ICTS, Bangalore, January 2018

Some Turbulence Fundamentals with emphasis on modeling and LES

Charles Meneveau
Johns Hopkins University

JOHNS HOPKINS UNIVERSITY

JOHNS HOPKINS Center for Environmental & Applied Fluid Mechanics

JHU Mechanical Engineering

idies Institute for Data Intensive Engineering and Science

ICTS, Bangalore, January 2018

Sreeni's PhD student, Yale U in the late 1980s

Ashvin Chabbra

CM KRS Ramshankar

2008

Symposium on Fluid Science & Turbulence

A Celebration of K.R. Sreenivasan's career on his 60th birthday/Baltimore, May 30 & 31, 2008

1. R. Narasimha
2. S.H. Davis
3. K.R. Sreenivasan
4. M.E. Fisher
5. M.J. Feigenbaum
6. D.D. Joseph
7. J.A. Yorke
8. C. Meneveau
9. I. Procaccia
10. L.P. Kadanoff
11. Y. Kaneda
12. S. Chen
13. D. Lathrop
14. G. Eyink
15. U. Frisch
16. R. Benzi
17. A. Chhabra
18. G. Stolovitzky
19. J. Palis

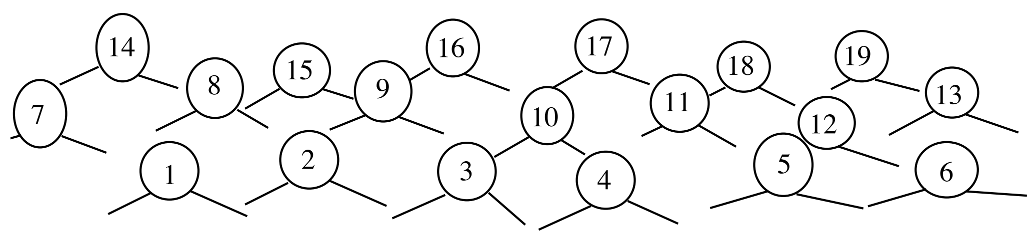


Figure 1. Symposium photograph taken on May 30, 2008 in front of Garland Hall, Homewood campus, Johns Hopkins University, Baltimore, MD.

ICTS, Bangalore, January 2018

Some Turbulence Fundamentals with emphasis on modeling and LES

Contents:

- Nature of Turbulence
- RANS, DNS and LES
- A fluid mechical view of eddy viscosity
- The dynamic Smagorisky model and variants

Johns Hopkins University

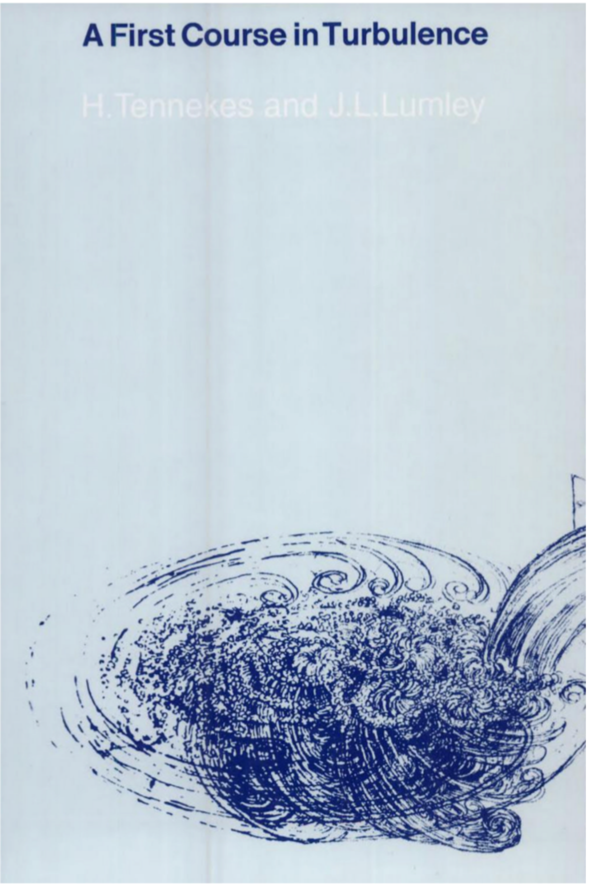
Johns Hopkins Center for Environmental & Applied Fluid Mechanics

JHU Mechanical Engineering

idies Institute for Data Intensive Engineering and Science

The Nature of Turbulence

(borrowring from list in Tennekes & Lumley's book)



Turbulence is ...

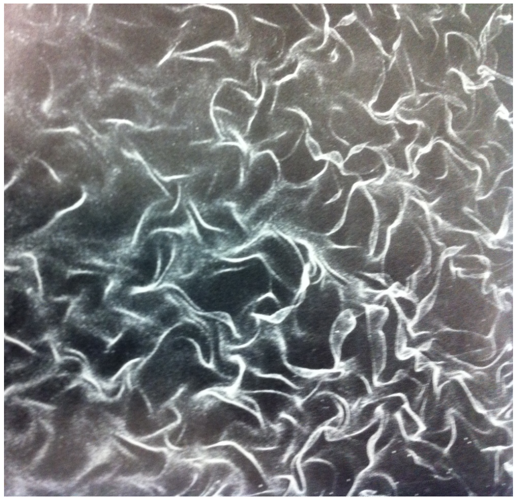
Turbulence is diffusive

http://listedmag.com/wp-content/uploads/2012/09/Lynn-Stout-Excerpt-Crop-11.jpg

Turbulence is diffusive:
also continuum, multiscale, high Re

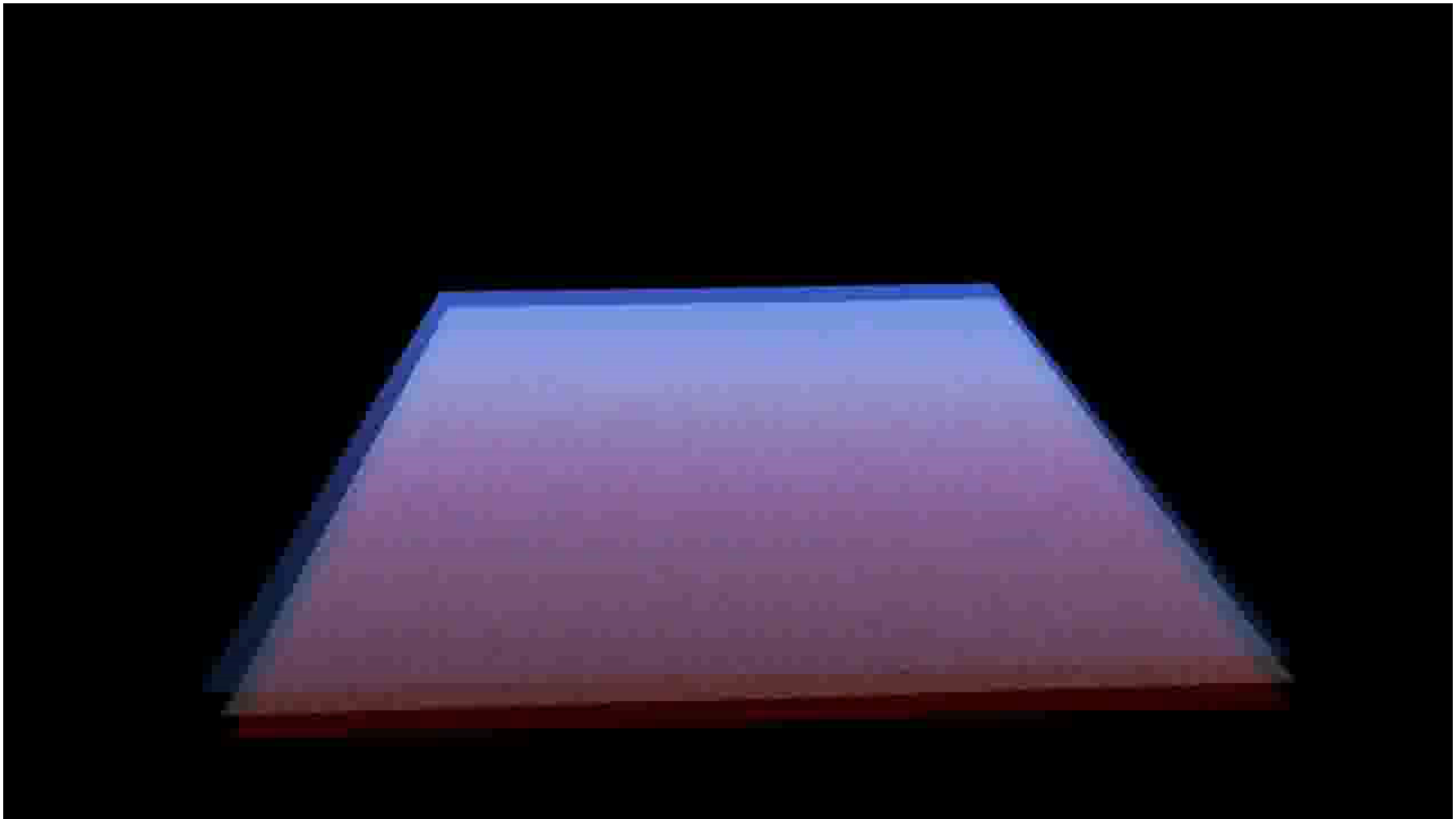
http://listedmag.com/wp-content/uploads/2012/09/Lynn-Stout-Excerpt-Crop-11.jpg

Turbulence is diffusive



M. Karweit (MS Thesis, & Album of Fluid Motion, CUP)

Dr. Kai Buerger (TUM, summer 2011)



Turbulence is dissipative

(but focus on decay of kinetic energy in the eddies)

https://cdn.theculturetrip.com/wp-content/uploads/2015/08/12220508645_5fd0871488_k-1024x683.jpg

Turbulence is irregular, rough (fractal)

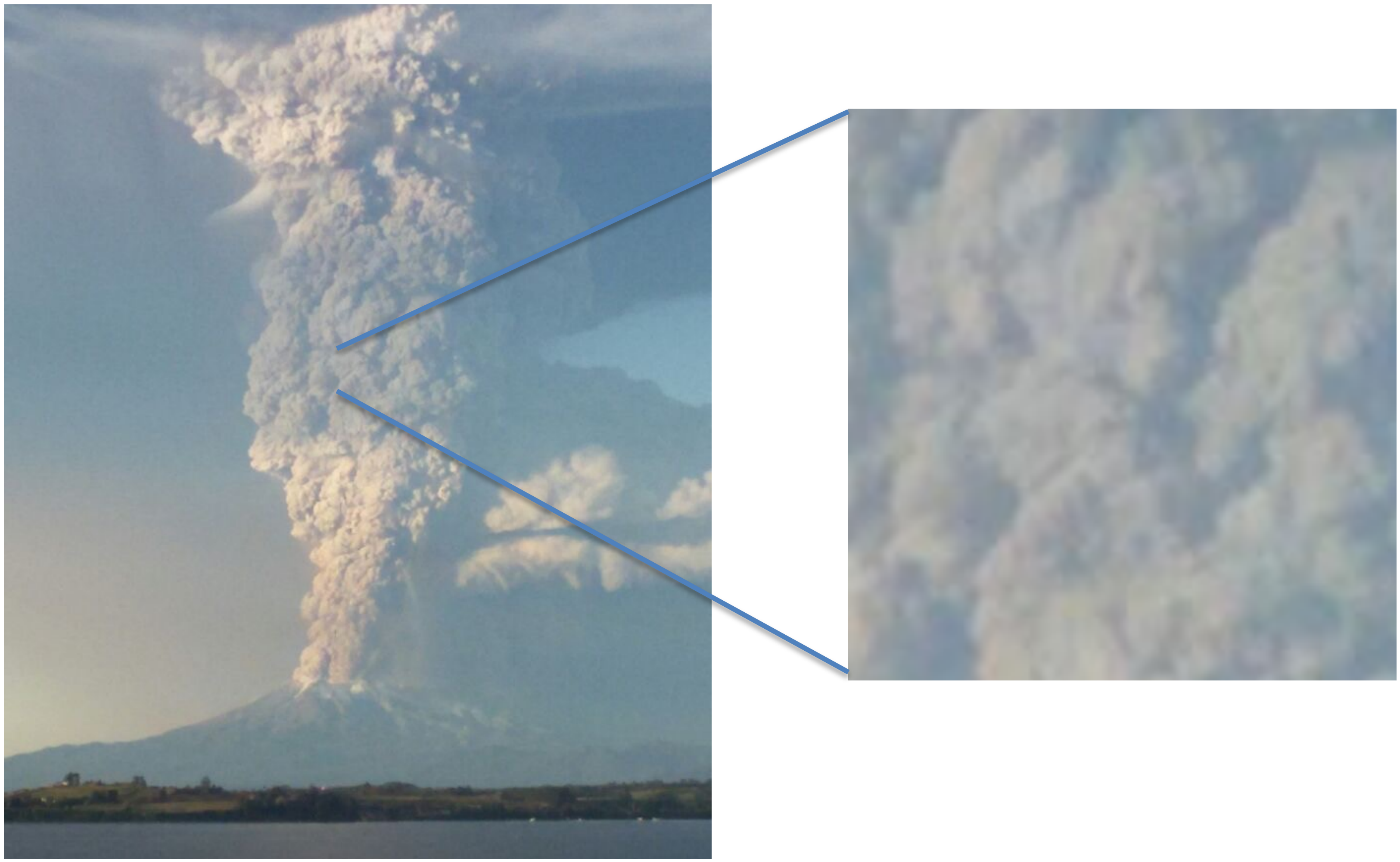
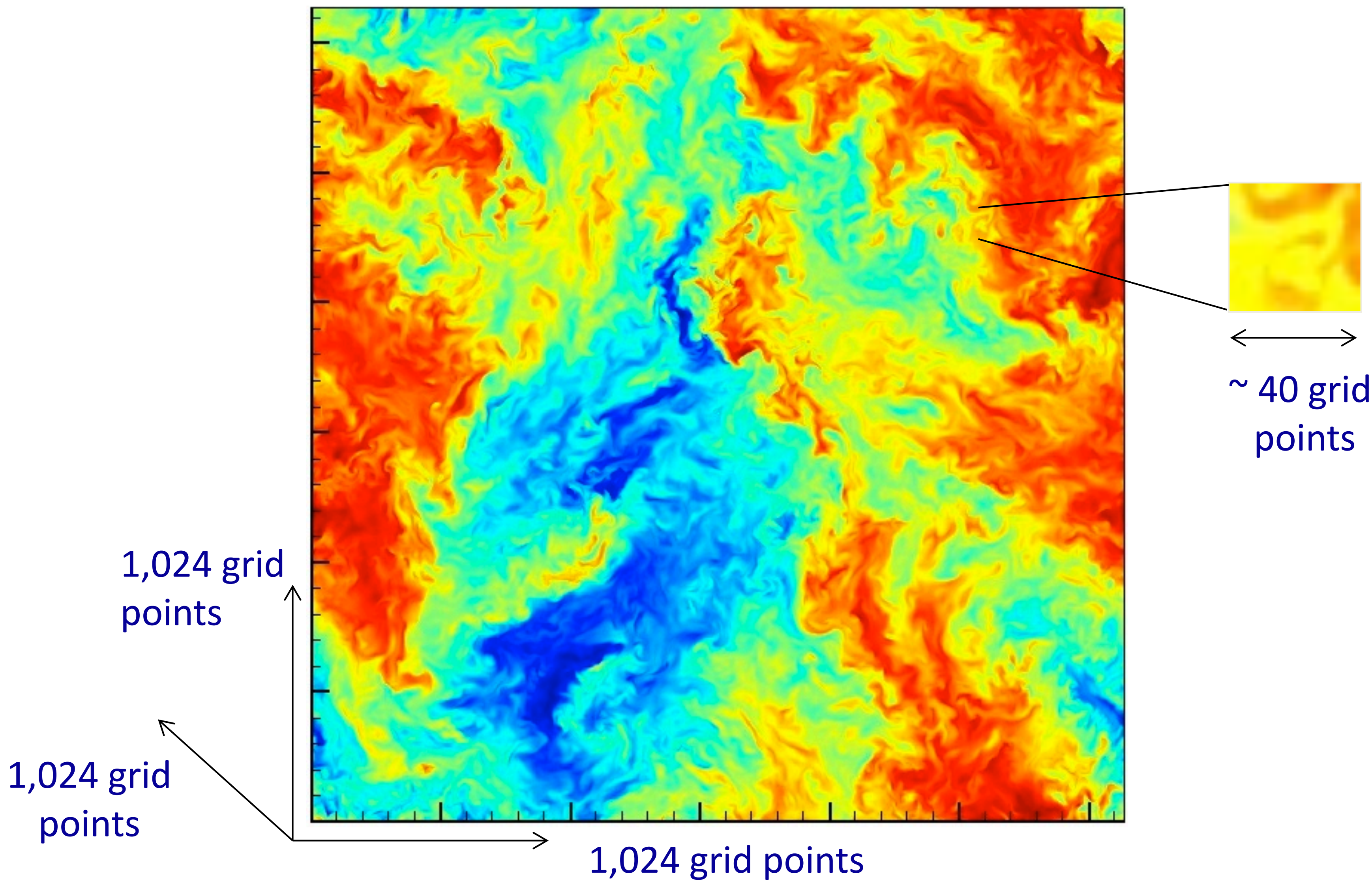


