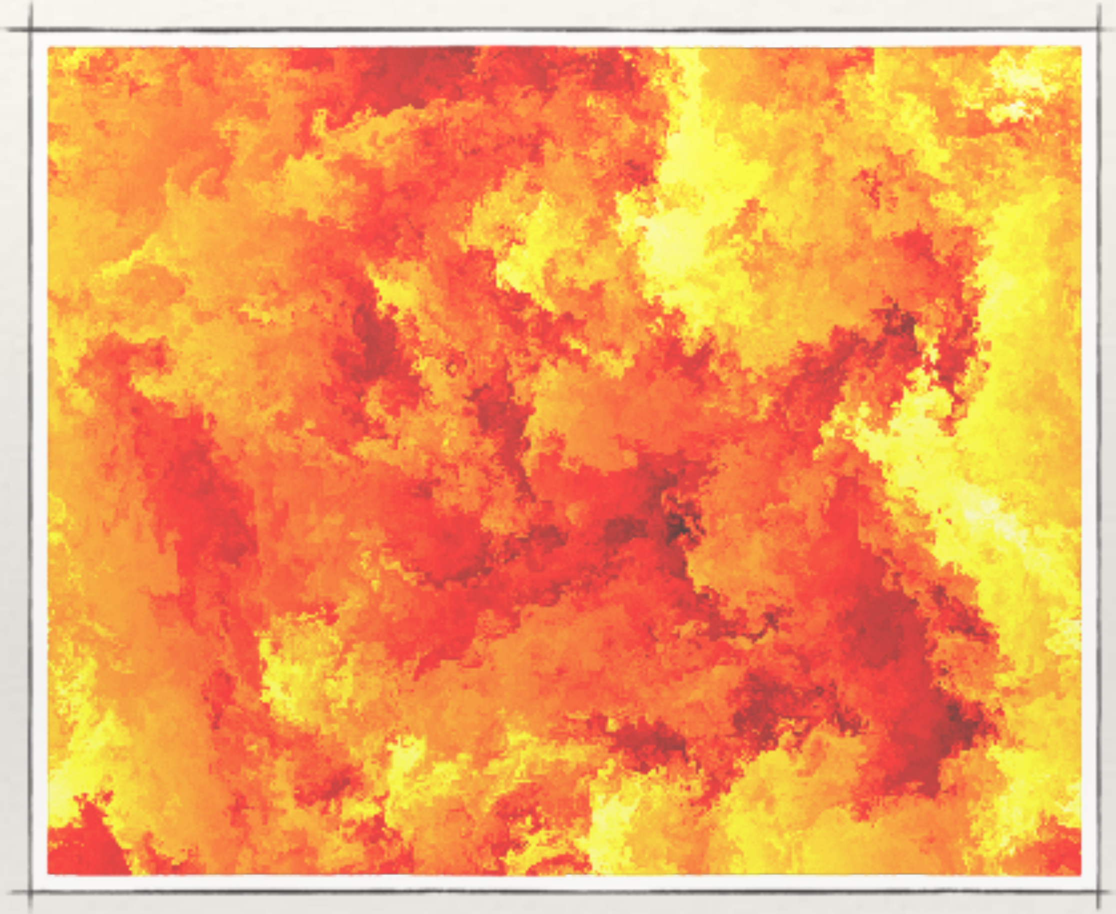
Lagrangian flows and turbulent irreversibility

Work in progress: *Simple motivations, many questions, few answers…*



Turbulence from Angstroms to light years, ICTS, Bangalore, 01/2018

Turbulence and inviscid flows

One possible mathematical formulation of the "turbulence problem":

1 - consider solutions to the randomly forced Navier–Stokes equation

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f}^\omega \qquad \nabla\cdot\boldsymbol{u} = 0 \qquad \boldsymbol{u}(\boldsymbol{x},t_0) = \boldsymbol{u}_0(\boldsymbol{x})$$

2 - define the probability measure μ_t^ν such that $\mathbb{E}_\omega \mathcal{F}[\boldsymbol{u}] = \int \mathcal{F}[\boldsymbol{u}]\,\mu_{t_0}^\nu(\mathrm{d}\boldsymbol{u})$

3 - construct the stationary distribution $\mu_\infty^\nu = \lim_{t_0\to-\infty} \mu_{t_0}^\nu$

4 - characterize the limit $\mu_{\mathrm{Turb}} = \lim_{\nu\to 0} \mu_\infty^\nu$

The limits $t_0 \to -\infty$ and $\nu \to 0$ do not necessarily commute (even if they do so for Burgers equation).

One is however tempted to interpret μ_{Turb} in terms of dissipative (non-classical) solutions to the Euler equation.

For large, finite Reynolds numbers, is there a signature of Euler equation in developed inertial-range dynamics?

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Uniqueness of weak solution to Euler?

- **Anomalous dissipation** requires that the limit is described by spatially irregular velocity fields (Onsager's conjecture) ⇒ **weak solutions**

- Such solutions are **not unique** (as for Burgers equation)
  They can be for instance compactly supported in time
  Scheffer (1993); Shnirelman (1997, 2000); De Lellis and Székelyhidi (2009)

- **Admissible weak solutions:** they obey the criterion of decreasing energy. The initial data with infinitely many admissible solutions are **dense** in L_2!
  Brenier et al. (2011); Wiedemann and Székelyhidi (2012)

No straightforward way to ensure unicity!
Can turbulent data give further constraints on physical solutions?
Is energy dissipation the only relevant signature of irreversibility?

Spontaneous stochasticity

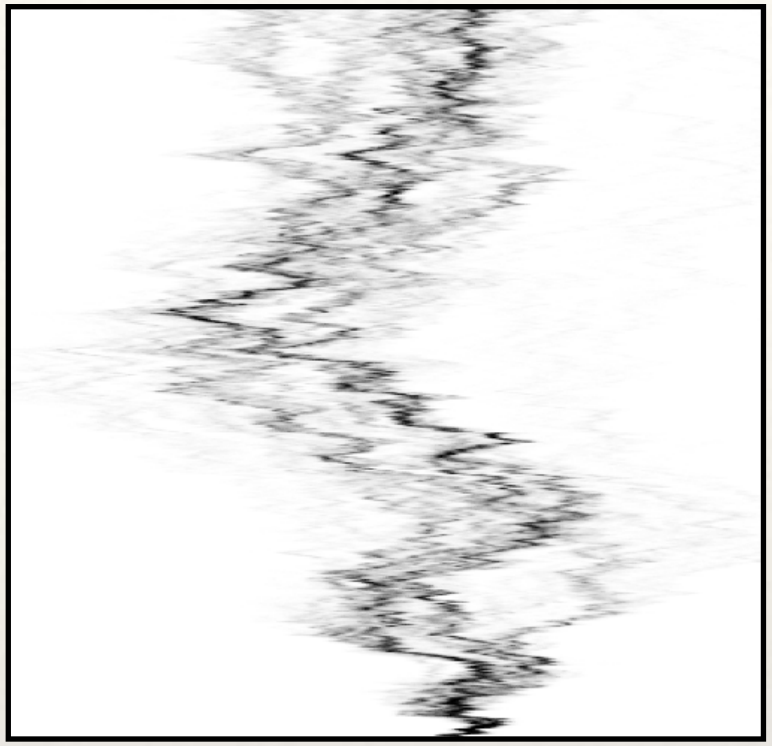
- At infinite Reynolds number, tracers have an **explosive behavior** (Richardson's dispersion)

- Phenomenology: for $R > 0$

$$\mathrm{d}R = (R+\eta)^{1/3} + \sqrt{2\kappa}\,\mathrm{d}W \text{ with } R(0) = 0 \text{ and}$$

UV cut-off (→ η)   diffusion (→ $\sqrt{2\kappa}$)

$$p(R,t|0,0) \neq \delta(R) \text{ when } \eta \to 0\text{, and then } \kappa \to 0$$



- The Lagrangian flow cannot be described by a map $\boldsymbol{x}_0 \mapsto \boldsymbol{X}(t;\boldsymbol{x}_0,t_0)$

$\Rightarrow$ A **probabilistic description** is necessary *(Bernard et al. 1998; Eyink 2008)*

Could this alter turbulent advection?

Does a velocity field make sense to describe turbulent motions?

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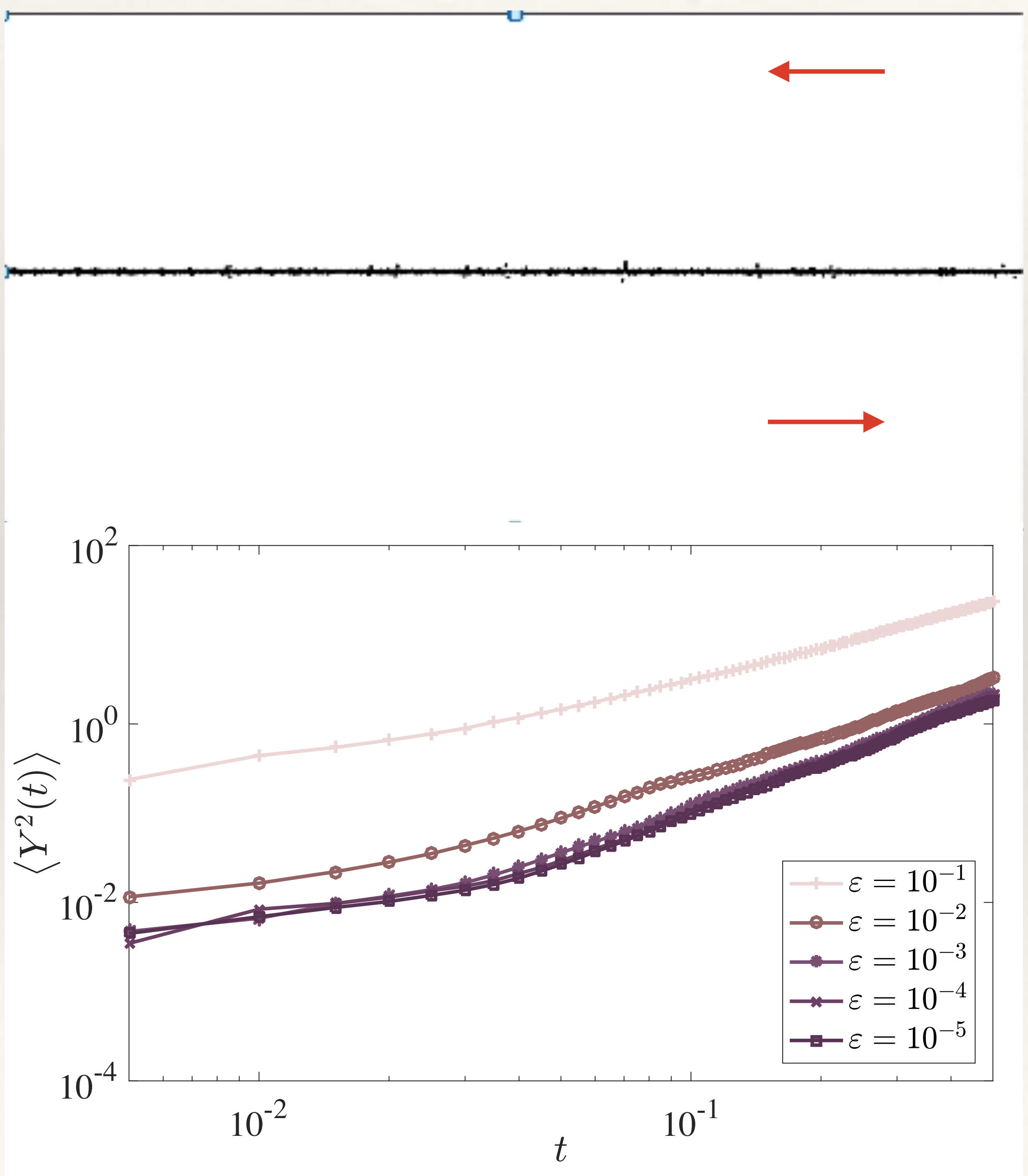
Intrinsic stochasticity of Euler flow