Photo credit: Guillermo Inio (Pta de Ilque, Chile)

Turbulence is property of continuum

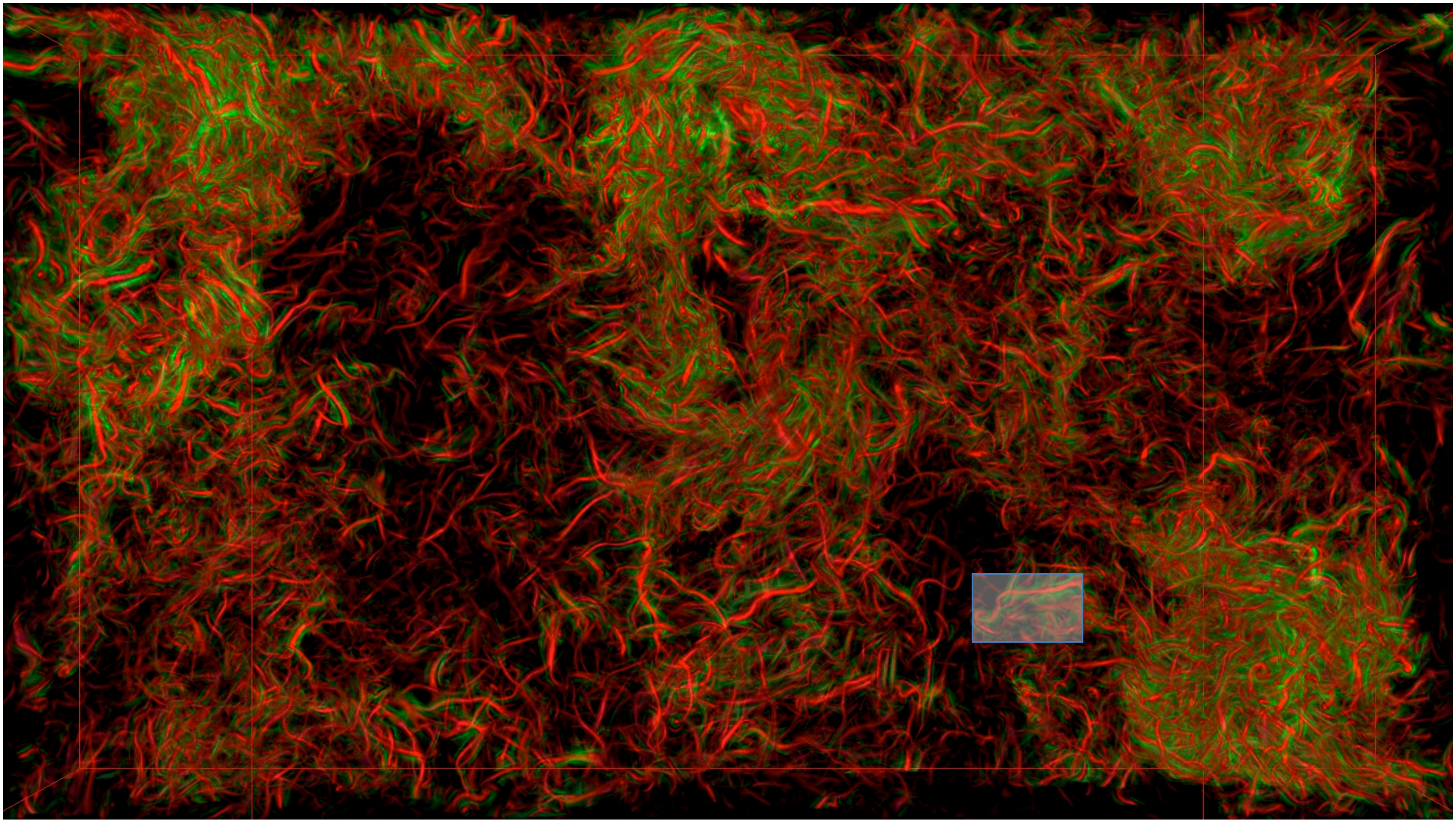
1,024[3] DNS: iso-velocity filled contours

$$u_1(x, y, z_0, t_0)$$

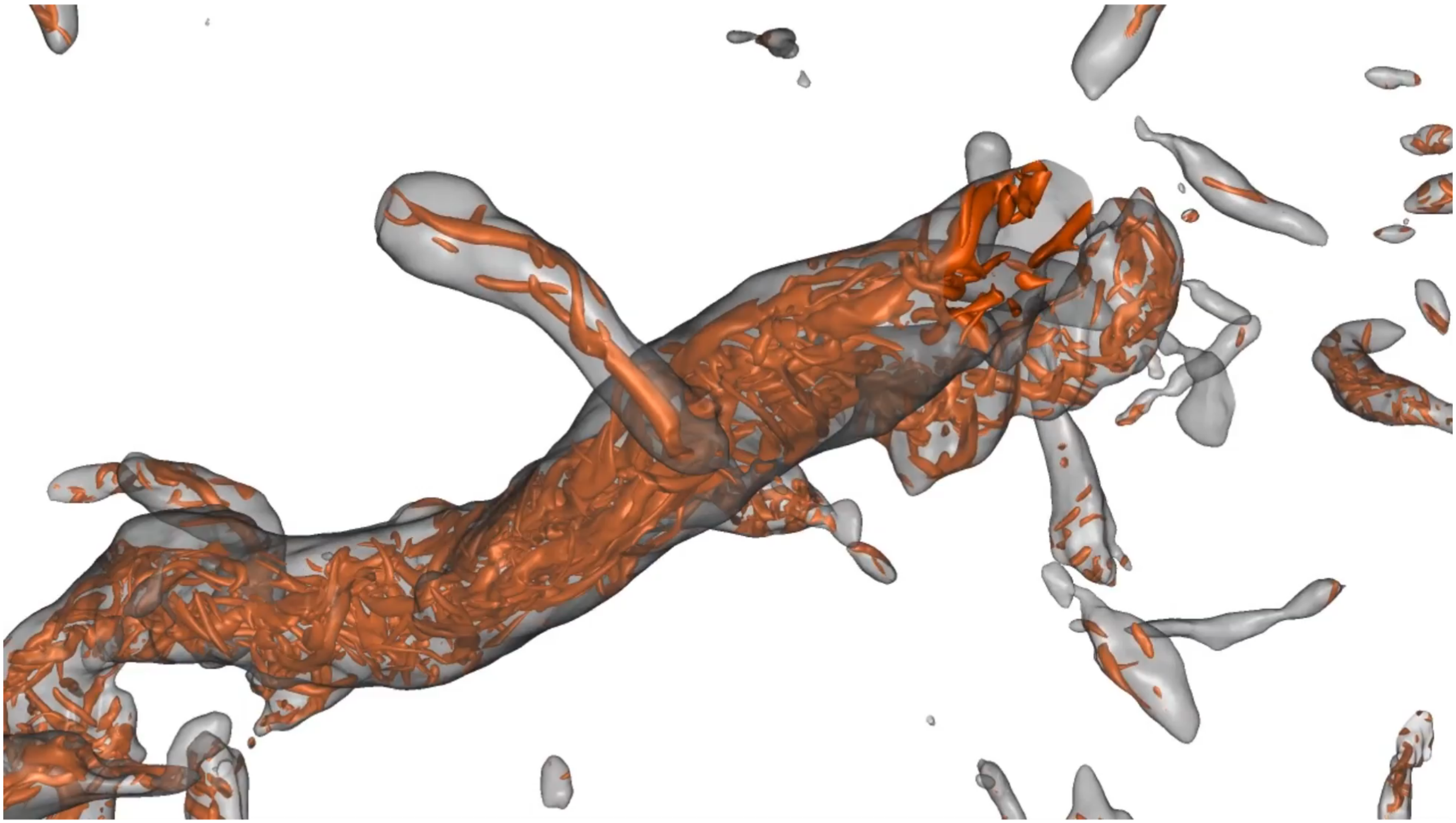


Turbulence is vortical (3D vorticity flucts.)

$$\left\|(\nabla\times\mathbf{u})^2\right\|$$



Turbulence is vortical (3D vorticity flucts.)



Turbulence =

eddies of many sizes

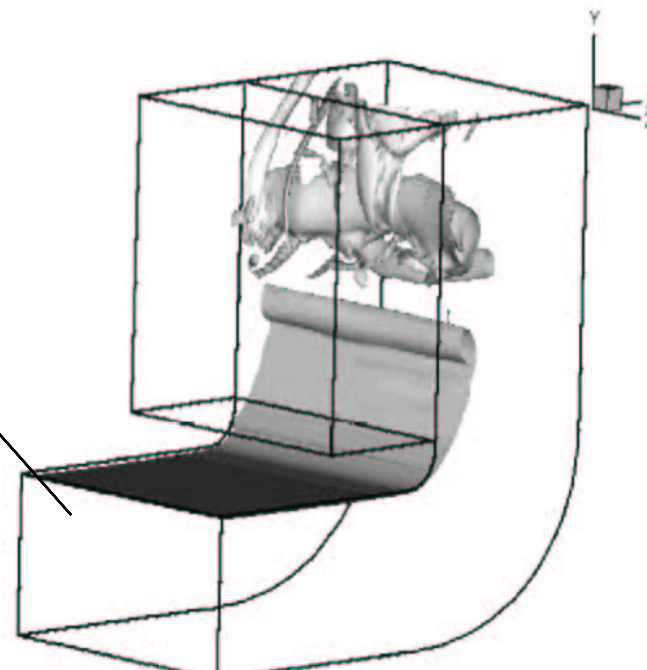
+ large-scale coherent structures

From: Multimedia Fluid Mechanics, Cambridge Univ. Press

Turbulence in aerospace systems:

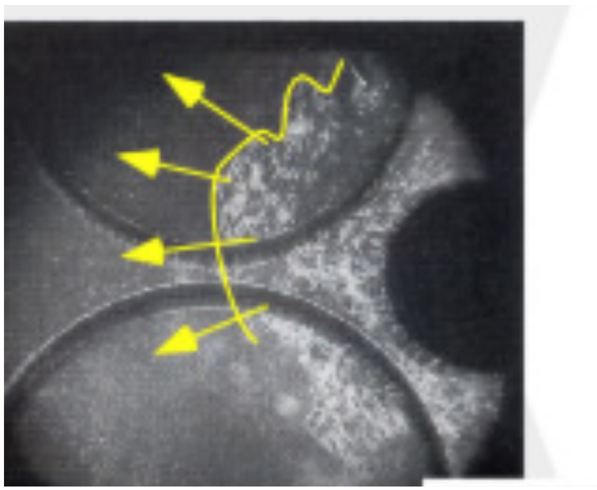
Jets and eddies during shuttle launch

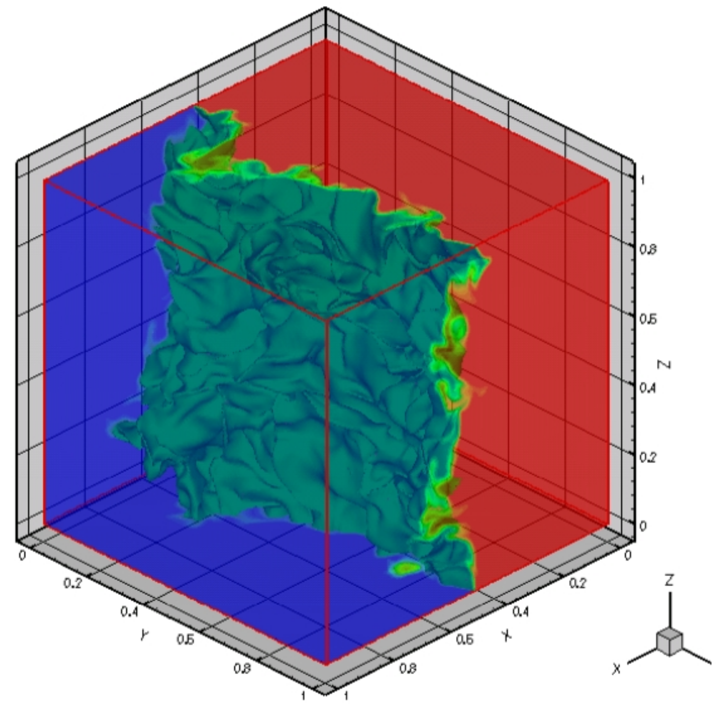
Large Eddy Simulation of flow in thrust-reversers
Blin, Hadjadi & Vervisch (2002)
J. of Turbulence.