- Possible consequence: solutions to Euler equation can themselves be **spontaneously stochastic** *(Mailybaev 2015, 2016)*
- Example: Kelvin–Helmholtz



Birkhoff–Rott

$$Z(s,t) = X(s,t) + i\,Y(s,t)$$

infinitesimal white noise ↓

$$\partial_t \overline{Z}(s,t) = \frac{1}{2\pi i} \fint \frac{\Gamma\left[1 + \varepsilon\eta(s')\right] \mathrm{d}s'}{Z(s,t) - Z(s',t)}$$

↑ regularization the integral (P.V.)

Finite-amplitude instability in the limit UV cutoff $\to 0$, and then $\varepsilon \to 0$

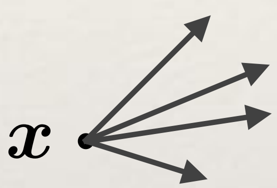
Advocates again a probabilistic description of the velocity

Necessity to think probabilistic?

To summarize:

- Unicity of weak (non-differentiable) solutions seems hard to prove
- Tracers explosive separation might be incompatible with classical advection
- The velocity can itself be spontaneously stochastic

$\Rightarrow$ Suggests to relax the notion of "velocity field"



$p(\boldsymbol{x}', t+\delta t \,|\, \boldsymbol{x}, t)$ $\quad\Longleftrightarrow\quad$ $\boldsymbol{u}(\boldsymbol{x},t) \longrightarrow \gamma_{(\boldsymbol{x},t)}(\mathrm{d}\boldsymbol{u})$ Young measure

DiPerna and Majda (1987): distributional solutions

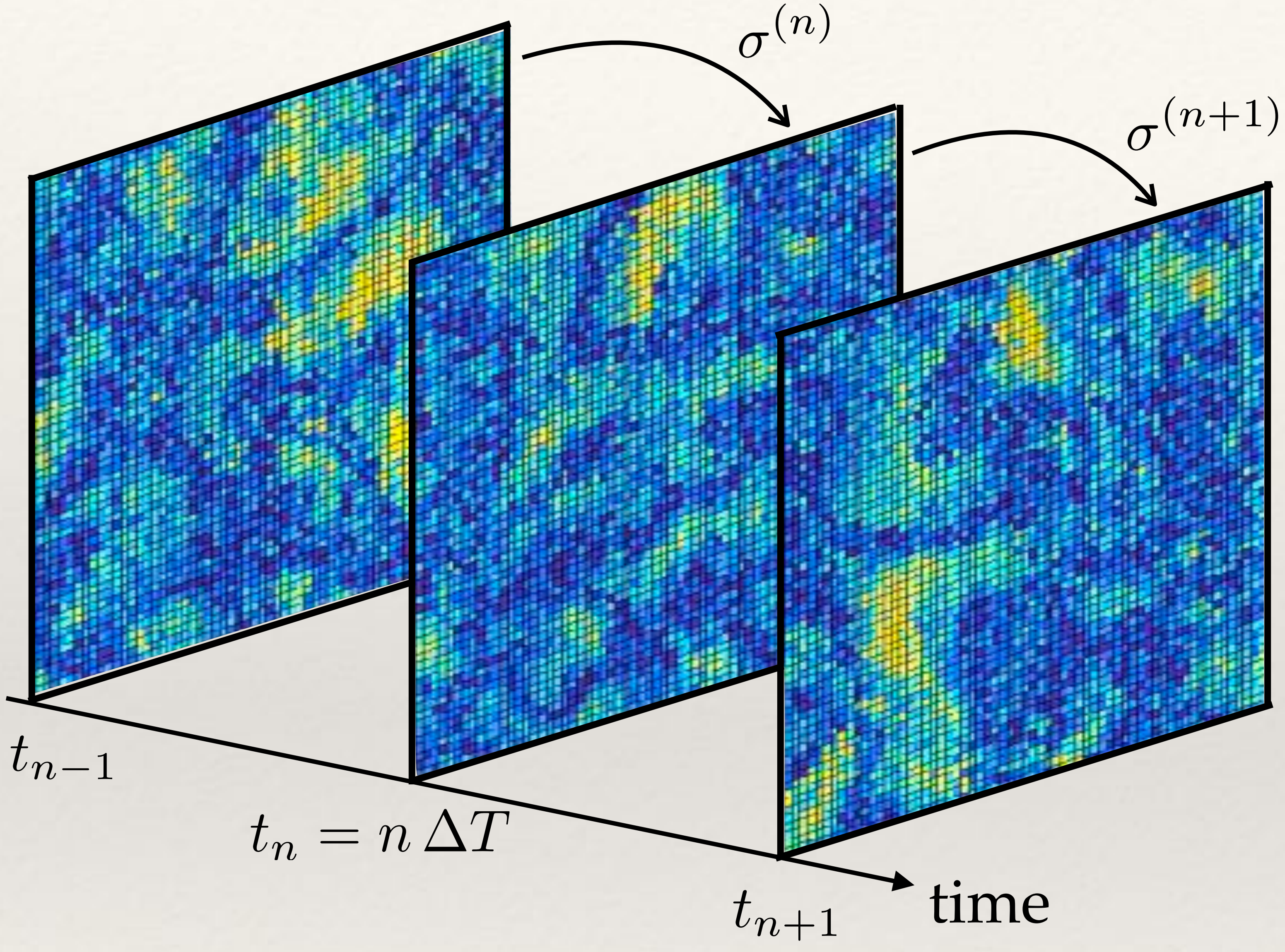
$$\partial_t \langle \boldsymbol{u} \rangle_\gamma + \nabla \cdot \langle \boldsymbol{u} \otimes \boldsymbol{u} \rangle_\gamma = -\nabla p, \qquad \nabla \cdot \langle \boldsymbol{u} \rangle_\gamma = 0 \qquad \langle f(\boldsymbol{u}) \rangle_\gamma = \int f(\boldsymbol{u})\, \gamma_{(\boldsymbol{x},t)}(\mathrm{d}\boldsymbol{u})$$

They can be obtained as the limit when $\nu \to 0$ of viscous solutions

Is it of relevance for turbulence? How to detect this from data?

Turbulent viewpoint: coarse-graining

Assume the domain is divided in cells $\{\mathcal{V}_i\}_i$ of size $\ell \gg \eta_{\mathrm{K}}$



Lagrangian flow approximated as permutations $\mathcal{V}_i \mapsto \mathcal{V}_{\sigma_i^{(n)}}$ of small volumes between discrete times

Transition probability written as a doubly stochastic matrix with elements

$$p_{i,j}^{(n)} = \Pr\left(\sigma_i^{(n)} = j\right)$$

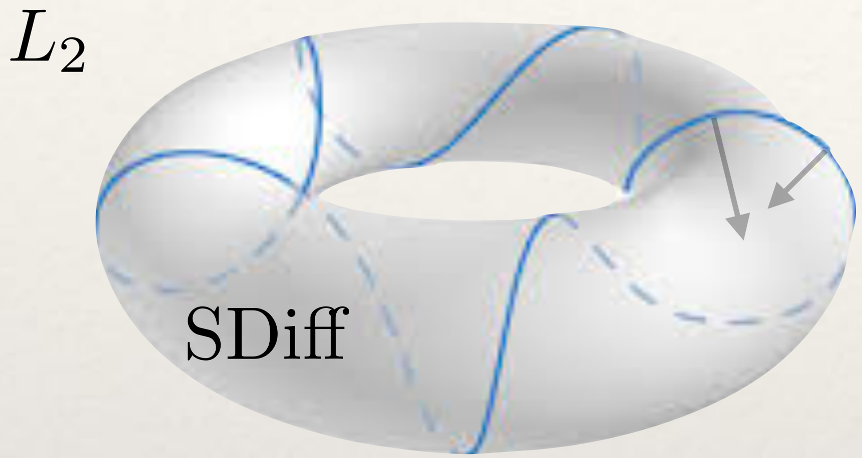
$$\sum_i p_{i,j}^{(n)} = \sum_j p_{i,j}^{(n)} = 1$$

Can we reconstruct an inviscid flow dynamics between different snapshots?

$\Rightarrow$ **Brenier's *Generalized flows***

Lagrangian variational principle

Arnol'd (1966): **regular** inviscid, incompressible flow are geodesic on the manifold of volume-preserving maps.