Turbulence in reacting flows:

Premixed flame in I.C. engine, combustion





Numerical simulation of flame propagation in decaying isotropic turbulence

Turbulence in renewable energy

From J.N. Sørensen, Annual Rev. Fluid Mech. 2011:

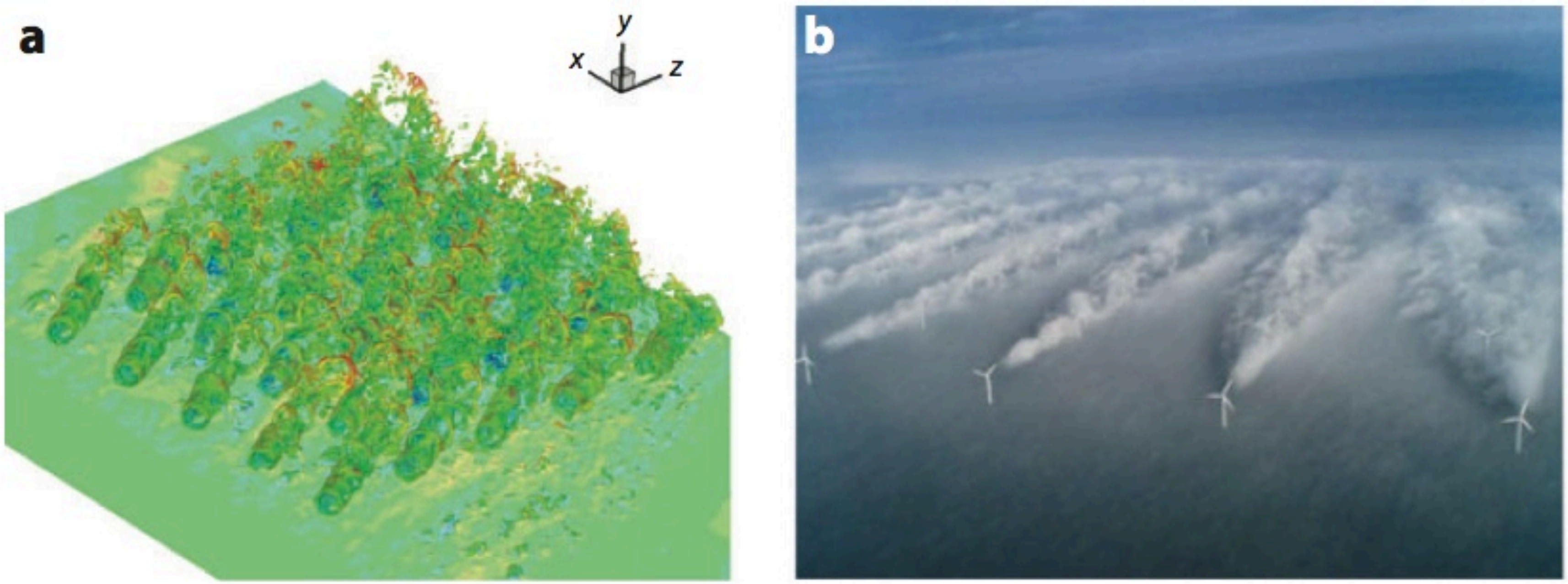
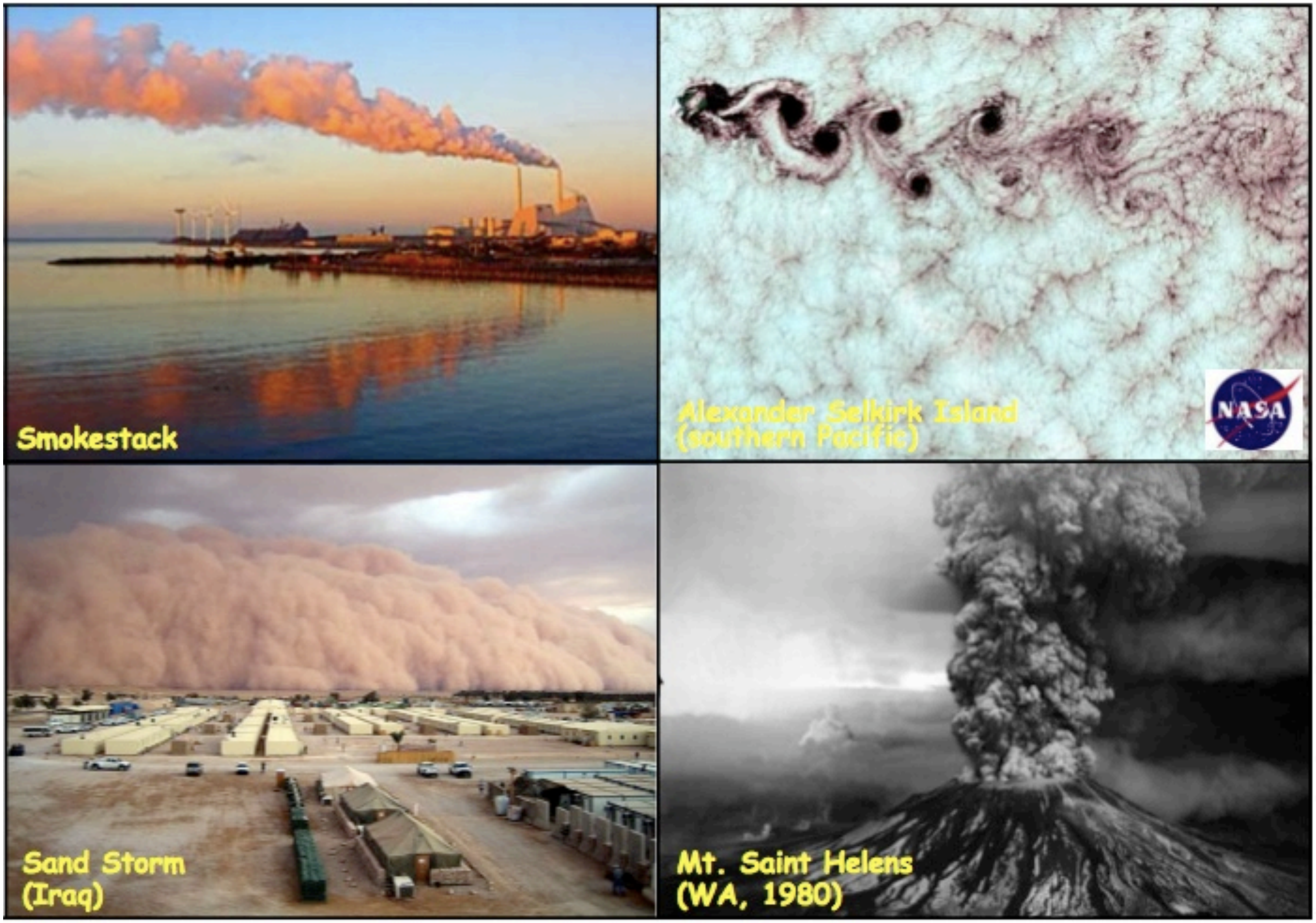


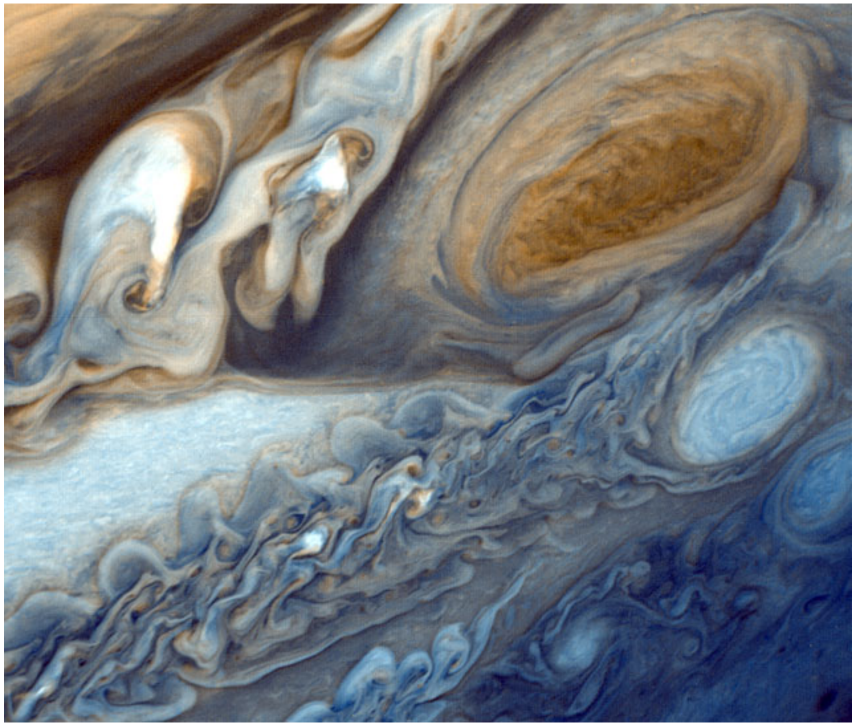
Figure 6

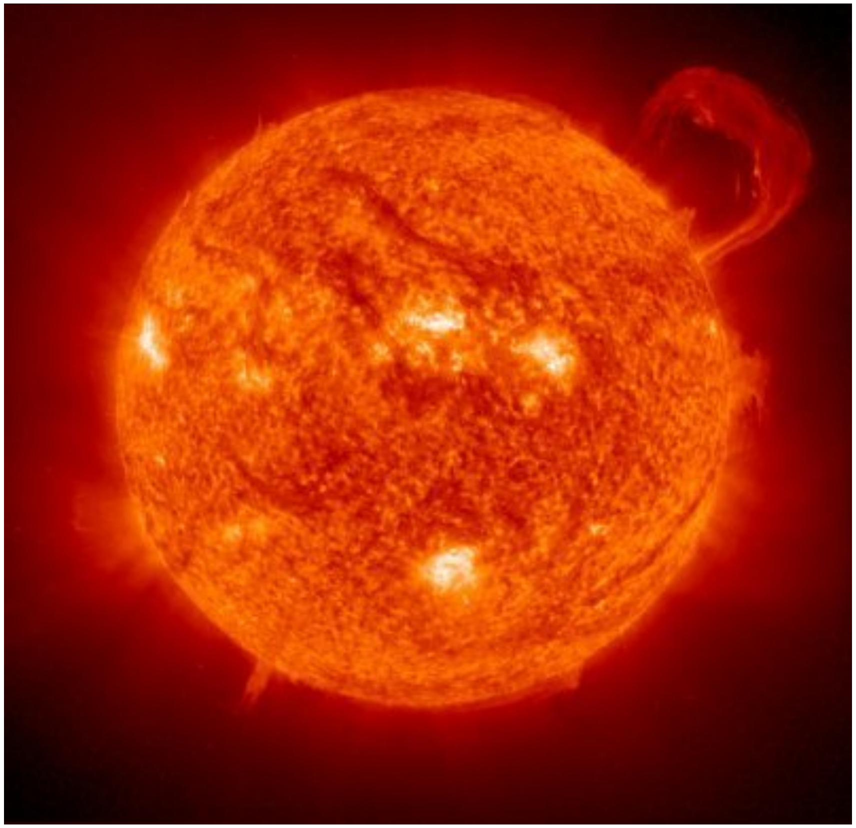
(*a*) Actuator disc computation of a wind farm consisting of 5 × 5 wind turbines. (*b*) Photograph showing the flow field around the Horns Rev wind farm.

Turbulence in environment and geophysics

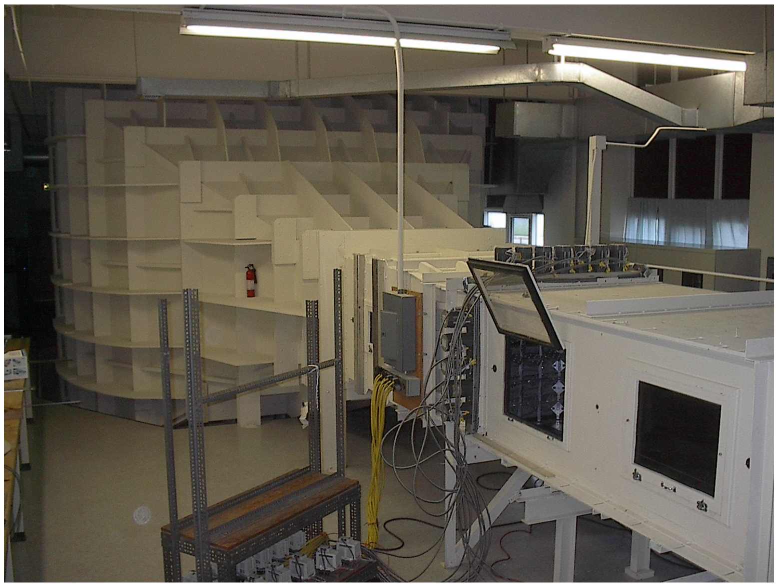


Turbulence in astrophysics



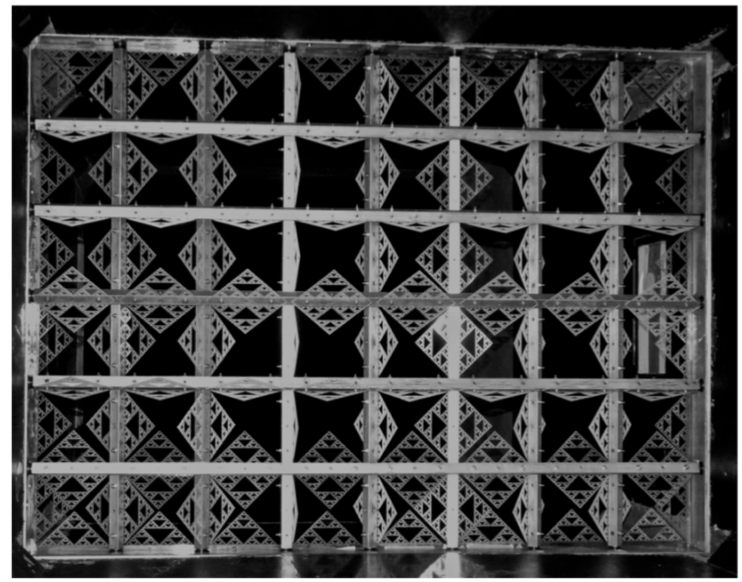


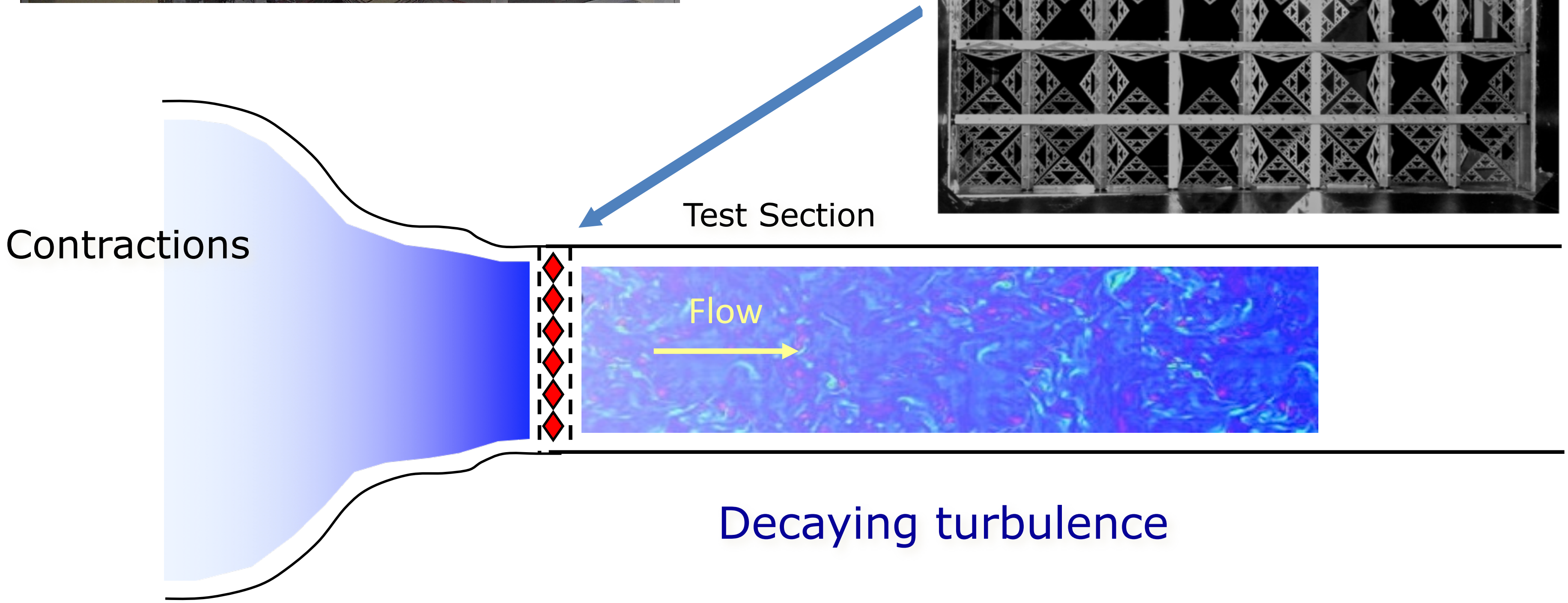
Simplest turbulence: Isotropic turbulence



Corrsin wind tunnel
at the Johns Hopkins University

Fractal Active Grid $M = 6$ "



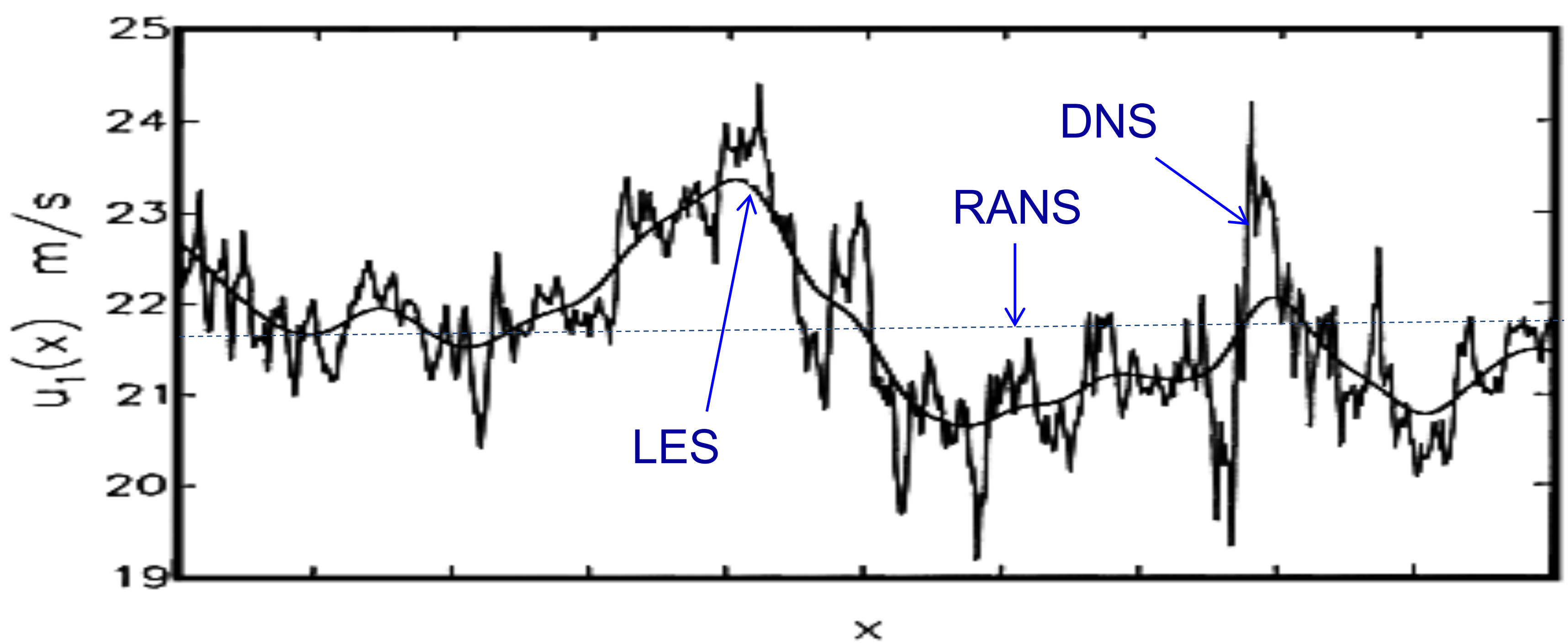


Navier-Stokes equations, incompressible, Newtonian

$$\begin{cases} \dfrac{\partial u_j}{\partial x_j} = 0 \\ \underbrace{\dfrac{\partial u_j}{\partial t} + \dfrac{\partial u_k u_j}{\partial x_k} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_j} + \nu \nabla^2 u_j + g_j}_{a_j = \frac{F_j}{m}} \end{cases}$$

Averaging and filtering: turbulence closure

$$
\begin{cases}
\dfrac{\partial u_j}{\partial x_j} = 0 \\
\dfrac{\partial u_j}{\partial t} + \dfrac{\partial u_k u_j}{\partial x_k} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_j} + \nu \nabla^2 u_j + g_j
\end{cases}
$$



Traditional approach: Reynolds decomposition

$$
\begin{cases}
\dfrac{\partial u_j}{\partial x_j} = 0 \\[2ex]
\dfrac{\partial u_j}{\partial t} + \dfrac{\partial u_k u_j}{\partial x_k} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_j} + \nu \nabla^2 u_j + g_j
\end{cases}
$$

Reynolds' equations:

t

$$
\begin{cases}
\dfrac{\partial \overline{u}_j}{\partial x_j} = 0 \\[2ex]
\dfrac{\partial \overline{u}_j}{\partial t} + \dfrac{\partial \overline{u}_j \overline{u}_k}{\partial x_k} = -\dfrac{1}{\rho}\dfrac{\partial \overline{p}}{\partial x_j} + \nu \nabla^2 \overline{u}_j + g_j - \dfrac{\partial}{\partial x_k}\left(\overline{u_j u_k} - \overline{u}_j \overline{u}_k\right)
\end{cases}
$$

- **Kinematic Reynolds stress (minus):**

$$
\begin{cases}
\dfrac{\partial \bar{u}_j}{\partial x_j} = 0 \\[2ex]
\dfrac{\partial \bar{u}_j}{\partial t} + \dfrac{\partial \bar{u}_j \bar{u}_k}{\partial x_k} = -\dfrac{1}{\rho}\dfrac{\partial \bar{p}}{\partial x_j} + \nu \nabla^2 \bar{u}_j + g_j - \dfrac{\partial}{\partial x_k}\left(\underbrace{\overline{u_j u_k} - \bar{u}_j \bar{u}_k}\right)
\end{cases}
$$

$$
\sigma_{jk}^{R} = \overline{u_j u_k} - \bar{u}_j \bar{u}_k
$$

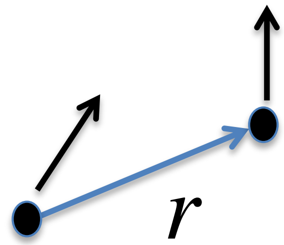
Written as velocity co-variance tensor:

$$
\sigma_{jk}^{R} = \overline{(\bar{u}_j + u'_j)(\bar{u}_k + u'_k)} - \bar{u}_j \bar{u}_k = \bar{\bar{u}}_j \bar{\bar{u}}_k + \bar{\bar{u}}_j \cancelto{0}{\overline{u'_k}} + \bar{\bar{u}}_k \cancelto{0}{\overline{u'_j}} + \overline{u'_j u'_k} - \bar{u}_j \bar{u}_k = \overline{u'_j u'_k}
$$

0

Eddies (shapes, sizes, correlations) determine R-stress:
Spatial "structure" of turbulence important – 2 point stats

Turbulence has eddies at many scales
Characterizing 2-point structure:



$$\delta u(r) = \left[\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})\right] \bullet \frac{\mathbf{r}}{r}$$

$$D_{LL}(r) = \left\langle \delta u(r)^2 \right\rangle$$

$$\Theta_{ij}(\mathbf{k}) = \left\langle \hat{u}_i(\mathbf{k}) \hat{u}_j^*(\mathbf{k}) \right\rangle$$

Isotropic turbulence

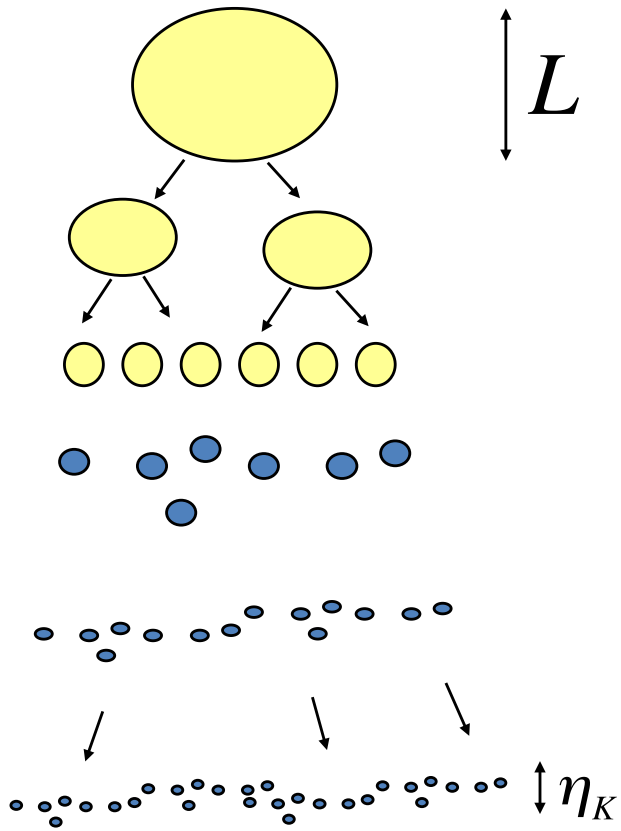
$$\Theta_{ij}(\mathbf{k}) = \frac{E(k)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)$$

$$E(k) = 4\pi k^2 \left\langle \hat{u}_i(\mathbf{k}) \hat{u}_i^*(\mathbf{k}) \right\rangle_{k=|\mathbf{k}|}$$

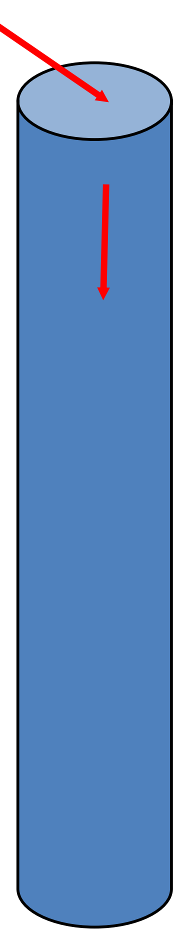
• **Turbulence Physics: the energy cascade**
(Richardson 1922, Kolmogorov 1941…)

$$u' \equiv \sqrt{\frac{1}{3}\overline{u_i' u_i'}}$$

$$\frac{u'^2}{Time} \sim \frac{u'^2}{L/u'} \sim \frac{u'^3}{L}$$



Injection of kinetic energy into turbulence (from mean flow)

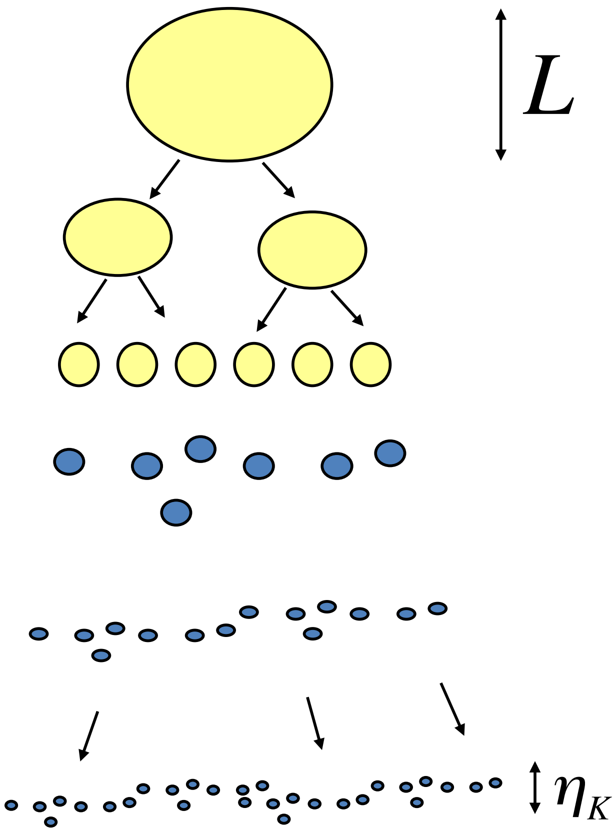


$$\varepsilon \sim \frac{u'^3}{L}$$

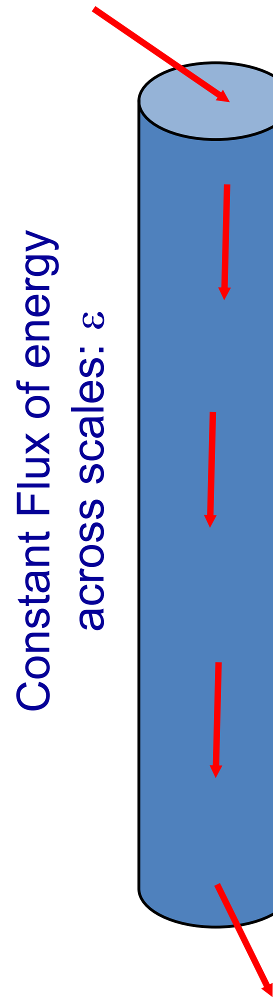
- **Turbulence Physics: the energy cascade**
  (Richardson 1922, Kolmogorov 1941…)

$$u' \equiv \sqrt{\frac{1}{3}\overline{u_i{}'u_i{}'}}$$

$$\frac{u'^2}{Time} \sim \frac{u'^2}{L/u'} \sim \frac{u'^3}{L}$$



Injection of kinetic energy into turbulence (from mean flow)



$$\varepsilon \sim \frac{u'^3}{L}$$

$$\varepsilon \sim \frac{\delta u(r_1)^3}{r_1}$$

$$\varepsilon \sim \frac{\delta u(r_2)^3}{r_2}$$

From Navier-Stokes:

$$\langle \varepsilon \rangle = -\frac{5}{4}\frac{\langle \delta u(r)^3 \rangle}{r}$$

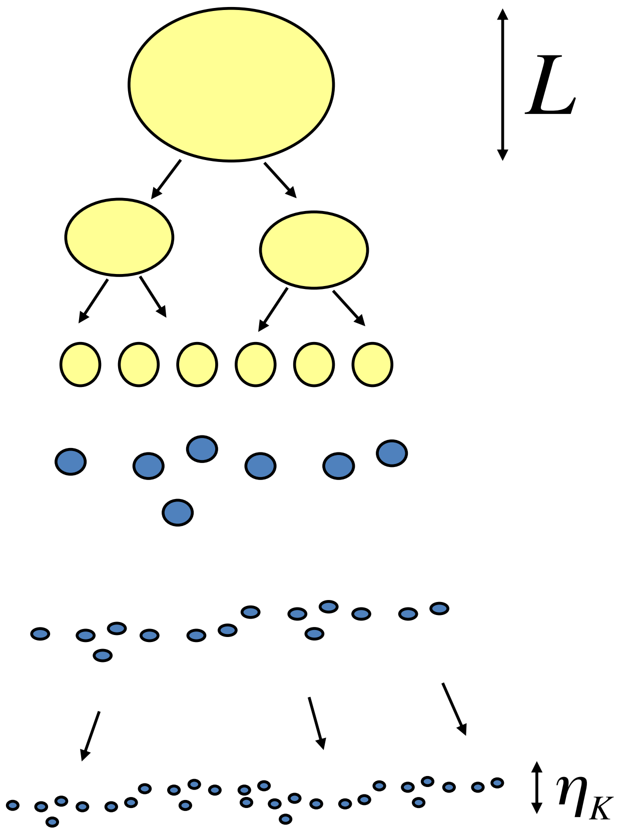
$$\varepsilon = 2\nu \overline{\frac{\partial u_i'}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}$$