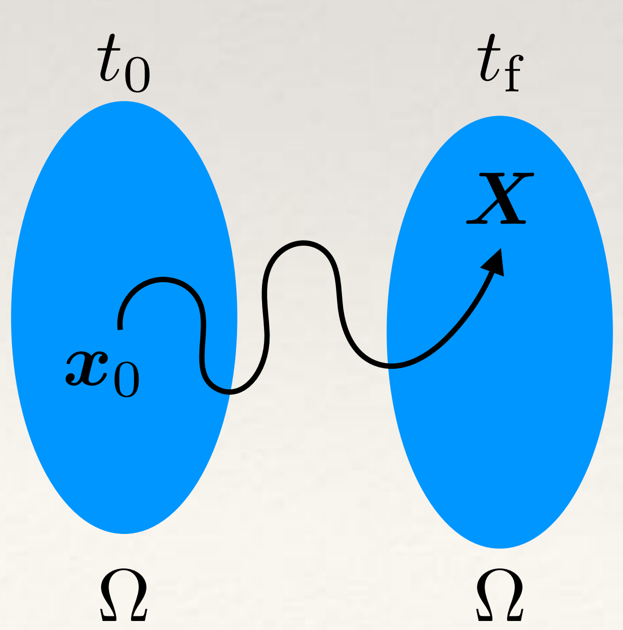
Lagrangian map: $\boldsymbol{x}_0 \mapsto \boldsymbol{X}(t;\boldsymbol{x}_0,t_0)$

$$\partial_t \boldsymbol{X}(t;\boldsymbol{x}_0,t_0) = \boldsymbol{u}(\boldsymbol{X}(t;\boldsymbol{x}_0,t_0),t)$$

$$\boldsymbol{X}(t_0;\boldsymbol{x}_0,t_0) = \boldsymbol{x}_0$$

$$\partial_t^2 \boldsymbol{X} = -\nabla p$$

Boundary-value problem: for a regular map $\boldsymbol{X}(t_\mathrm{f};\boldsymbol{x}_0,t_0)$ between t_0 and t_f, reconstruct the full Euler flow at intermediate times $t_0 \le t \le t_\mathrm{f}$



The solution minimizes the action

$$\int_{t_0}^{t_\mathrm{f}} \int_\Omega \|\partial_t \boldsymbol{X}(t;\boldsymbol{x}_0,t_0)\|^2 \, \mathrm{d}^3 x_0 \, \mathrm{d}t$$

over all smooth curves of $\mathrm{SDiff}(\Omega)$ (diffeomorphisms with unit Jacobian) that satisfy the B.C.

Generalized variational principle

Y. Brenier (1989): probabilistic version of the variational principle:

Find a probability measure $\rho(\mathrm{d}\boldsymbol{X})$ on the set of continuous paths which:

- preserves Lebesgue in average: $\forall t, \int \varphi(\boldsymbol{X}(t))\,\rho(\mathrm{d}\boldsymbol{X}) = \int_\Omega \varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$
- satisfies BC, in average: $\int \varphi(\boldsymbol{X}(t_0), \boldsymbol{X}(t_\mathrm{f}))\,\rho(\mathrm{d}\boldsymbol{X}) = \int_{\Omega\times\Omega} \varphi(\boldsymbol{x}, \boldsymbol{y})\,\eta(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{y})$

  with, e.g., $\eta(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{y}) = \delta(\boldsymbol{y} - \boldsymbol{X}(t_\mathrm{f}; x, t_0))\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}$
- minimizes the generalized action $\mathcal{A}[\rho] = \int \int_{t_0}^{t_\mathrm{f}} \|\dot{\boldsymbol{X}}\|^2\,\mathrm{d}t\,\rho(\mathrm{d}\boldsymbol{X})$

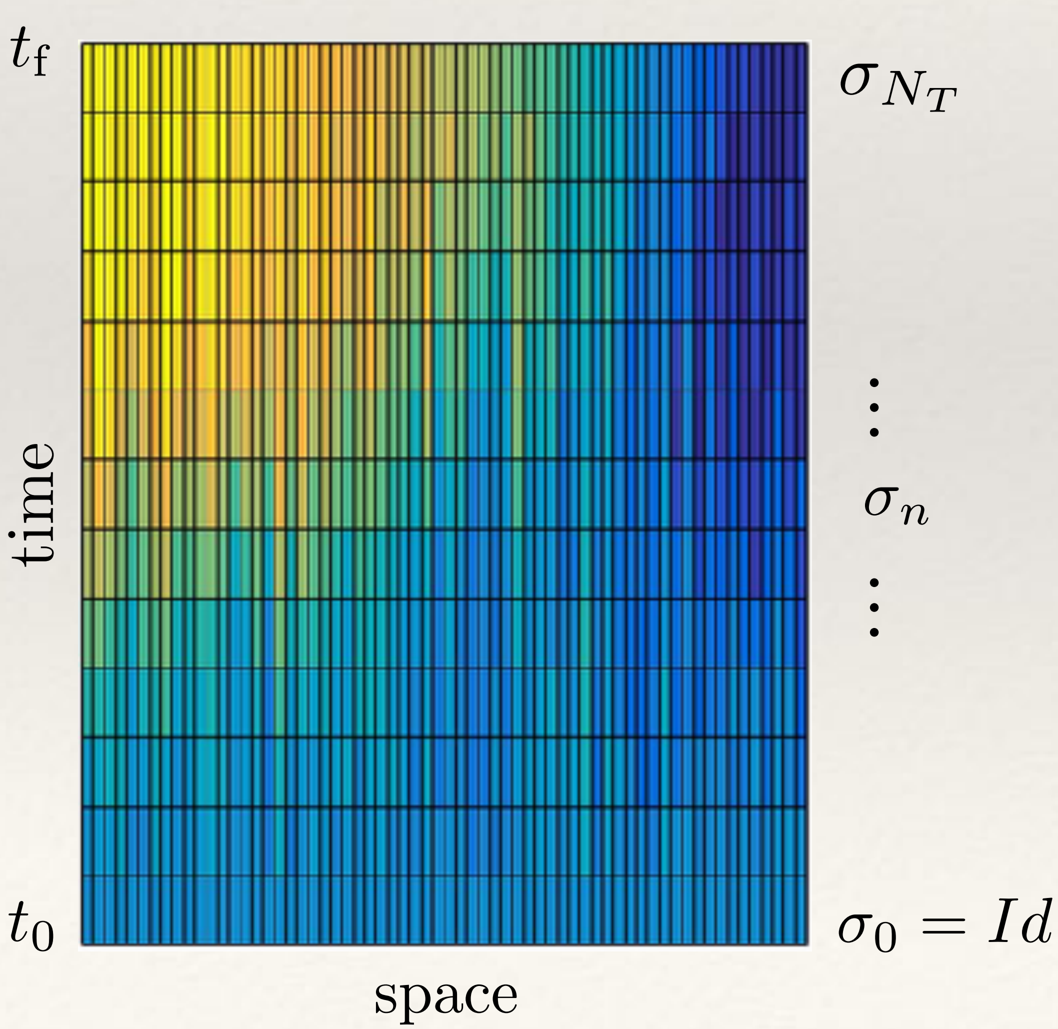
1. Solutions exist for all doubly-stochastic final configurations $\eta(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{y})$
2. For final maps given by a classical solution to Euler, the solution is concentrated if the final time t_f is small enough
3. **DiPerna–Majda distributional solutions can be formally represented in terms of a generalized variational principle, again for small t_f**

Is this framework relevant for turbulent Lagrangian maps?

A stat-mech approach

Numerics: combinatorics *(Brenier 2008)*
semi-discrete optimal transport *(Mérigot-Mirebeau 2016)*

Goals: - Access the full distribution of maps
- Enforce the incompressibility condition (make use of permutations)
- Introduce thermal fluctuations (formally equivalent to viscosity?)



Idea: Gibbs measure on the set of maps

$$p_\beta(\{\sigma_n\}_n) = \frac{1}{Z(\beta)}\, e^{-\beta\, \mathcal{E}[\{\sigma_n\}_n]}$$

$$\mathcal{E}[\{\sigma_n\}_n] = \sum_{n=1}^{N_T} \sum_i \|\sigma_n(i) - \sigma_{n-1}(i)\|^2$$

with the B.C. $\sigma_0(i) = i$ and $\sigma_{N_T} =$ target

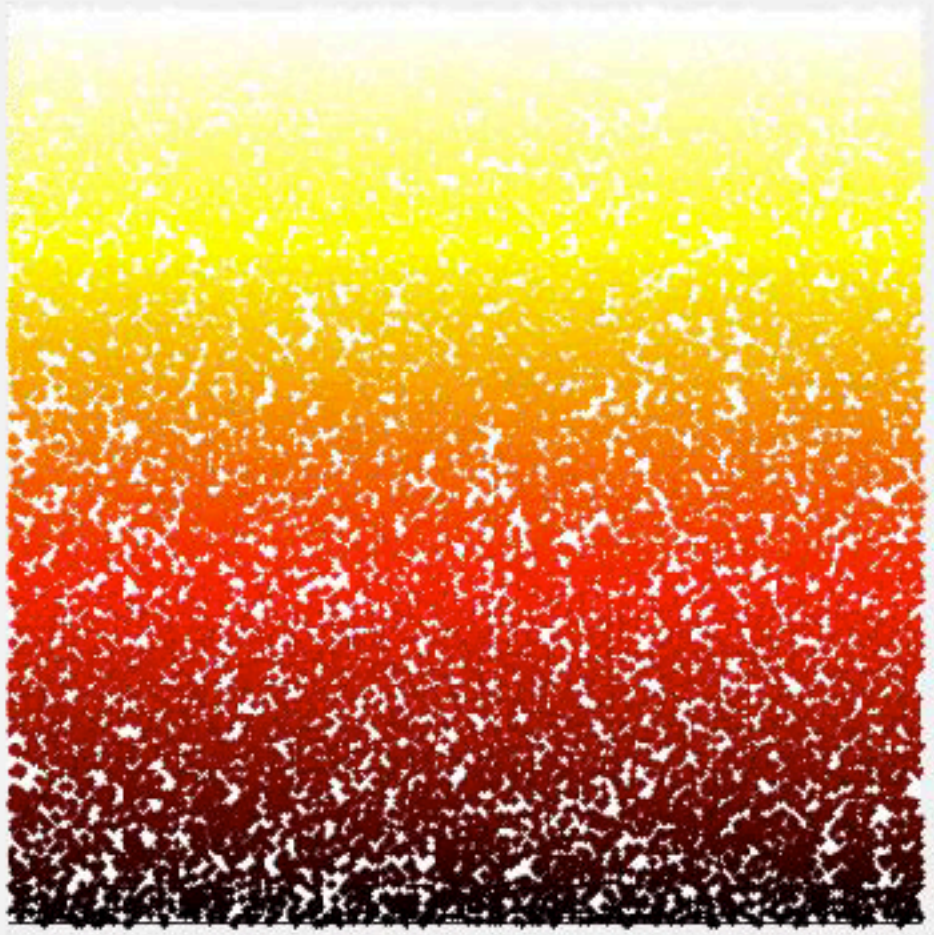
Metropolis Monte Carlo + small temperature