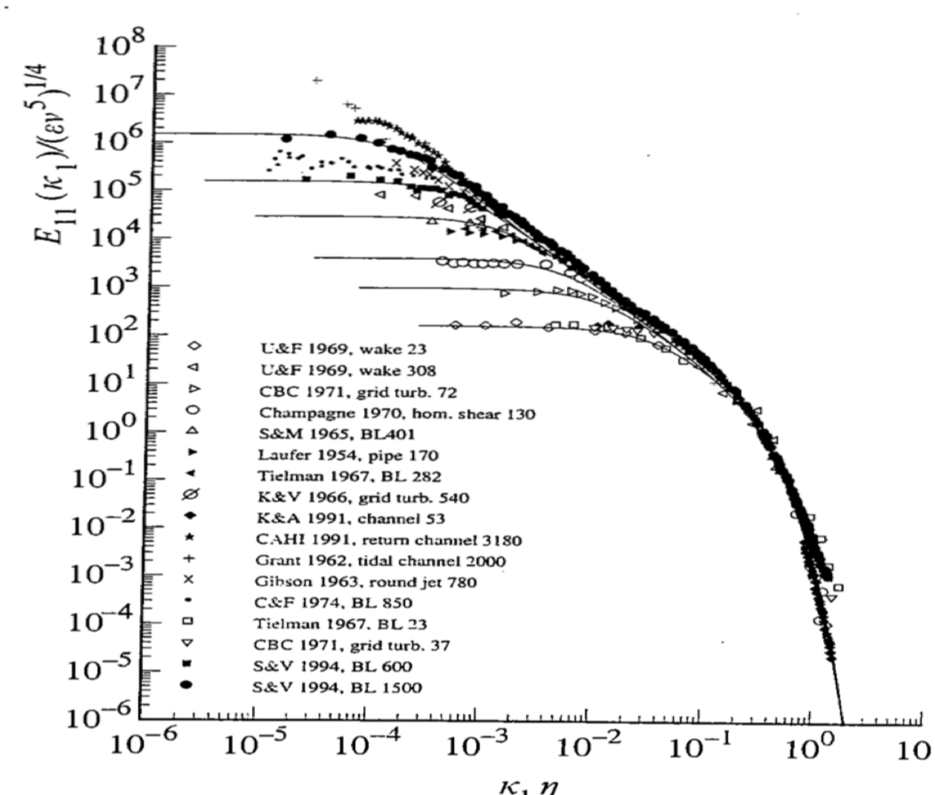
Dissipation of kinetic energy into heat (due to molecular friction)

• Turbulence Physics: the energy cascade
(Richardson 1922, Kolmogorov 1941)

$$\varepsilon \sim \frac{\delta u(r)^3}{r} \quad \rightarrow \quad \delta u(r)^2 \sim \varepsilon^{2/3} r^{2/3}$$

$$E(k) = C_K \varepsilon^{2/3} k^{-5/3} f(k\eta)$$





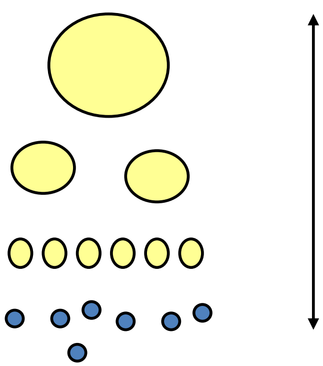
From: Saddoughi&Veeravalli, JFM

Direct Numerical Simulation:

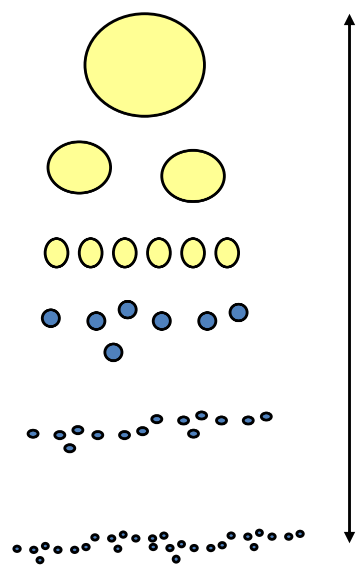
N-S equations:

$$\frac{\partial u_j}{\partial t} + \frac{\partial u_k u_j}{\partial x_k} = -\frac{\partial p}{\partial x_j} + \nu \nabla^2 u_j + g_j \qquad \frac{\partial u_j}{\partial x_j} = 0$$

Moderate Re
(~ 10^3),
DNS possible



High Re
(~ 10^7),
DNS impossible

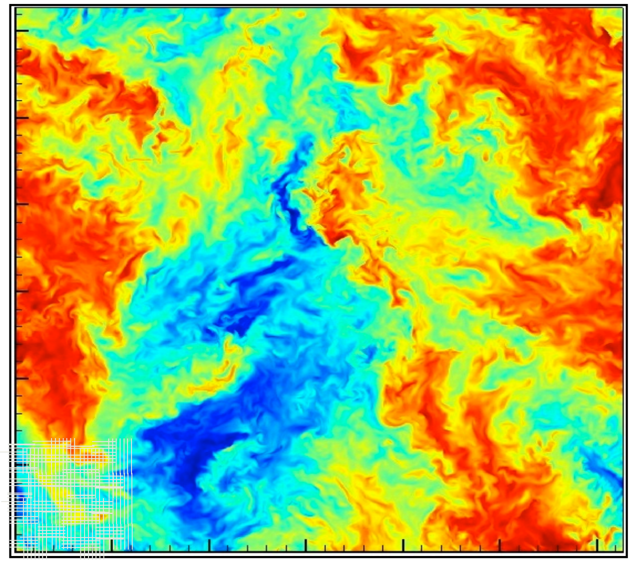


Coarse-graining - Large-Eddy-Simulation (LES):

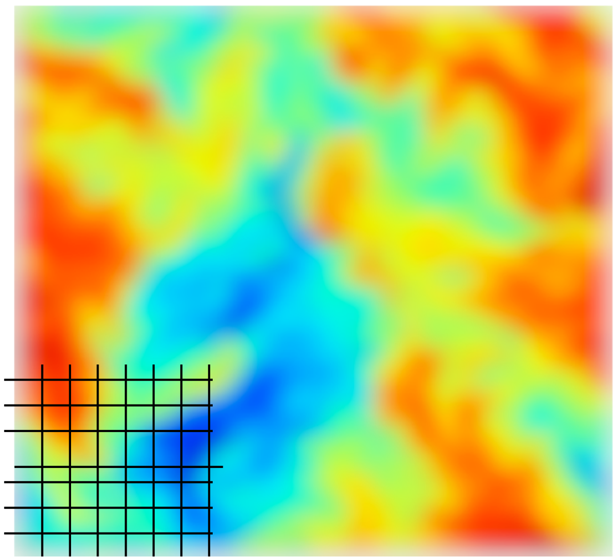
Coarse-graining for more affordable simulations

$u_1(x, y, z_0, t_0)$

4x10^9 d.o.f.



$\tilde{u}_1(x, y, z_0, t_0)$



10^5 d.o.f.

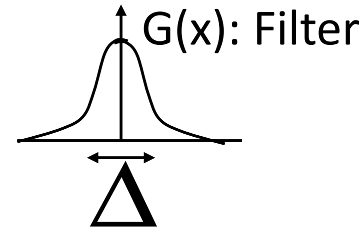
Ongoing Research Questions:

- How do small-scales affect large scale motions (and vice-versa)?
- How can we replace the effects of small scales on large scales (SGS modeling)?

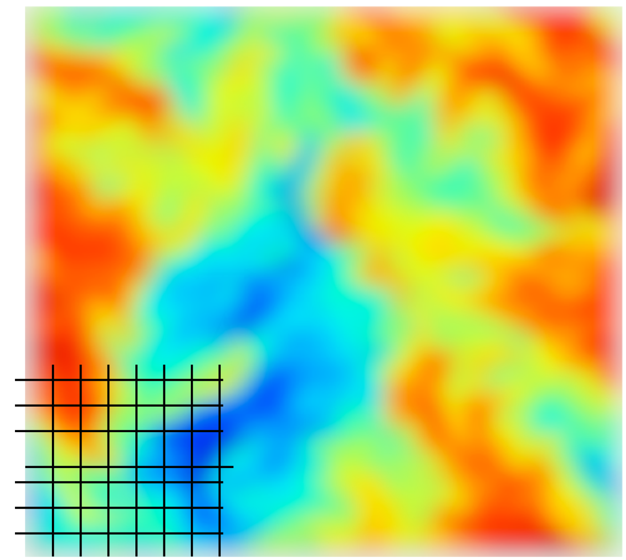
Large-eddy-simulation (LES) and filtering:

N-S equations:

$$\frac{\partial u_j}{\partial t} + \frac{\partial u_k u_j}{\partial x_k} = -\frac{\partial p}{\partial x_j} + \nu \nabla^2 u_j \qquad \frac{\partial u_j}{\partial x_j} = 0$$



$$\tilde{u}_1(x, y, z_0, t_0)$$

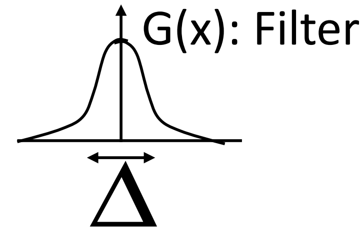


$$\tilde{u}_i(\mathbf{x},t) = G_\Delta * u_i = \int G_\Delta(\mathbf{x}-\mathbf{x}')\,u_i(\mathbf{x}')\,d^3\mathbf{x}'$$

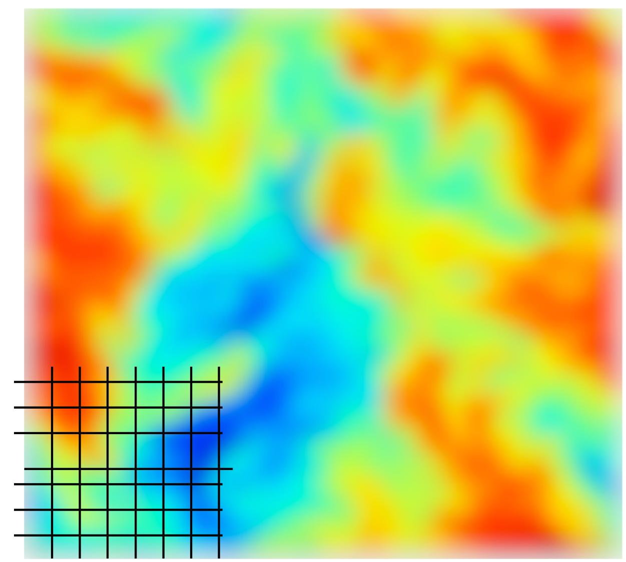
Large-eddy-simulation (LES) and filtering:

N-S equations:

$$\frac{\partial u_j}{\partial t} + \frac{\partial u_k u_j}{\partial x_k} = -\frac{\partial p}{\partial x_j} + \nu \nabla^2 u_j \qquad \frac{\partial u_j}{\partial x_j} = 0$$



$$\tilde{u}_1(x, y, z_0, t_0)$$



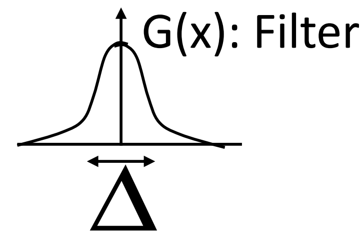
Filtered N-S equations:

$$\frac{\partial \tilde{u}_j}{\partial t} + \frac{\partial \widetilde{u_k u_j}}{\partial x_k} = -\frac{\partial \tilde{p}}{\partial x_j} + \nu \nabla^2 \tilde{u}_j$$

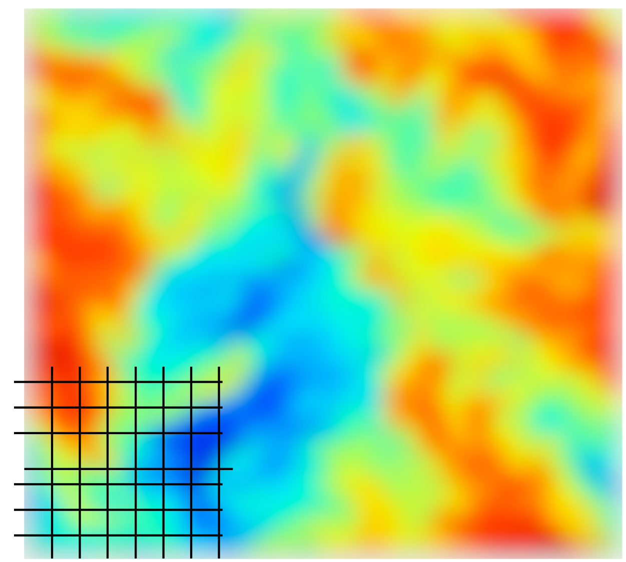
Large-eddy-simulation (LES) and filtering:

N-S equations:

$$\frac{\partial u_j}{\partial t} + \frac{\partial u_k u_j}{\partial x_k} = -\frac{\partial p}{\partial x_j} + \nu \nabla^2 u_j \qquad \frac{\partial u_j}{\partial x_j} = 0$$



$$\tilde{u}_1(x, y, z_0, t_0)$$



Filtered N-S equations:

$$\frac{\partial \tilde{u}_j}{\partial t} + \frac{\partial \widetilde{u_k u_j}}{\partial x_k} = -\frac{\partial \tilde{p}}{\partial x_j} + \nu \nabla^2 \tilde{u}_j$$

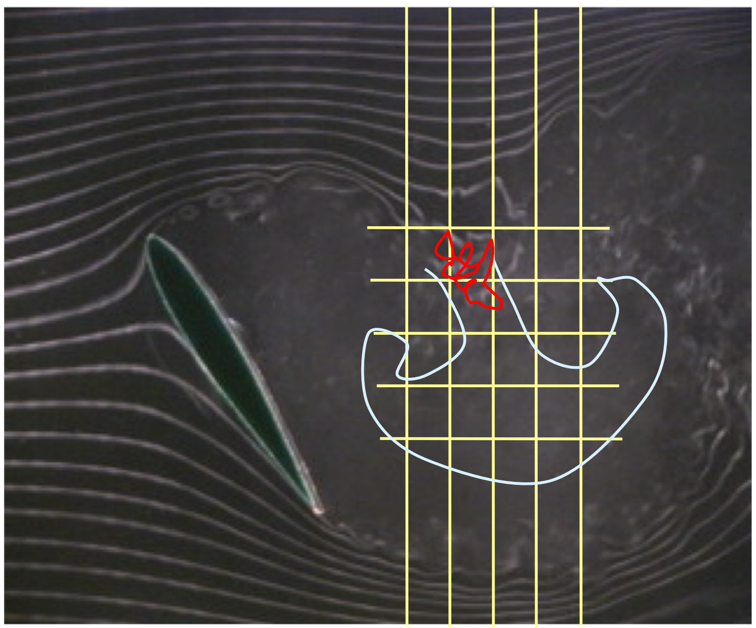
$$\frac{\partial \tilde{u}_j}{\partial t} + \tilde{u}_k \frac{\partial \tilde{u}_j}{\partial x_k} = -\frac{\partial \tilde{p}}{\partial x_j} + \nu \nabla^2 \tilde{u}_j - \frac{\partial}{\partial x_k} \tau_{jk}$$

where SGS stress tensor is:

$$\tau_{ij} = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j$$

Most common modeling approach: eddy-viscosity

$$\tau_{ij}^{d} = -\nu_{sgs}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) = -2\nu_{sgs}\tilde{S}_{ij}$$



$$\nu_{sgs} = l \times vel \sim \Delta \times (\Delta\,|\tilde{\mathbf{S}}|)$$

$$\nu_{sgs} = (c_s\Delta)^2\,|\tilde{S}|$$

c_s: "Smagorinsky coefficient"