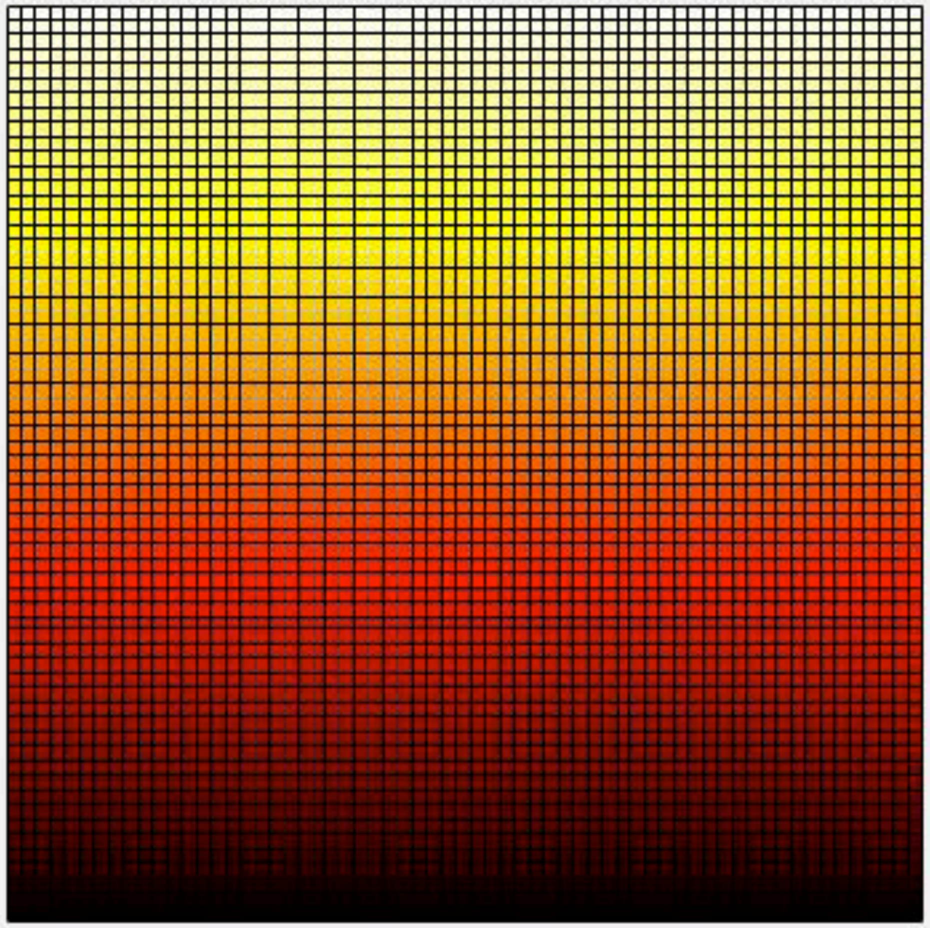
Equivalent to the **entropic regularization** of *Benamou, Carlier & Nenna (2016)*

1st test case: 2D Beltrami flow

Objective: Test ideas in a flow where everything is supposedly under control

Stationary solution $\boldsymbol{u} = \begin{pmatrix} \sin \pi x \; \cos \pi y \\ -\cos \pi x \; \sin \pi y \end{pmatrix}$ $\quad N_X \times N_Y = 64 \times 64$





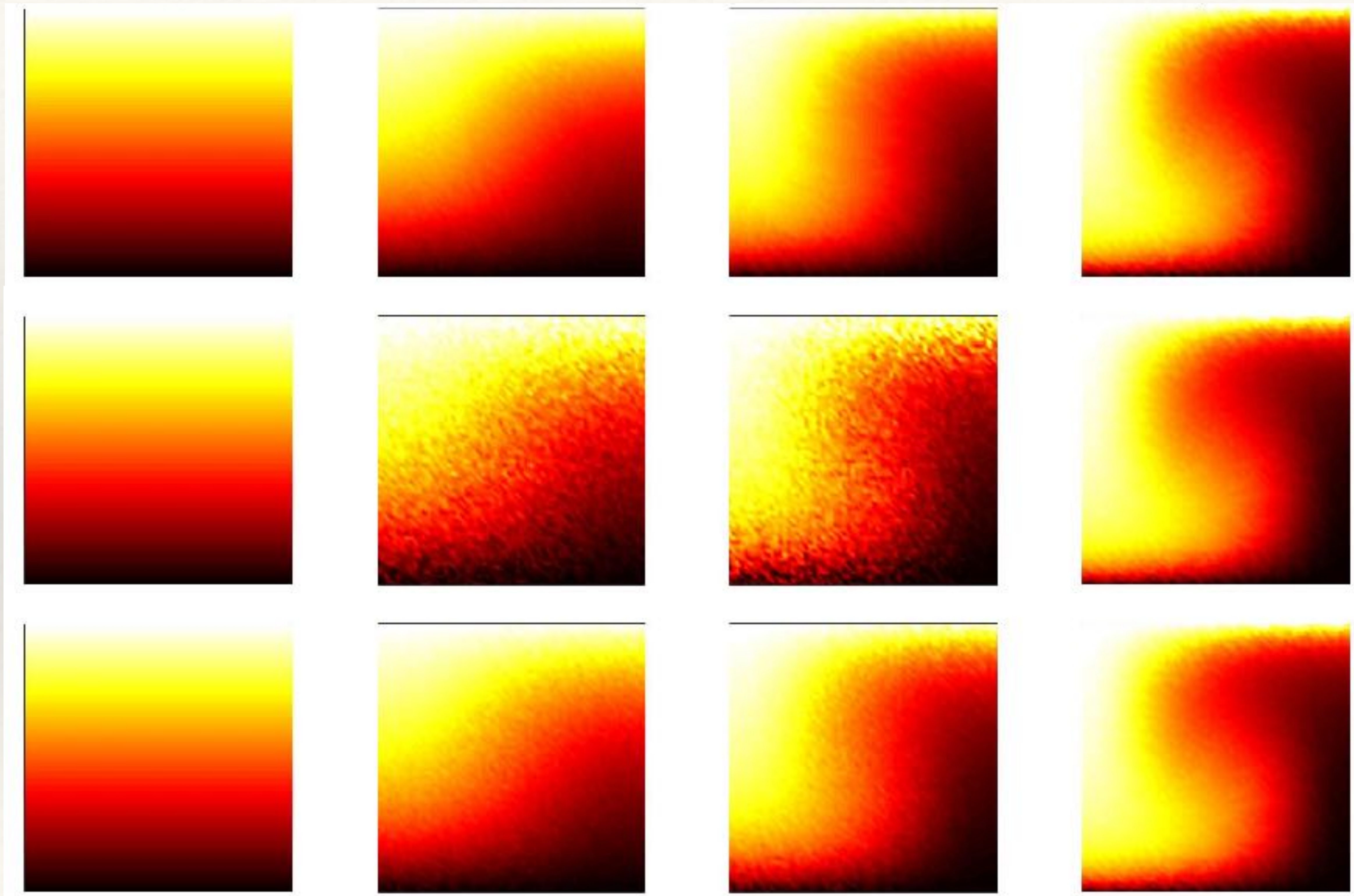
Convergence at infinite temperature

$\Delta t = 0.125$ $\quad t = 0 \quad t = 0.25 \quad t = 0.5 \quad t = t_{\mathrm{f}} = 0.75$