HISTORY: 1960s, 1970s

- J Smagorinsky
- DK Lilly
- J Deardorff

Effects of τ_{ij} upon resolved motions: **Energetics (kinetic energy):**

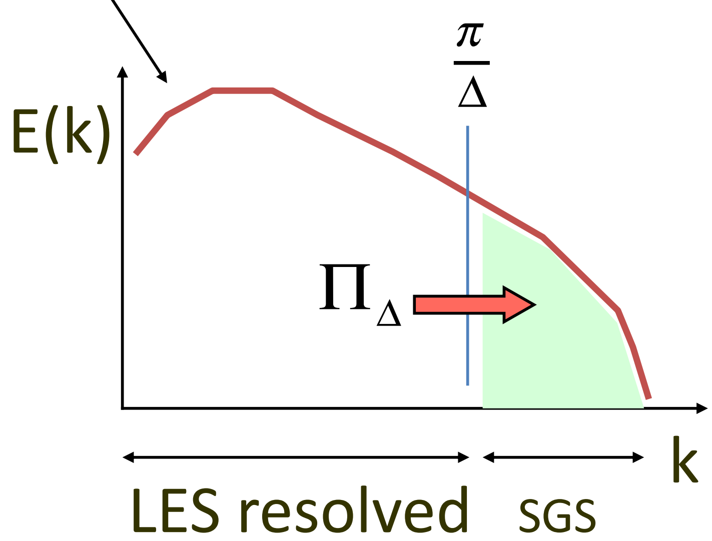
$$\frac{\partial \frac{1}{2}\tilde{u}_j\tilde{u}_j}{\partial t} + \tilde{u}_k \frac{\partial \frac{1}{2}\tilde{u}_j\tilde{u}_j}{\partial x_k} = -\frac{\partial}{\partial x_j}(....) - 2\nu \tilde{S}_{jk}\tilde{S}_{jk} - \left(-\tau_{jk}\tilde{S}_{jk}\right)$$

$$\tilde{S}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right)$$

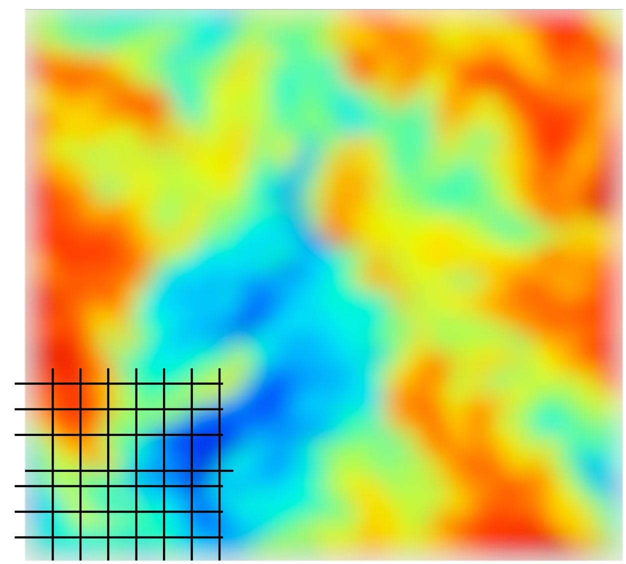
$$\Pi_\Delta = -\left\langle \tau_{jk}\tilde{S}_{jk} \right\rangle$$

Inertial-range flux

$$\varepsilon = \frac{u'^3}{L}$$



$$\tilde{u}_1(x, y, z_0, t_0)$$



Δ

Two-point structure of coarse-grained NS:

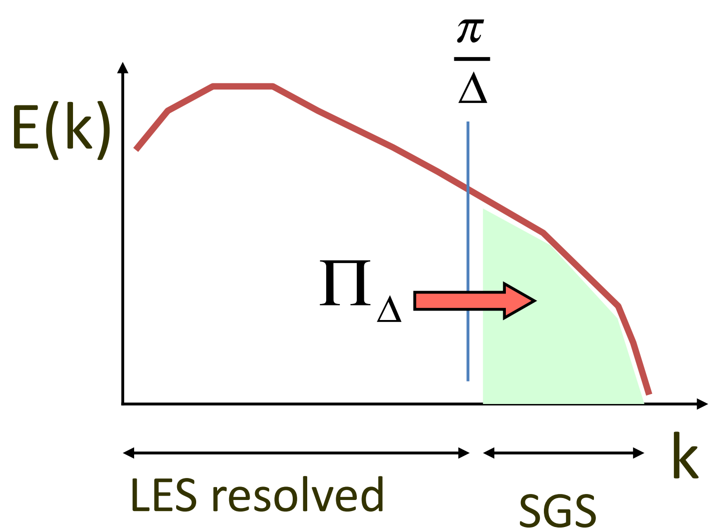
$$\frac{\partial \frac{1}{2}\tilde{u}_j\tilde{u}_j}{\partial t} + \tilde{u}_k \frac{\partial \frac{1}{2}\tilde{u}_j\tilde{u}_j}{\partial x_k} = -\frac{\partial}{\partial x_j}(....) - 2\nu\tilde{S}_{jk}\tilde{S}_{jk} - \left(-\tau_{jk}\tilde{S}_{jk}\right)$$

$$\Pi_\Delta = -\left\langle \tau_{jk}\tilde{S}_{jk} \right\rangle$$

Similarly to von Karman-Howarth and Kolmogorov equation:

For isotropic turbulence, in inertial rang:

$$\left\langle \delta\tilde{u}(r)^3 \right\rangle + 6\left\langle \tau_{LL}(x)\delta\tilde{u}(r) \right\rangle = -\frac{4}{5}\Pi_\Delta r$$



$$\tau_{ij} = -2(c_s\Delta)^2 \,|\tilde{S}|\, \tilde{S}_{ij}$$

Simplest model that can “control” the “dissipation”

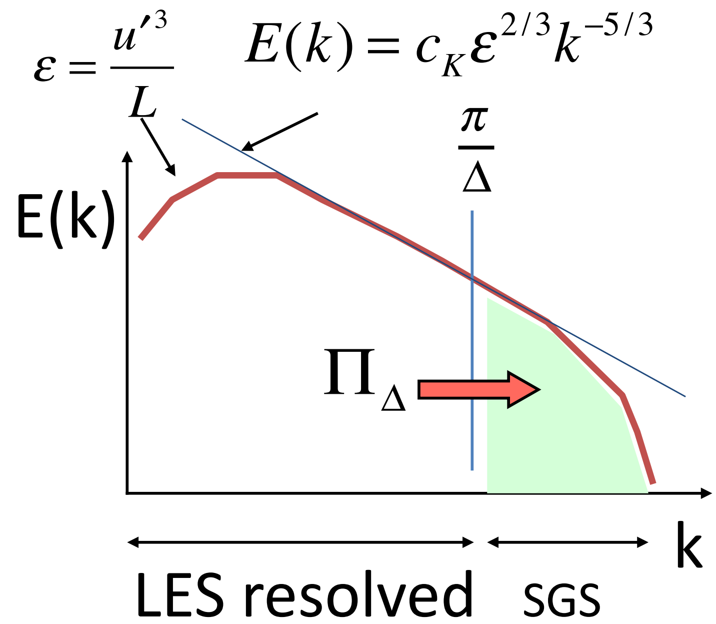
But how much is c_s?

Theoretical calibration of c_s (D.K. Lilly, 1967):

$$\Pi_\Delta = \varepsilon = -\left\langle \tau_{ij}\tilde{S}_{ij}\right\rangle \qquad \tau_{ij} = -2(c_s\Delta)^2 \,|\,\tilde{S}\,|\,\tilde{S}_{ij}$$

$$\varepsilon = c_s^2\Delta^2 2\left\langle |\,\tilde{S}\,|\,\tilde{S}_{ij}\tilde{S}_{ij}\right\rangle$$

$$\varepsilon \approx c_s^2\Delta^2 2^{3/2}\left\langle \tilde{S}_{ij}\tilde{S}_{ij}\right\rangle^{3/2}$$

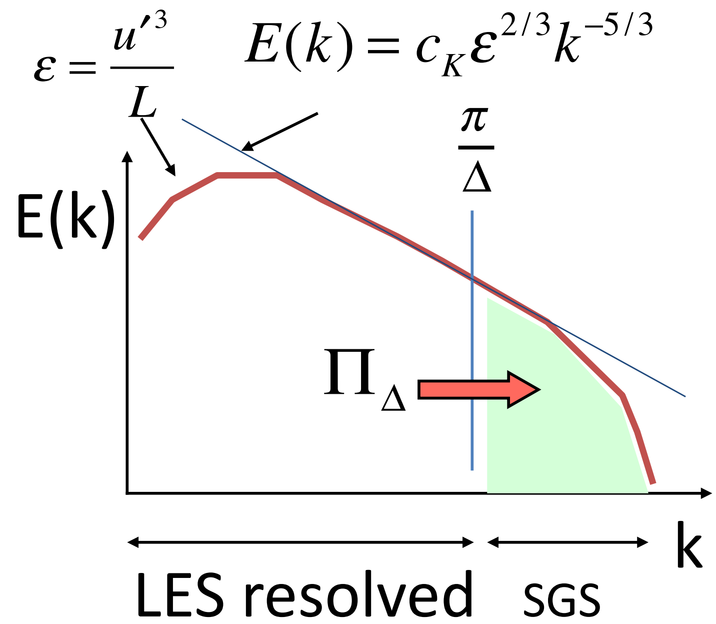


Theoretical calibration of c_s (D.K. Lilly, 1967):

$$\Pi_\Delta = \varepsilon = -\langle \tau_{ij}\tilde{S}_{ij}\rangle \qquad \tau_{ij} = -2(c_s\Delta)^2 |\tilde{S}|\tilde{S}_{ij}$$

$$\varepsilon = c_s^2\Delta^2 2\langle |\tilde{S}|\tilde{S}_{ij}\tilde{S}_{ij}\rangle$$

$$\varepsilon \approx c_s^2\Delta^2 2^{3/2}\langle \tilde{S}_{ij}\tilde{S}_{ij}\rangle^{3/2}$$



$$\langle \tilde{S}_{ij}\tilde{S}_{ij}\rangle = \frac{1}{2}\left\langle \frac{\partial \tilde{u}_i}{\partial x_j}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right)\right\rangle =$$

$$= \frac{1}{2}\iiint_{|\mathbf{k}|<\pi/\Delta}[k_j{}^2\Theta_{ii}(\mathbf{k}) + k_ik_j\Theta_{ij}(\mathbf{k})]d^3\mathbf{k} = \frac{1}{2}\iiint_{|\mathbf{k}|<\pi/\Delta}[k^2(\frac{E(k)}{4\pi k^2}(\delta_{ii} - \frac{k^2}{k^2})) + 0]d^3\mathbf{k}$$

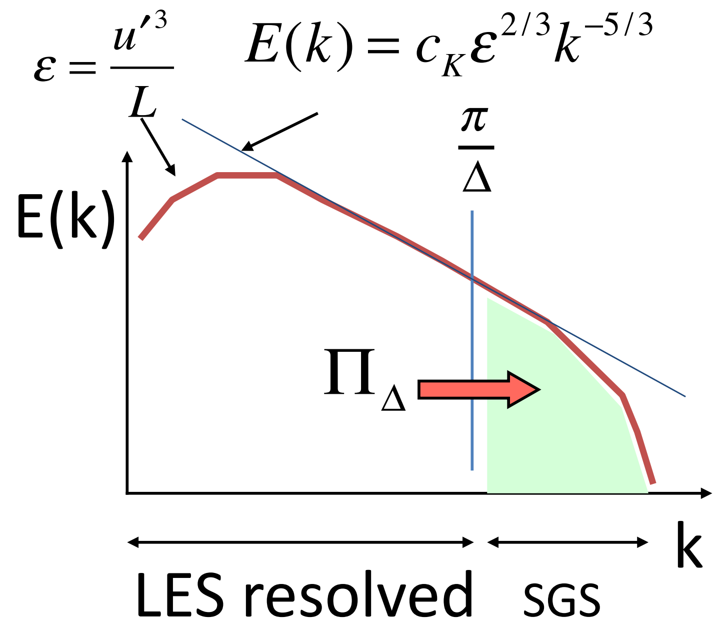
$$= c_K\varepsilon^{2/3}\frac{1}{2}\int_0^{\pi/\Delta}k^{-5/3+2}\frac{3-1}{4\pi k^2}4\pi k^2 dk = c_K\varepsilon^{2/3}\int_0^{\pi/\Delta}k^{1/3}dk = c_K\varepsilon^{2/3}\frac{3}{4}\left(\frac{\pi}{\Delta}\right)^{4/3}$$

Theoretical calibration of c_s (D.K. Lilly, 1967):

$$\Pi_\Delta = \varepsilon = -\left\langle \tau_{ij}\tilde{S}_{ij}\right\rangle \qquad \tau_{ij} = -2(c_s\Delta)^2\,|\tilde{S}|\,\tilde{S}_{ij}$$

$$\varepsilon = c_s^2\Delta^2 2\left\langle |\tilde{S}|\,\tilde{S}_{ij}\tilde{S}_{ij}\right\rangle$$

$$\varepsilon \approx c_s^2\Delta^2 2^{3/2}\left\langle \tilde{S}_{ij}\tilde{S}_{ij}\right\rangle^{3/2}$$



$$\left\langle \tilde{S}_{ij}\tilde{S}_{ij}\right\rangle = \frac{1}{2}\left\langle \frac{\partial\tilde{u}_i}{\partial x_j}\left(\frac{\partial\tilde{u}_i}{\partial x_j}+\frac{\partial\tilde{u}_j}{\partial x_i}\right)\right\rangle =$$

$$= \frac{1}{2}\iiint_{|\mathbf{k}|<\pi/\Delta}[k_j{}^2\Theta_{ii}(\mathbf{k}) + k_ik_j\Theta_{ij}(\mathbf{k})]d^3\mathbf{k} = \frac{1}{2}\iiint_{|\mathbf{k}|<\pi/\Delta}[k^2(\frac{E(k)}{4\pi k^2}(\delta_{ii}-\frac{k^2}{k^2})) + 0]d^3\mathbf{k}$$

$$= c_K\varepsilon^{2/3}\frac{1}{2}\int_0^{\pi/\Delta} k^{-5/3+2}\frac{3-1}{4\pi k^2}4\pi k^2 dk = c_K\varepsilon^{2/3}\int_0^{\pi/\Delta} k^{1/3}dk = c_K\varepsilon^{2/3}\frac{3}{4}\left(\frac{\pi}{\Delta}\right)^{4/3}$$