Classical solution

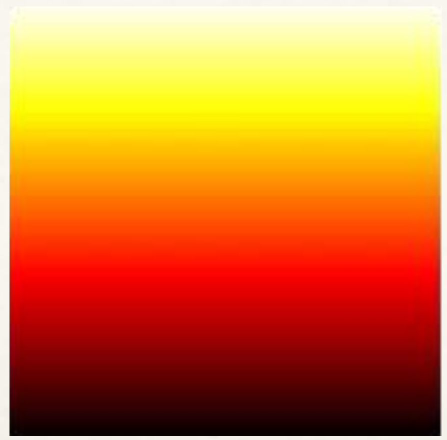
$\beta = 0.1$

$\beta = 1$

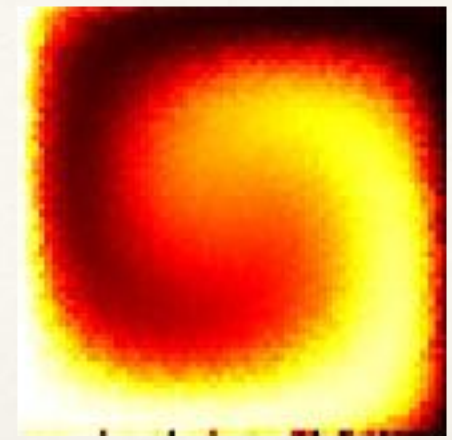


Brenier's generalized flows

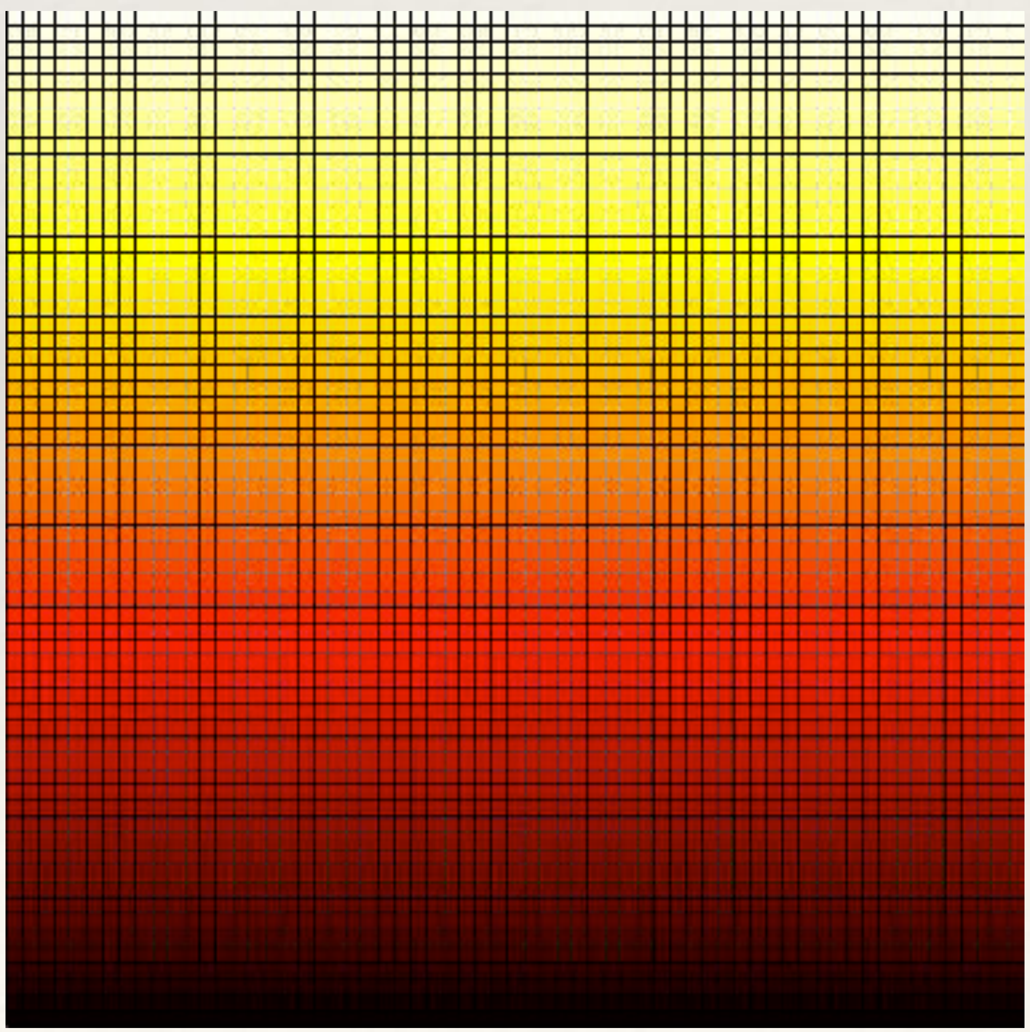
$t = 0$



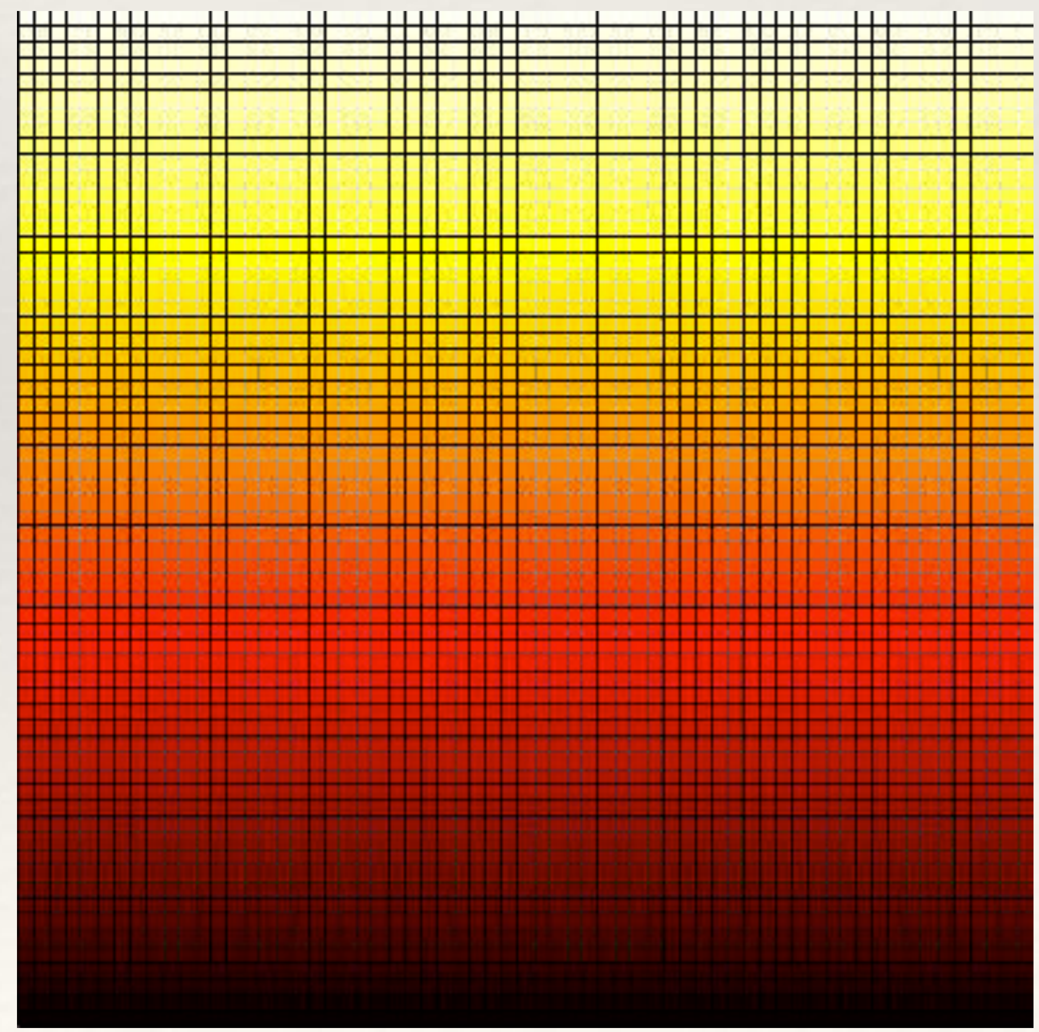
$t_{\mathrm{f}} > t_{\star} = 1$



Classical solution



Generalized flow



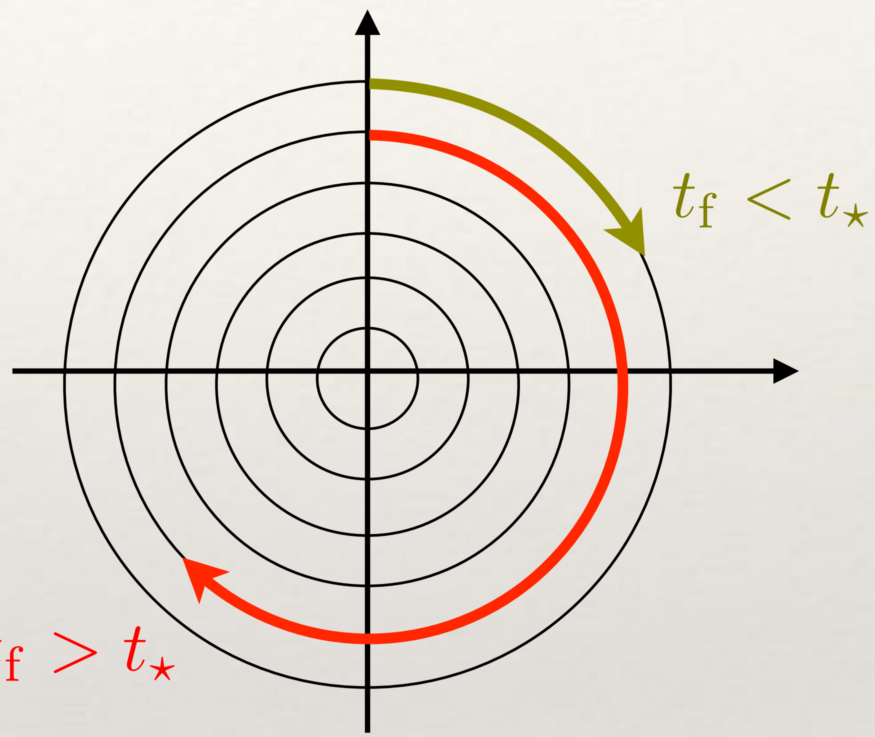
Maximal final time

Brenier 1999: classical solutions minimizers only if $t_{\mathrm{f}} < t_\star = \dfrac{\pi}{\sup_{\boldsymbol{x},t} \| \mathrm{Hess}(p) \|^{1/2}}$

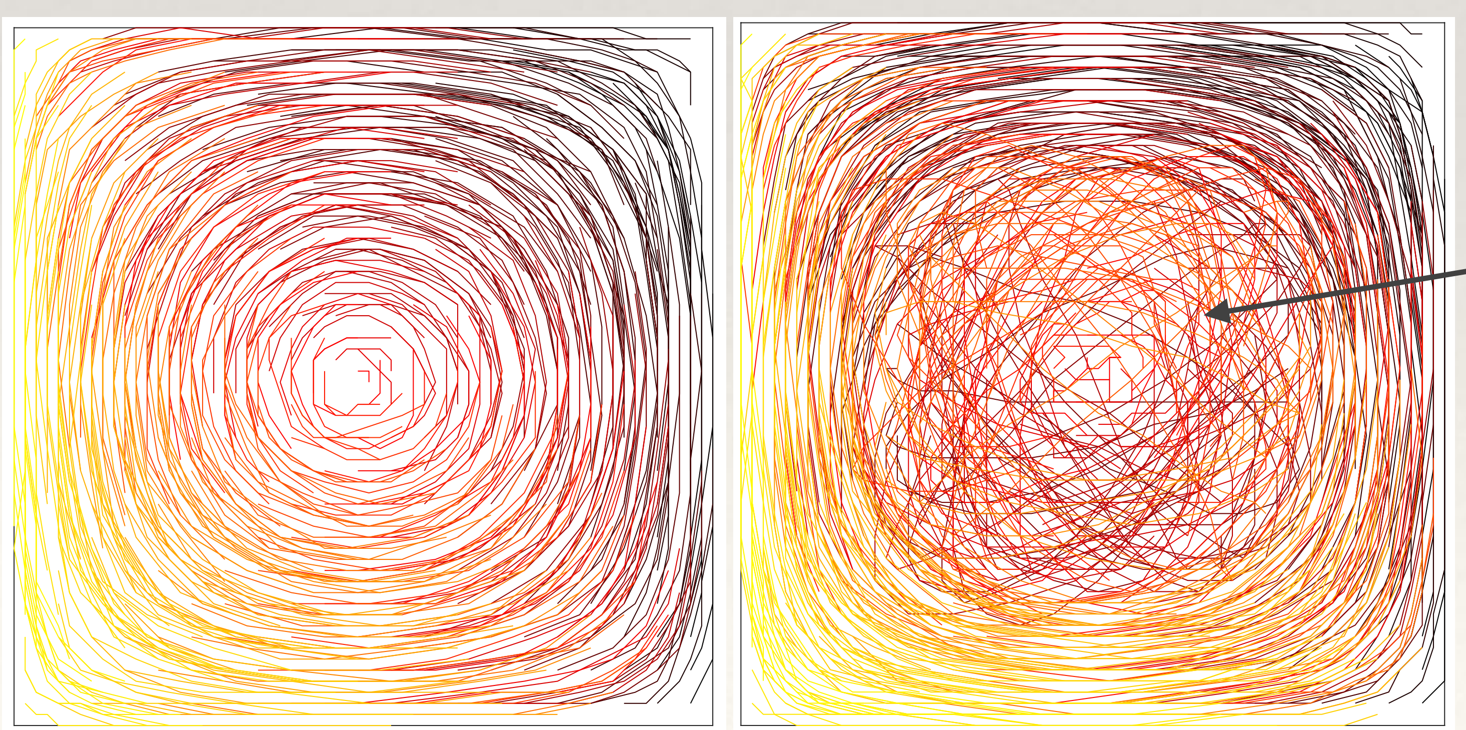
Geometrical interpretation: Jacobian matrix $\mathbb{J} \equiv \nabla_{\boldsymbol{x}_0} \boldsymbol{X}(t; \boldsymbol{x}_0, t_0)$

$\partial_t^2 \boldsymbol{X} = -\nabla p \;\Rightarrow\; \partial_t^2 \mathbb{J} = -\mathbb{H}\,\mathbb{J}$
with $\mathbb{H}_{i,j} = \partial_i \partial_j p$ pressure Hessian

At $t = t_\star$, the fluid has locally rotated of a half turn.
Shortcuts become cheaper.

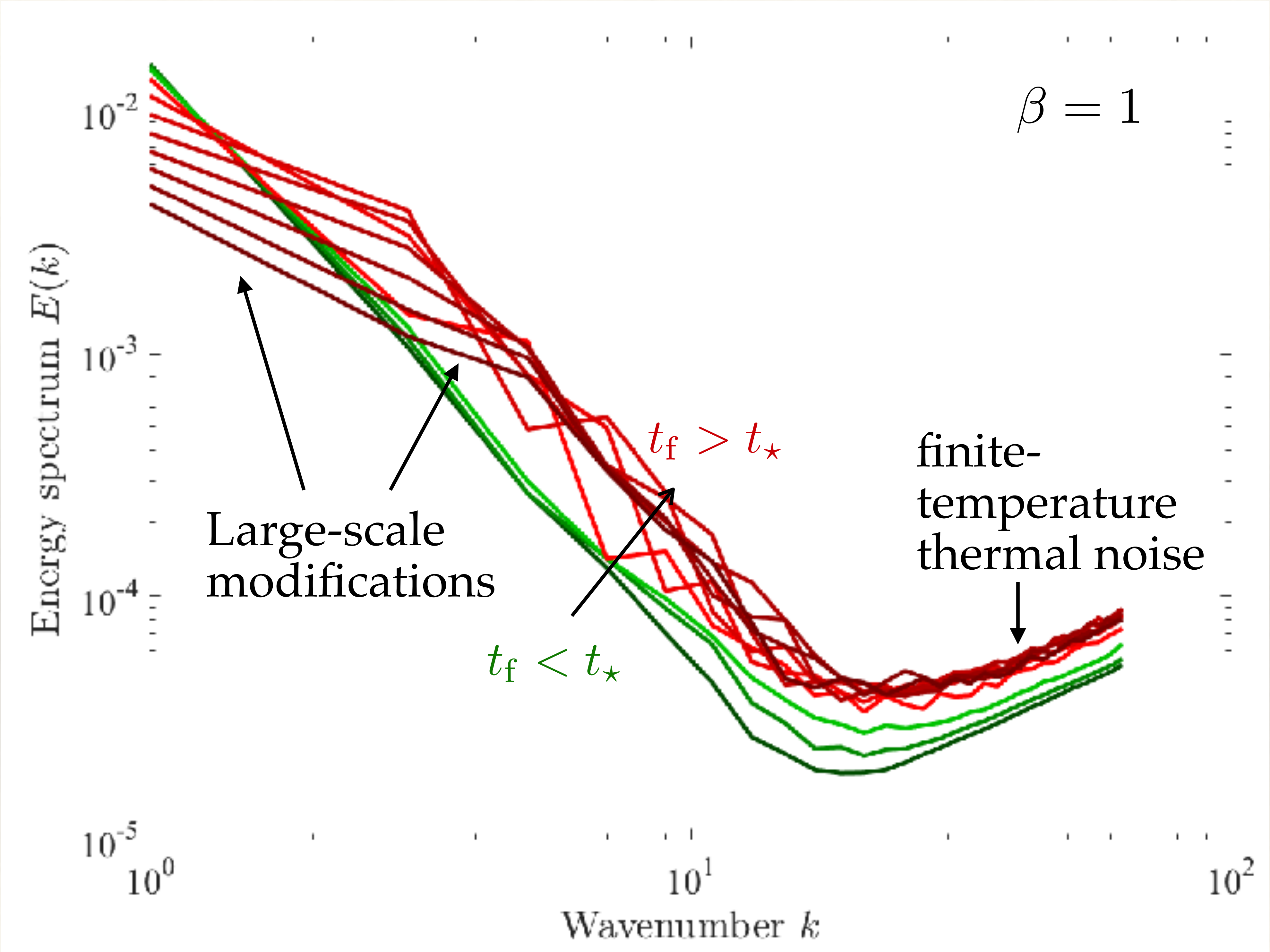


Beltrami



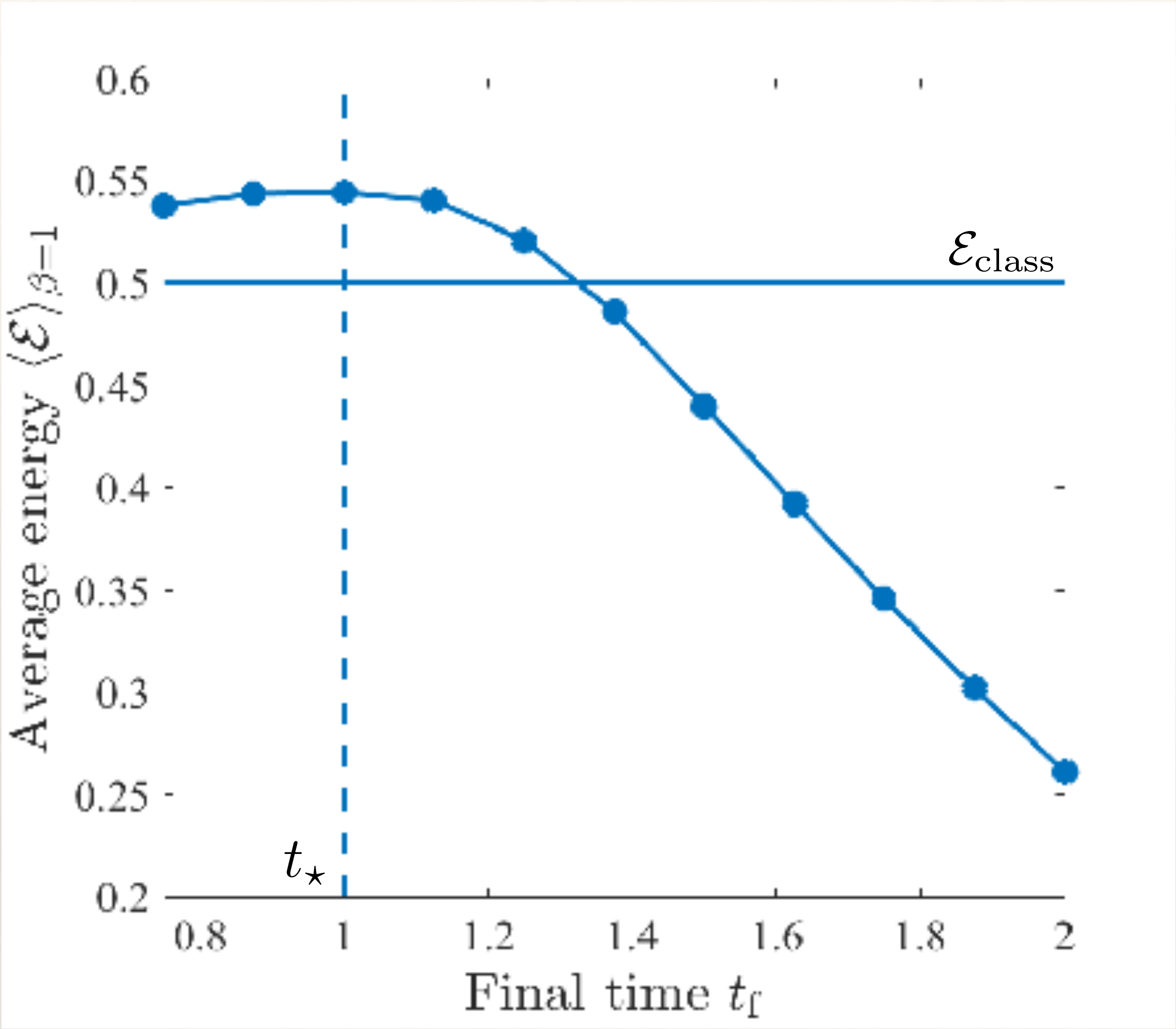
Physically irrelevant trajectories, even if not fully random

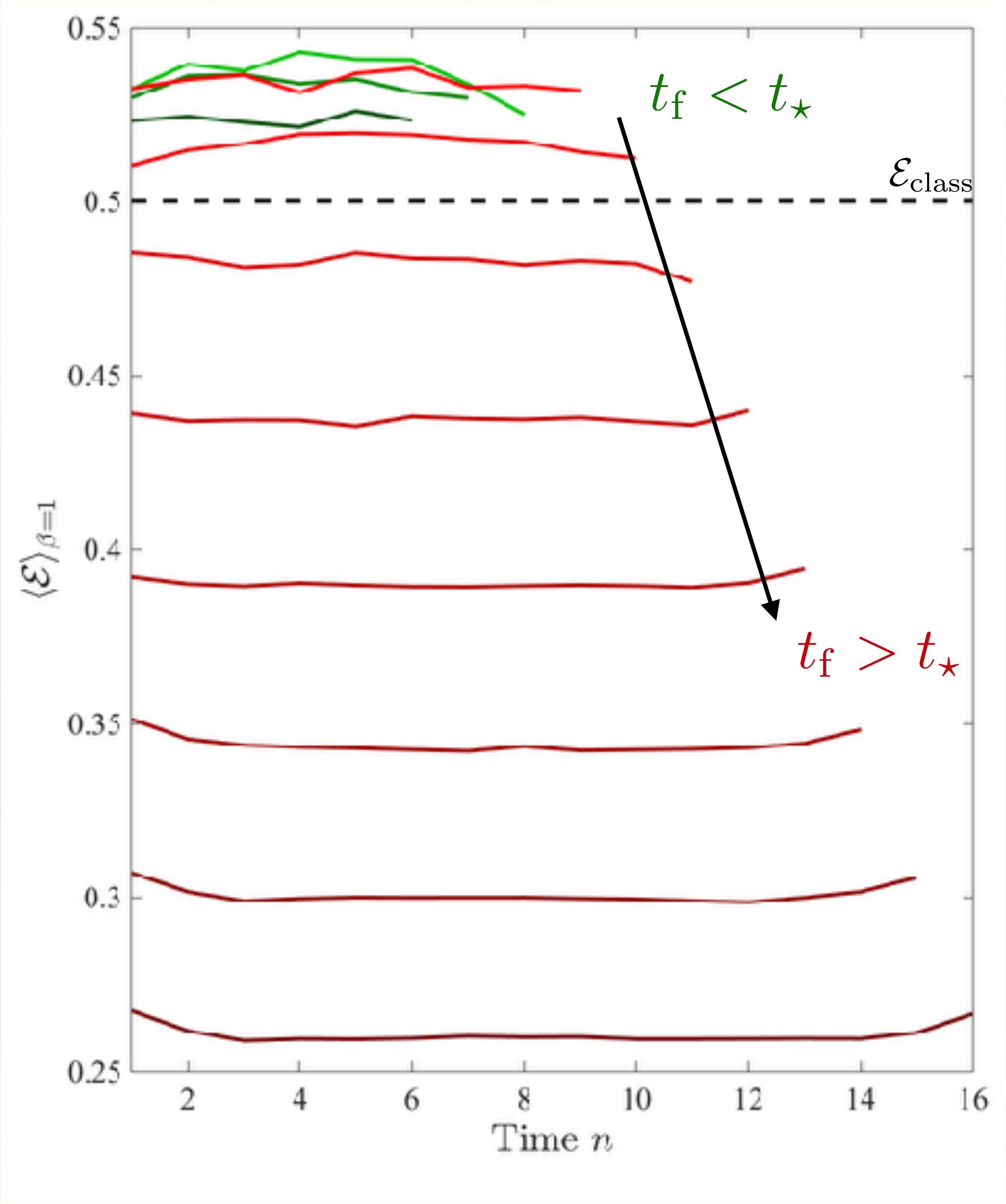
Reconstructed energy spectra



Generalized energy statistics

Decrease of the mean energy with t_f





Solutions at $t_\mathrm{f} > t_\star$ are reversible and do not conserve energy (even at the level of individual realizations)

$\Rightarrow$ unlikely turbulent candidates

A first move toward turbulence

Objective: Find a case where the inertial-range dynamics is described by Euler dynamics, but is not reversible.