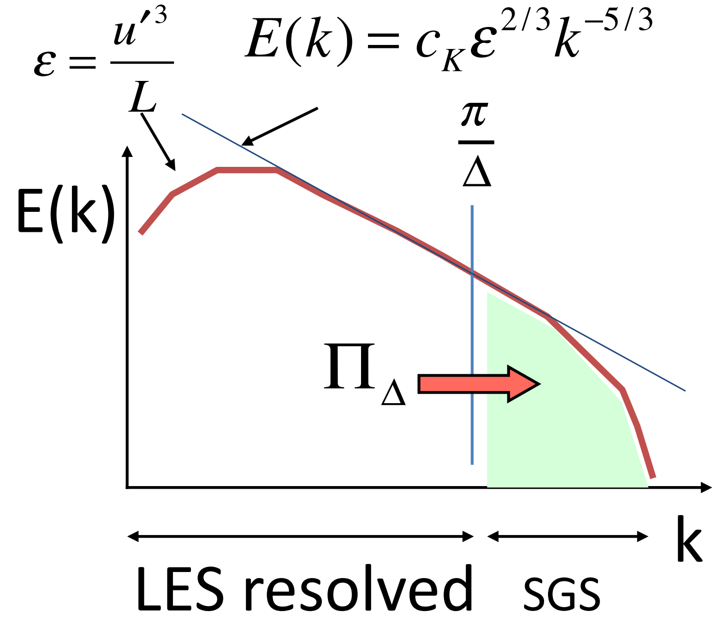
$$\varepsilon \approx c_s^2\Delta^2 2^{3/2}\left(c_K\varepsilon^{2/3}\frac{3}{4}\left(\frac{\pi}{\Delta}\right)^{4/3}\right)^{3/2}$$

Theoretical calibration of c_s (D.K. Lilly, 1967):

$$\Pi_\Delta = \varepsilon = -\left\langle \tau_{ij}\tilde{S}_{ij}\right\rangle \qquad \tau_{ij} = -2(c_s\Delta)^2 \,|\tilde{S}|\,\tilde{S}_{ij}$$

$$\varepsilon = c_s^2\Delta^2 2\left\langle |\tilde{S}|\,\tilde{S}_{ij}\tilde{S}_{ij}\right\rangle$$

$$\varepsilon \approx c_s^2\Delta^2 2^{3/2}\left\langle \tilde{S}_{ij}\tilde{S}_{ij}\right\rangle^{3/2}$$



$$\left\langle \tilde{S}_{ij}\tilde{S}_{ij}\right\rangle = \frac{1}{2}\left\langle \frac{\partial\tilde{u}_i}{\partial x_j}\left(\frac{\partial\tilde{u}_i}{\partial x_j}+\frac{\partial\tilde{u}_j}{\partial x_i}\right)\right\rangle =$$

$$= \frac{1}{2}\iiint_{|\mathbf{k}|<\pi/\Delta}[k_j{}^2\Theta_{ii}(\mathbf{k}) + k_ik_j\Theta_{ij}(\mathbf{k})]d^3\mathbf{k} = \frac{1}{2}\iiint_{|\mathbf{k}|<\pi/\Delta}[k^2(\frac{E(k)}{4\pi k^2}(\delta_{ii}-\frac{k^2}{k^2})) + 0]d^3\mathbf{k}$$

$$= c_K\varepsilon^{2/3}\frac{1}{2}\int_0^{\pi/\Delta}k^{-5/3+2}\frac{3-1}{4\pi k^2}4\pi k^2 dk = c_K\varepsilon^{2/3}\int_0^{\pi/\Delta}k^{1/3}dk = c_K\varepsilon^{2/3}\frac{3}{4}\left(\frac{\pi}{\Delta}\right)^{4/3}$$

$$\varepsilon \approx c_s^2\Delta^2 2^{3/2}\left(c_K\varepsilon^{2/3}\frac{3}{4}\left(\frac{\pi}{\Delta}\right)^{4/3}\right)^{3/2}$$

$$\Rightarrow 1 \approx c_s^2\pi^2\left(\frac{3c_K}{2}\right)^{3/2} \Rightarrow c_s = \left(\frac{3c_K}{2}\right)^{-3/4}\pi^{-1}$$

$$c_K = 1.6 \Rightarrow c_s \approx 0.16$$

some remarks on eddy-viscosity

$$\tau_{ij}^{d} = -\nu_{sgs}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) = -2\nu_{sgs}\tilde{S}_{ij}$$

Functional form in analogy to kinetic theory
of gases (Chapman-Enskog expansions, etc..
"Eddies ~ molecules" (???)

Limitations of basic eddy-viscosity:

$$\tau_{ij}^{d} = -\nu_{sgs}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) = -2\nu_{sgs}\tilde{S}_{ij}$$

Turbulence is not like a "can of sand"

Limitations of basic eddy-viscosity:

$$\tau_{ij}^{d} = -\nu_{sgs}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) = -2\nu_{sgs}\tilde{S}_{ij}$$

Turbulence is not like a “can of sand”

but more like a
“can of worms”

A “fluid-mechanical” rationale for basic eddy-viscosity:

$$\tau_{ij}^{d} \equiv \left(\widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j\right)^d \approx \left(\widetilde{u_i' u_j'}\right)^d$$

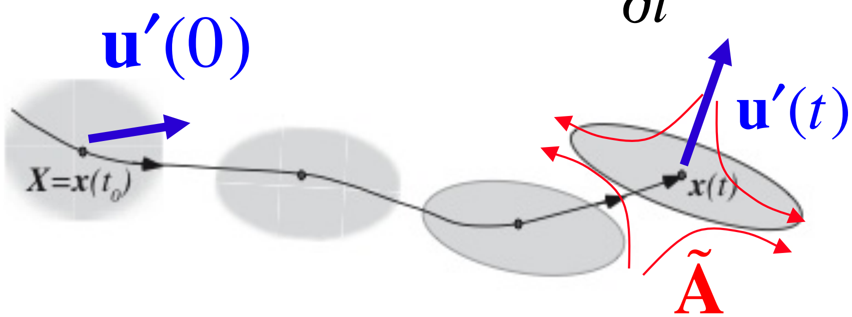
$$\left\langle \widetilde{\mathbf{u}'\mathbf{u}'^{T}} \,\middle|\, \tilde{\mathbf{A}} \right\rangle$$

A “fluid-mechanical” rationale for basic eddy-viscosity:

$$\tau_{ij}^d \equiv \left(\widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j\right)^d \approx \left(\widetilde{u_i' u'}\right)^d$$

$$\left\langle \widetilde{\mathbf{u}'\mathbf{u}'^T} \middle| \tilde{\mathbf{A}} \right\rangle$$

$$\frac{\partial(\tilde{u}_i + u_i')}{\partial t} + (\tilde{u}_k + u_k')\frac{\partial(\tilde{u}_i + u_i')}{\partial x_k} = forces$$

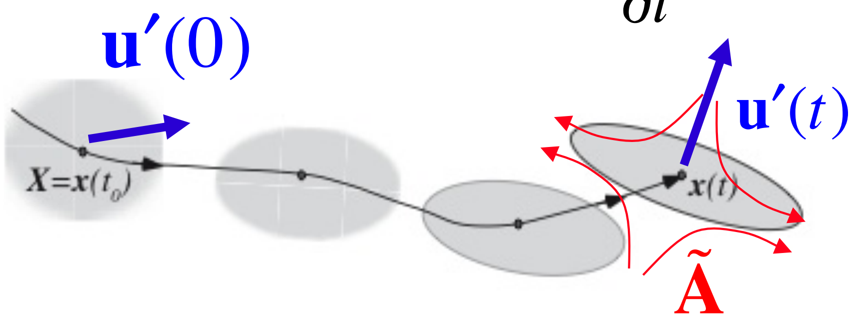


A "fluid-mechanical" rationale for basic eddy-viscosity:

$$\tau_{ij}^{d} \equiv \left(\widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j\right)^d \approx \left(\widetilde{u_i' u_i'}\right)^d$$

$$\left\langle \widetilde{\mathbf{u}'\mathbf{u}'^T} \middle| \tilde{\mathbf{A}} \right\rangle$$

$$\frac{\partial(\tilde{u}_i + u_i')}{\partial t} + (\tilde{u}_k + u_k')\frac{\partial(\tilde{u}_i + u_i')}{\partial x_k} = forces$$

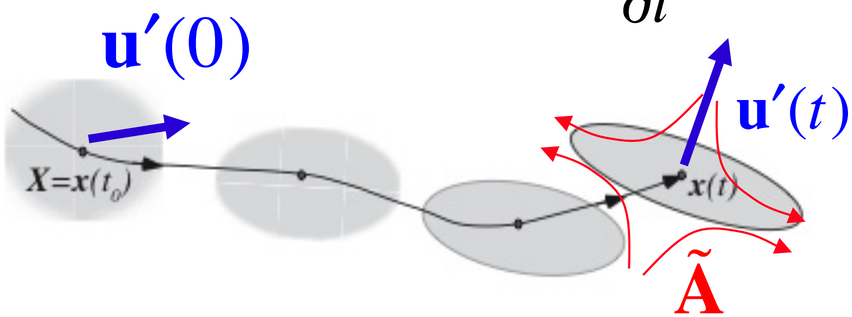


$$\frac{du_i'}{dt} = -\frac{\partial \tilde{u}_i}{\partial x_k} u_k' + \cancel{\left[forces' - \nabla(u'u') \right]}$$

$$\tau_{ij}^{d} \equiv \left(\widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j\right)^{d} \approx \left(\widetilde{u_i' u'}\right)^{d}$$

$$\left\langle \widetilde{\mathbf{u}'\mathbf{u}'^{T}} \middle| \tilde{\mathbf{A}} \right\rangle$$

$$\frac{\partial(\tilde{u}_i + u_i')}{\partial t} + (\tilde{u}_k + u_k')\frac{\partial(\tilde{u}_i + u_i')}{\partial x_k} = forces$$



$$\frac{du_i'}{dt} = -\frac{\partial \tilde{u}_i}{\partial x_k} u_k' + \left[\cancel{forces' - \nabla(u'u')}\right]$$

"Production-only" approximation:

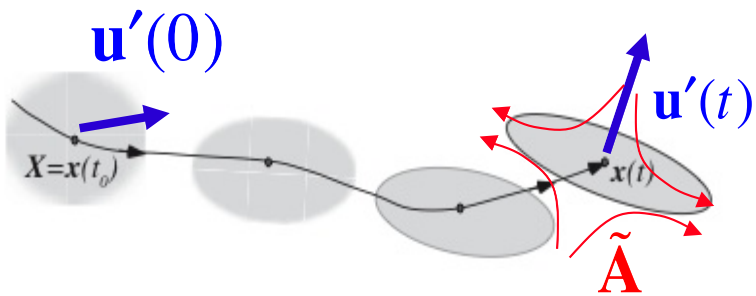
$$\frac{d\mathbf{u}'}{dt} \approx -\tilde{\mathbf{A}} \bullet \mathbf{u}',$$

Li, Chevillard, Eyink & CM
Phys Rev. E, 2009.

(stretching and tilting of vel.
fluctuation by large-scale velocity gradients
- consistent with vortex stretching)

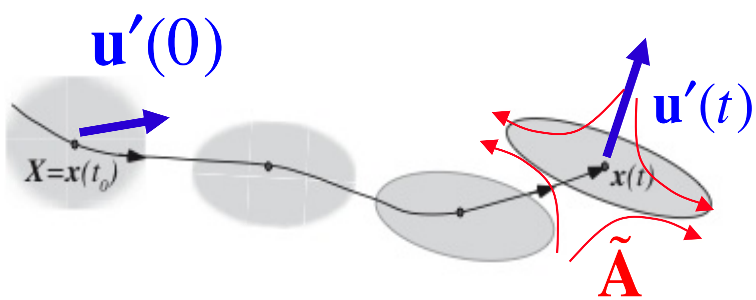
$$\tilde{A}_{ik} = \frac{\partial \tilde{u}_i}{\partial x_k}$$

$$\frac{d\mathbf{u}'}{dt} \approx -\tilde{\mathbf{A}} \bullet \mathbf{u}'$$



$$\mathbf{u}'(t) = \mathbf{exp}(-\tilde{\mathbf{A}}t) \bullet \mathbf{u}'(0)$$

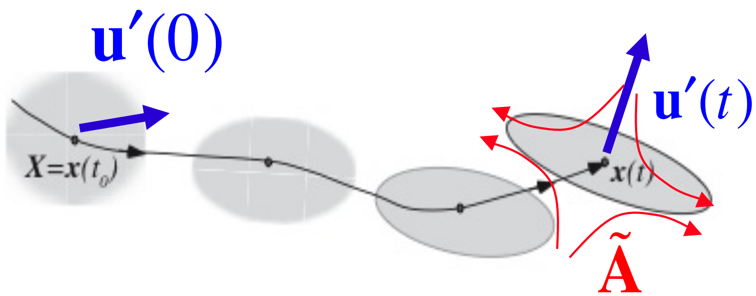
$$\frac{d\mathbf{u}'}{dt} \approx -\tilde{\mathbf{A}} \bullet \mathbf{u}'$$



$$\mathbf{u}'(t) = \mathbf{exp}(-\tilde{\mathbf{A}}t) \bullet \mathbf{u}'(0)$$

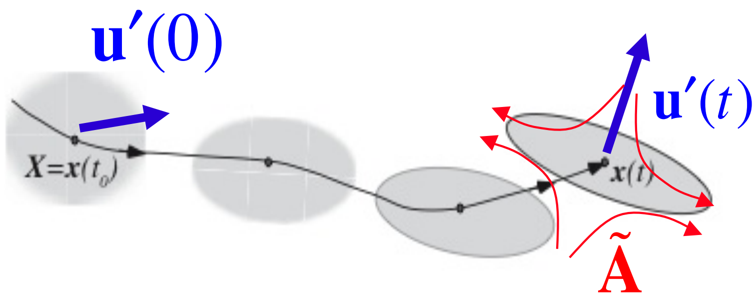
$$\frac{d\mathbf{u}'}{dt} \approx -\tilde{\mathbf{A}} \bullet \mathbf{u}'$$

$$\mathbf{u}'(t) \approx \left(\mathbf{I} - \tilde{\mathbf{A}}t + \frac{1}{2}(\tilde{\mathbf{A}}t)^2 - \ldots \right) \bullet \mathbf{u}'(0)$$



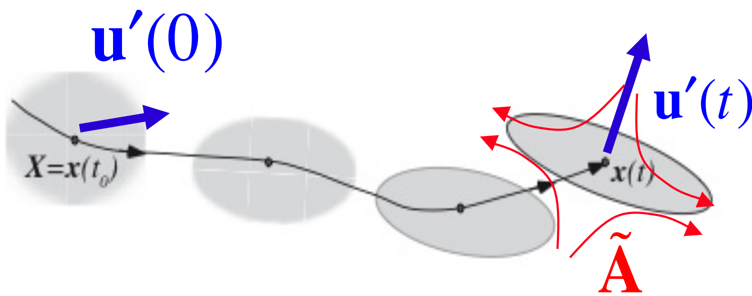
$$\frac{d\mathbf{u}'}{dt} \approx -\tilde{\mathbf{A}} \bullet \mathbf{u}' \quad \longrightarrow \quad \mathbf{u}'(t) \approx \left(\mathbf{I} - \tilde{\mathbf{A}}t + \frac{1}{2}(\tilde{\mathbf{A}}t)^2 - \ldots \right) \bullet \mathbf{u}'(0)$$

$$\mathbf{u}' \otimes \mathbf{u}' = \mathbf{u}'\mathbf{u}'^T \approx \left[\left(\mathbf{I} - \tilde{\mathbf{A}}t \right) \bullet \mathbf{u}'(0) \right] \left[\mathbf{u}'^T(0) \bullet \left(\mathbf{I} - \tilde{\mathbf{A}}^T t \right) \right]$$



$$\frac{d\mathbf{u}'}{dt} \approx -\tilde{\mathbf{A}} \bullet \mathbf{u}' \longrightarrow \mathbf{u}'(t) \approx \left(\mathbf{I} - \tilde{\mathbf{A}}t + \frac{1}{2}(\tilde{\mathbf{A}}t)^2 - \ldots \right) \bullet \mathbf{u}'(0)$$