⇒ **Possible candidate:** two-dimensional direct cascade

kinetic energy $E = \dfrac{1}{2}\displaystyle\int \|\boldsymbol{u}(\boldsymbol{x},t)\|^2\,\mathrm{d}^2x$ (almost) conserved

enstrophy $Z = \dfrac{1}{2}\displaystyle\int \omega^2(\boldsymbol{x},t)\,\mathrm{d}^2x$ cascades toward small scales

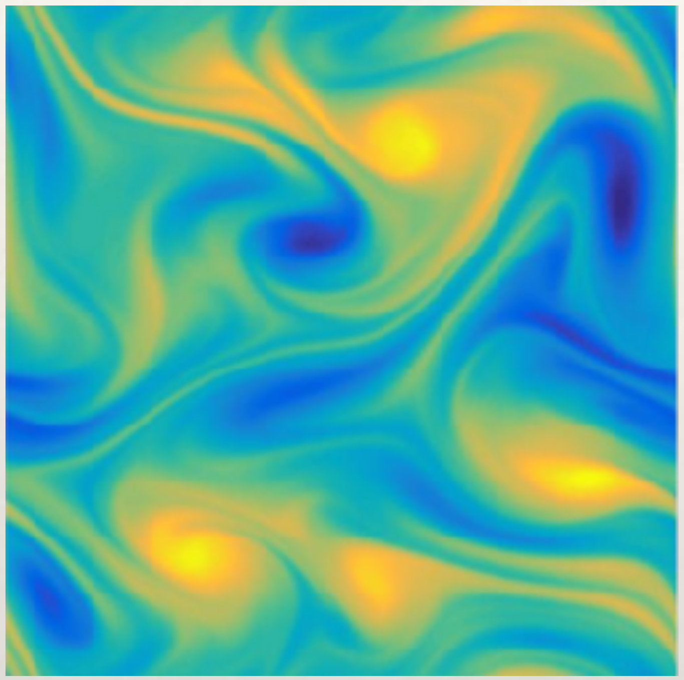
General questions:

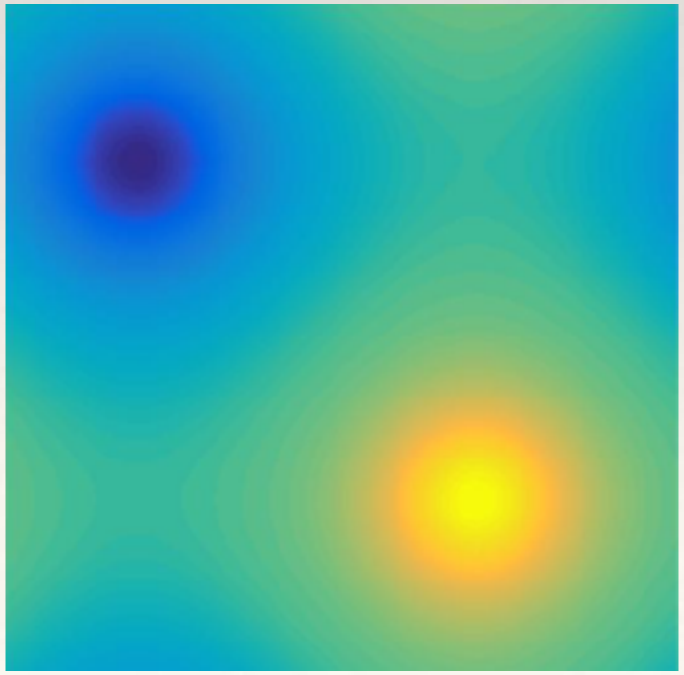
- Can the generalised variational principle reproduce such behaviors?
- How is the maximal time $t_\star$ obtained? Is it the turnover time?
- Is the information on **irreversibility entailed in the map** used as B.C.?

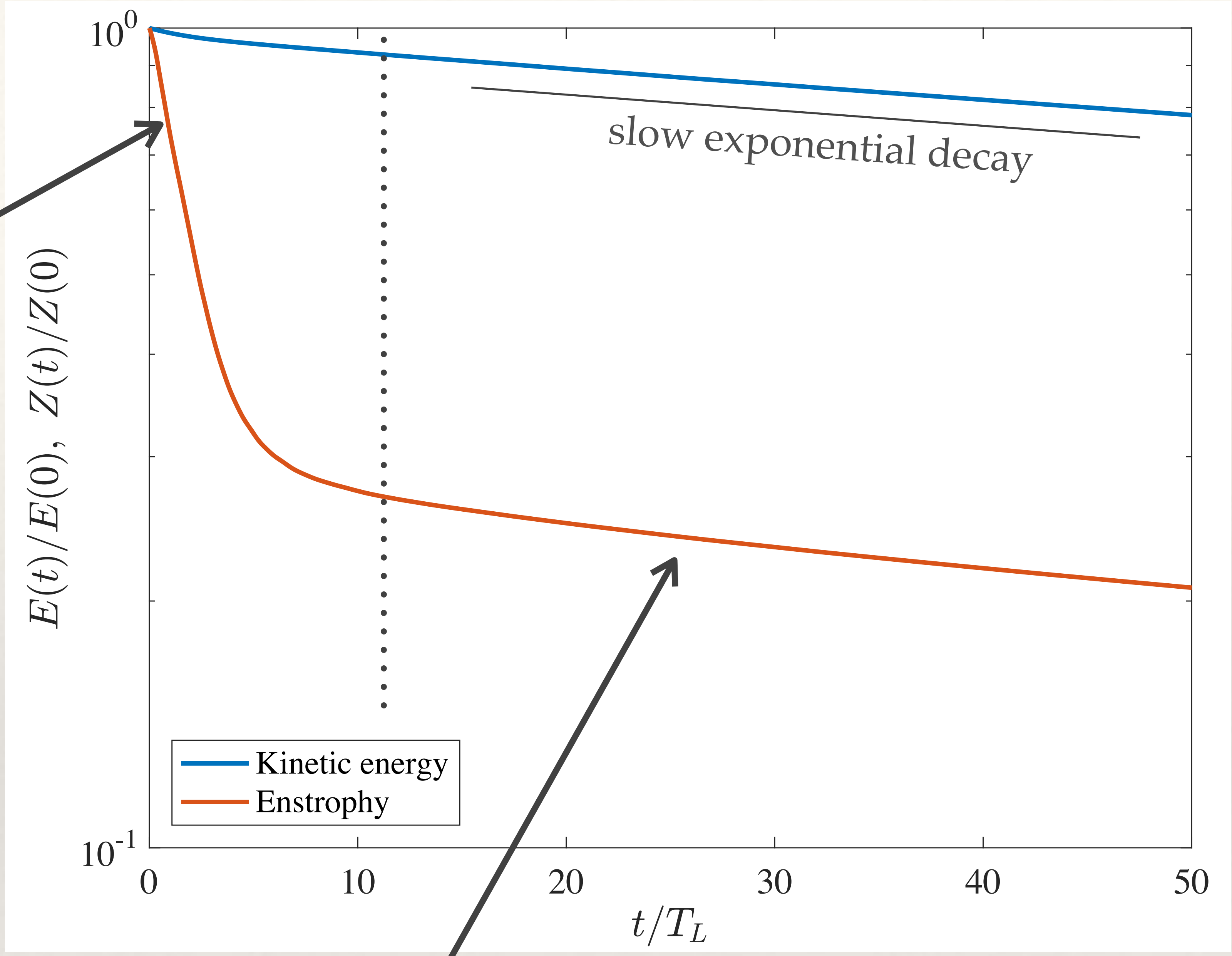
2nd test case: 2D turbulent decay

Two regimes:

1. Fast enstrophy decrease at early times (direct cascade)







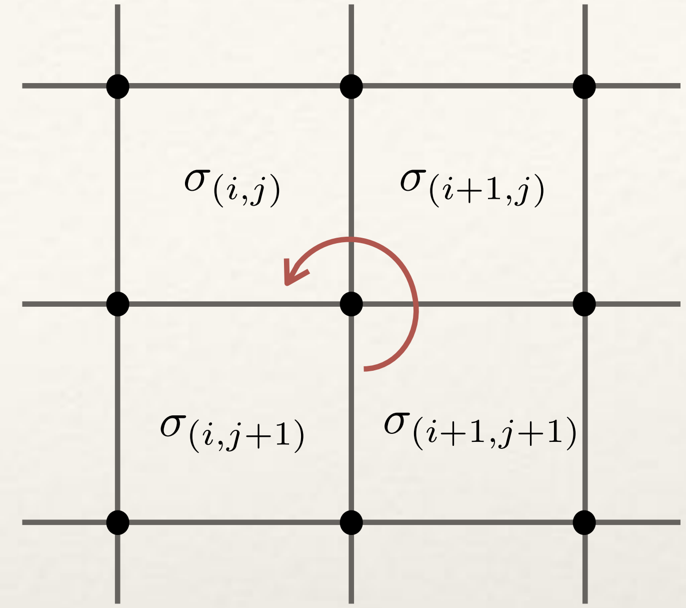
2. Slow energy decay at larger times (≃ steady Euler equation)

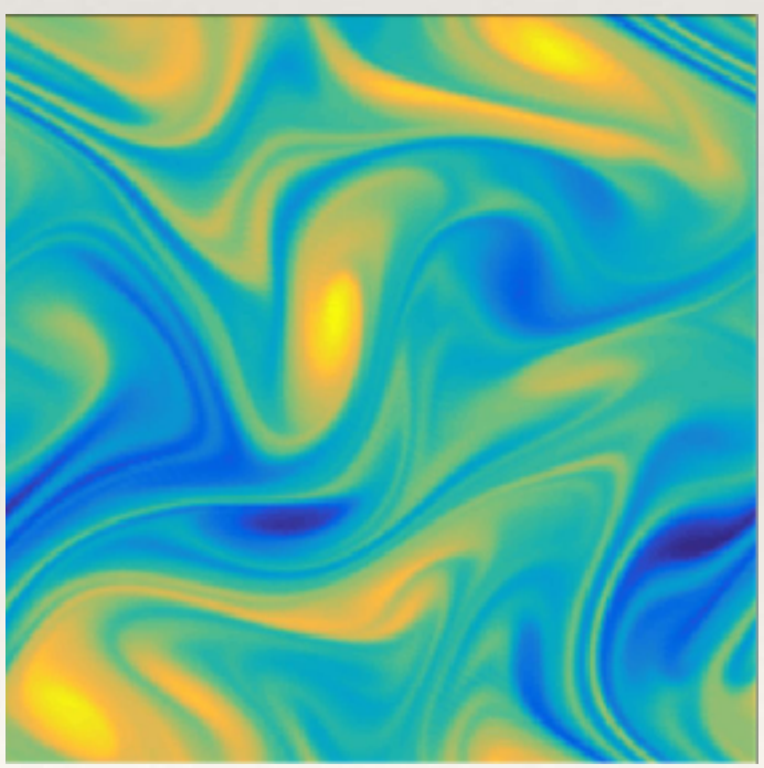
see, e.g., *Fang & Ouellette* (2017)

Vorticity reconstruction

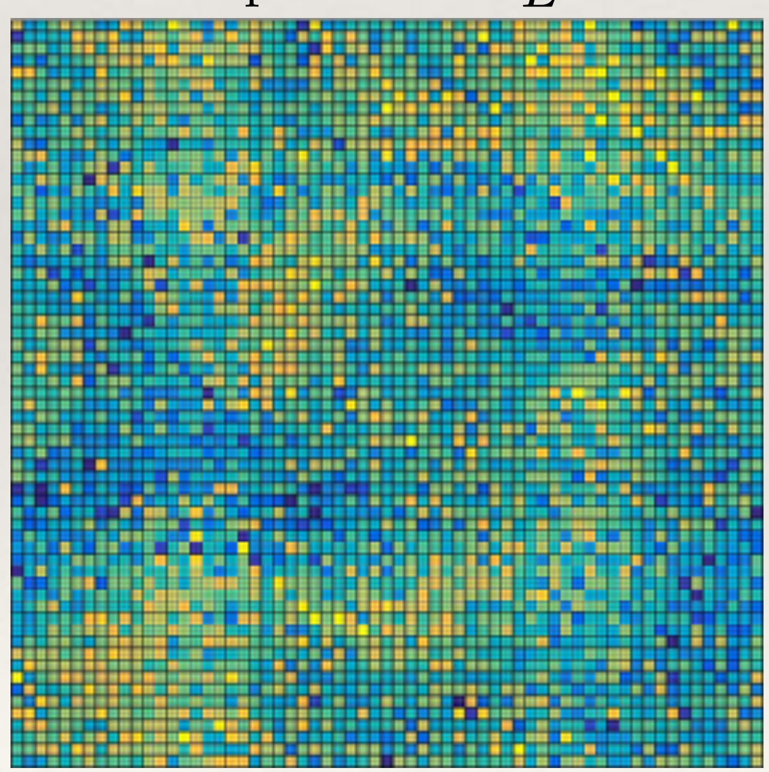
For classical solutions, the circulation is the Noether invariant associated to the relabelling symmetry

Coarse-grained vorticity is equivalent to a circulation on the dual lattice

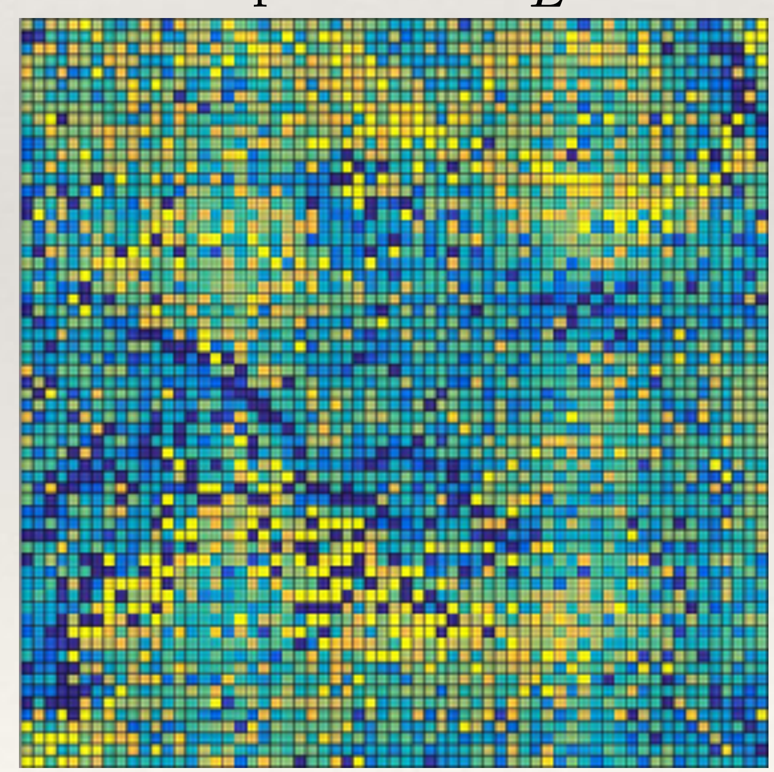




$t_{\mathrm{f}} = 0.2\,T_L$



$t_{\mathrm{f}} = 0.3\,T_L$

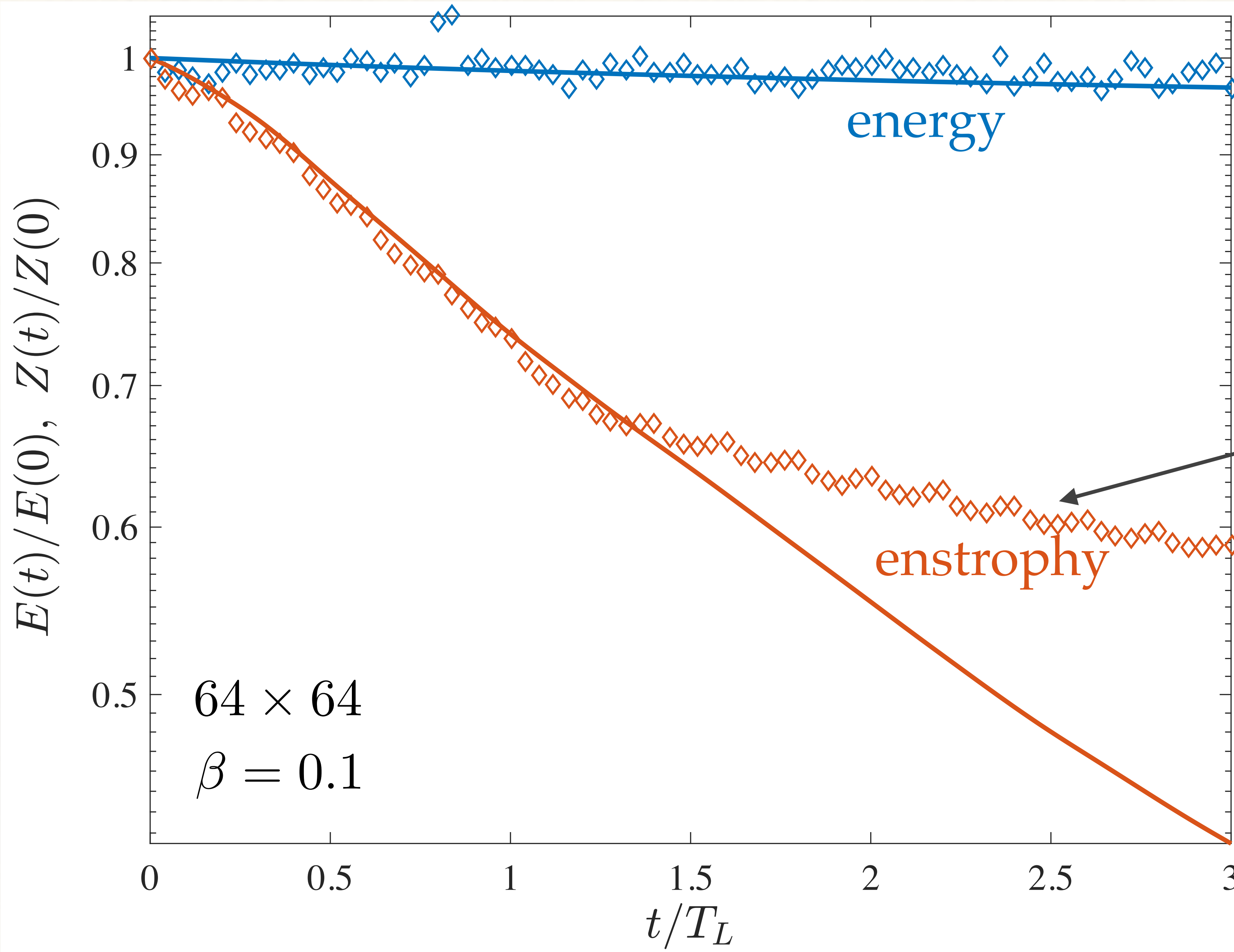


64×64

$\beta = 0.1$

$\Rightarrow$ time step $\approx 0.2\,T_L$

Enstrophy decay



Finite-temperature saturation?

Time decrease of enstrophy is entailed in the coarse-grained Lagrangian map.

Conclusions / Road map

Generalized Euler flows obtained from variational principle are physically relevant if the time boundary-value problem is considered over a time $t_{\mathrm{f}} < t_\star$. The critical time $t_\star$ is of the order of the smaller turnover time at the coarse-graining scale.

The information on time irreversibility included in the coarse-grained map might be enough to account for energy / enstrophy transfers. Still, reproducing long-term behaviors require very small temperatures.

Extensions to 3D:

- Generalize the approach to transition probability (doubly stochastic matrices) instead of maps (permutations).
- Monte-Carlo methods might be intractable: need to develop more effective optimization algorithms.
- Use this formulation to design new observables or conserved quantities, as for instance a generalized circulation (Sreenivasan *et al.* APS DFD 2017)