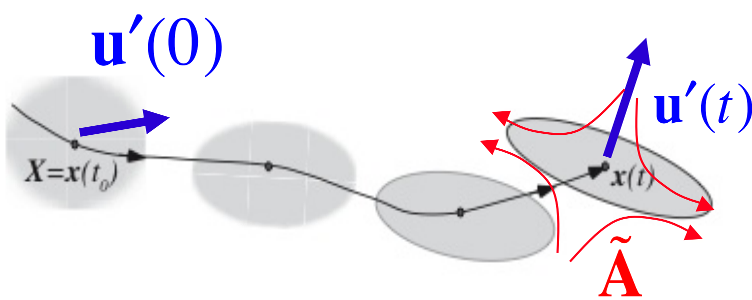
$$\mathbf{u}' \otimes \mathbf{u}' = \mathbf{u}'\mathbf{u}'^T \approx \left[\left(\mathbf{I} - \tilde{\mathbf{A}}t \right) \bullet \mathbf{u}'(0) \right] \left[\mathbf{u}'^T(0) \bullet \left(\mathbf{I} - \tilde{\mathbf{A}}^T t \right) \right]$$



$$\left\langle \mathbf{u}'\mathbf{u}'^T \middle| \tilde{\mathbf{A}} \right\rangle \approx \left(\mathbf{I} - \tilde{\mathbf{A}}t \right) \bullet \left\langle \mathbf{u}'\mathbf{u}'^T \middle| \tilde{\mathbf{A}} \right\rangle_{t=0} \bullet \left(\mathbf{I} - \tilde{\mathbf{A}}^T t \right)$$

$$\frac{d\mathbf{u}'}{dt} \approx -\tilde{\mathbf{A}}\bullet\mathbf{u}' \longrightarrow \mathbf{u}'(t) \approx \left(\mathbf{I} - \tilde{\mathbf{A}}t + \frac{1}{2}(\tilde{\mathbf{A}}t)^2 - \ldots\right)\bullet\mathbf{u}'(0)$$

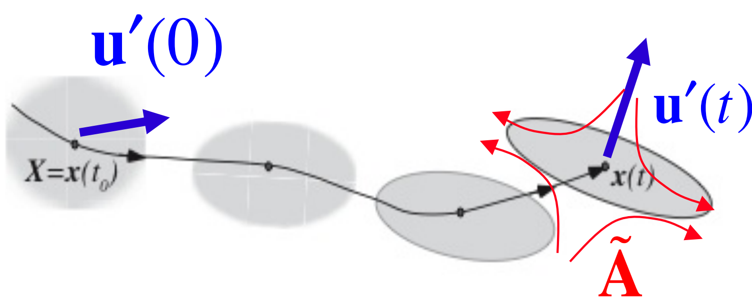
$$\mathbf{u}' \otimes \mathbf{u}' = \mathbf{u}'\mathbf{u}'^T \approx \left[\left(\mathbf{I} - \tilde{\mathbf{A}}t\right)\bullet\mathbf{u}'(0)\right]\left[\mathbf{u}'^T(0)\bullet\left(\mathbf{I} - \tilde{\mathbf{A}}^T t\right)\right]$$



$$\left\langle \mathbf{u}'\mathbf{u}'^T \middle| \tilde{\mathbf{A}} \right\rangle \approx \left(\mathbf{I} - \tilde{\mathbf{A}}t\right) \bullet \Big\langle \underbrace{\mathbf{u}'\mathbf{u}'^T \,\big|\, \tilde{\mathbf{A}}}_{isotropy:\ \left(c_e\Delta\left|\tilde{S}\right|\right)^2 \mathbf{I}} \Big\rangle_{t=0} \bullet \left(\mathbf{I} - \tilde{\mathbf{A}}^T t\right)$$

$$\frac{d\mathbf{u}'}{dt} \approx -\tilde{\mathbf{A}} \bullet \mathbf{u}' \quad \longrightarrow \quad \mathbf{u}'(t) \approx \left(\mathbf{I} - \tilde{\mathbf{A}}t + \frac{1}{2}(\tilde{\mathbf{A}}t)^2 - \ldots \right) \bullet \mathbf{u}'(0)$$

$$\mathbf{u}' \otimes \mathbf{u}' = \mathbf{u}'\mathbf{u}'^T \approx \left[\left(\mathbf{I} - \tilde{\mathbf{A}}t \right) \bullet \mathbf{u}'(0) \right] \left[\mathbf{u}'^T(0) \bullet \left(\mathbf{I} - \tilde{\mathbf{A}}^T t \right) \right]$$



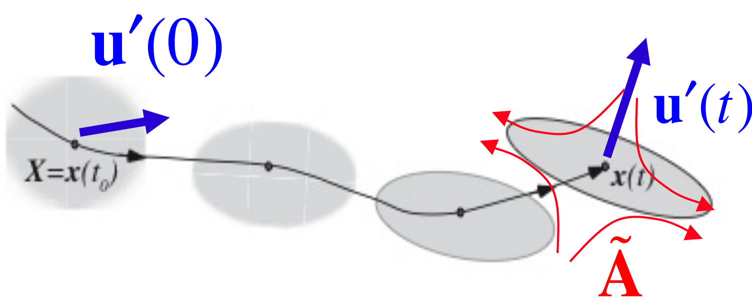
$$\left\langle \mathbf{u}'\mathbf{u}'^T \middle| \tilde{\mathbf{A}} \right\rangle \approx \left(\mathbf{I} - \tilde{\mathbf{A}}t \right) \bullet \underbrace{\left\langle \mathbf{u}'\mathbf{u}'^T \middle| \tilde{\mathbf{A}} \right\rangle_{t=0}}_{\textit{isotropy}:\ \left(c_e \Delta |\tilde{S}|\right)^2 \mathbf{I}} \bullet \left(\mathbf{I} - \tilde{\mathbf{A}}^T t \right)$$

$$\left\langle \mathbf{u}'\mathbf{u}'^T \middle| \tilde{\mathbf{A}} \right\rangle_{t_A} \approx \left(c_e \Delta |\tilde{S}|\right)^2 \left[(\mathbf{I} - \tilde{\mathbf{A}}t_A) \bullet (\mathbf{I} - \tilde{\mathbf{A}}^T t_A) \right] \approx \left(c_e \Delta |\tilde{S}|\right)^2 \left[\mathbf{I} - \underbrace{(\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T)}_{2\,\tilde{\mathbf{S}}}\, t_A + O(t^2) \right]$$

A "fluid-mechanical" rationale for basic eddy-viscosity:

$$\frac{d\mathbf{u}'}{dt} \approx -\tilde{\mathbf{A}}\bullet\mathbf{u}' \quad \longrightarrow \quad \mathbf{u}'(t) \approx \left(\mathbf{I} - \tilde{\mathbf{A}}t + \frac{1}{2}(\tilde{\mathbf{A}}t)^2 - \ldots\right)\bullet\mathbf{u}'(0)$$

$$\mathbf{u}'\otimes\mathbf{u}' = \mathbf{u}'\mathbf{u}'^T \approx \left[\left(\mathbf{I} - \tilde{\mathbf{A}}t\right)\bullet\mathbf{u}'(0)\right]\left[\mathbf{u}'^T(0)\bullet\left(\mathbf{I} - \tilde{\mathbf{A}}^T t\right)\right]$$



$$\left\langle \mathbf{u}'\mathbf{u}'^T \middle| \tilde{\mathbf{A}} \right\rangle \approx \left(\mathbf{I} - \tilde{\mathbf{A}}t\right) \bullet \underbrace{\left\langle \mathbf{u}'\mathbf{u}'^T \middle| \tilde{\mathbf{A}} \right\rangle_{t=0}}_{} \bullet \left(\mathbf{I} - \tilde{\mathbf{A}}^T t\right)$$

$$isotropy: \quad \left(c_e\Delta\left|\tilde{S}\right|\right)^2 \mathbf{I}$$

$$\left\langle \mathbf{u}'\mathbf{u}'^T \middle| \tilde{\mathbf{A}} \right\rangle_{t_A} \approx \left(c_e\Delta\left|\tilde{S}\right|\right)^2 \left[(\mathbf{I} - \tilde{\mathbf{A}}t_A)\bullet(\mathbf{I} - \tilde{\mathbf{A}}^T t_A)\right] \approx \left(c_e\Delta\left|\tilde{S}\right|\right)^2 \left[\mathbf{I} - \underbrace{(\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T)}_{2\,\tilde{\mathbf{S}}}\, t_A + O(t^2)\right]$$

$$\tau_{ij}^d = \left\langle \mathbf{u}'\mathbf{u}'^T \middle| \tilde{\mathbf{A}} \right\rangle_{t_A}^d \approx -2\underbrace{\left(c_e\Delta\left|\tilde{S}\right|\right)^2 t_A}_{\nu_{sgs}}\, \tilde{\mathbf{S}}$$

A "fluid-mechanical" rationale for basic eddy-viscosity:

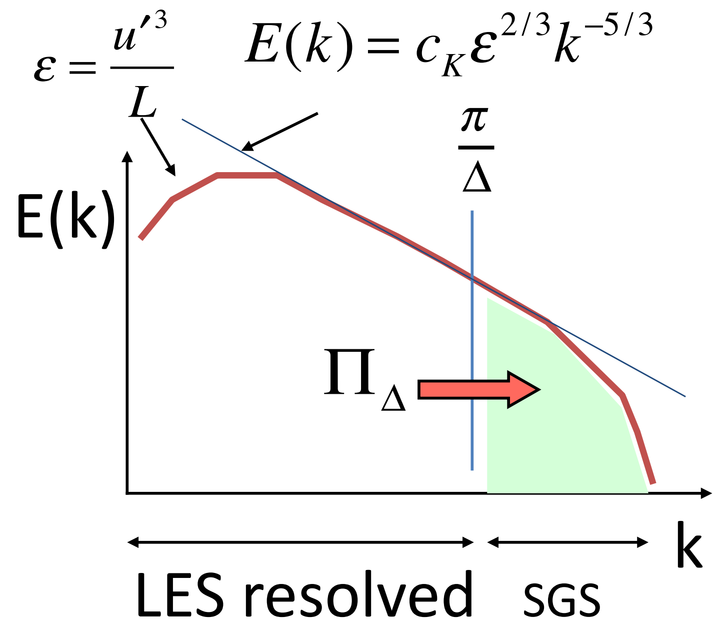
$$\tau_{ij}^{d} = \left\langle \mathbf{u}'\mathbf{u}'^{T} \right\rangle_{t_A}^{d} \approx -2\underbrace{\left(c_e \Delta \left|\tilde{S}\right|\right)^2 t_A}_{\nu_{sgs}} \tilde{\mathbf{S}}$$

choosing $\quad t_A \propto \dfrac{1}{\left|\tilde{S}\right|}$

$$\nu_{sgs} = \left(c_s \Delta\right)^2 \, |\tilde{S}|$$

c_s: "Smagorinsky coefficient"

Theoretical calibration of c_s (D.K. Lilly, 1967):



$$\tau_{ij} = -2(c_s\Delta)^2 |\tilde{S}| \tilde{S}_{ij}$$

$$\Pi_\Delta = \varepsilon = -\left\langle \tau_{ij}\tilde{S}_{ij} \right\rangle$$

$$\varepsilon = c_s^2 \Delta^2 2 \left\langle |\tilde{S}| \tilde{S}_{ij}\tilde{S}_{ij} \right\rangle$$

$$\varepsilon \approx c_s^2 \Delta^2 2^{3/2} \left\langle \tilde{S}_{ij}\tilde{S}_{ij} \right\rangle^{3/2}$$

$$\varepsilon \approx c_s^2 \Delta^2 2^{3/2} \left(c_K \varepsilon^{2/3} \frac{3}{4} \left(\frac{\pi}{\Delta}\right)^{4/3} \right)^{3/2}$$

$$\Rightarrow 1 \approx c_s^2 \pi^2 \left(\frac{3c_K}{2}\right)^{3/2} \quad \Rightarrow c_s = \left(\frac{3c_K}{2}\right)^{-3/4} \pi^{-1}$$

$$c_K = 1.6 \Rightarrow c_s \approx 0.16$$

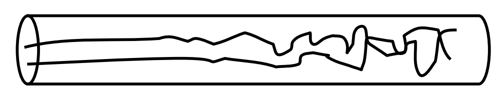
c_s=0.16 works well for isotropic, high Reynolds number turbulence

But in practice (complex flows)

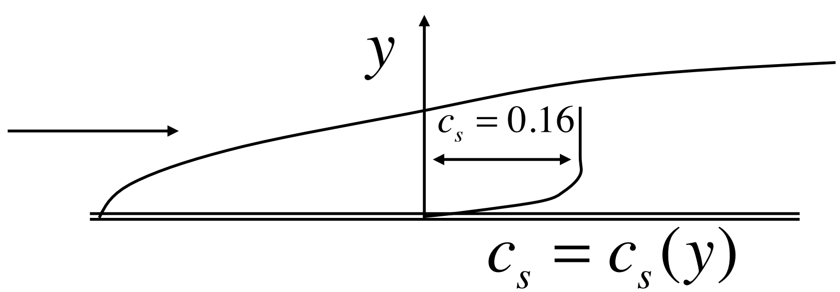
$$c_s = c_s(\mathbf{x}, t)$$

Ad-hoc tuning?

Examples: Transitional pipe flow: from 0 to 0.16



Near wall damping for wall boundary layers (Piomelli et al 1989)



How does c_s vary under realistic conditions? Interrogate data:

Measure: $$\Pi_\Delta = -\langle \tau_{jk} \tilde{S}_{jk} \rangle$$

Measure: $$\frac{\Pi_\Delta^{Smag}}{c_s^2} = 2\Delta^2 \langle |\tilde{S}| \tilde{S}_{ij} \tilde{S}_{ij} \rangle$$

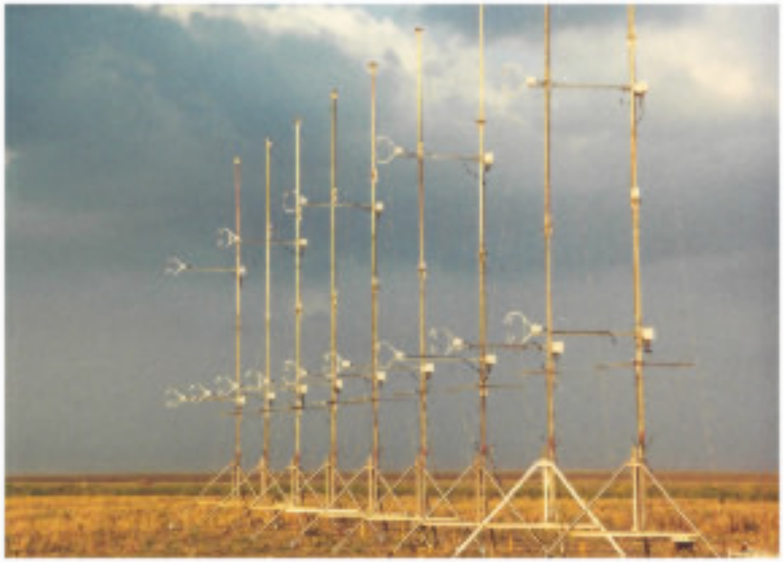
Obtain"empirical" Smagorinsky coefficient = *f(x,conditions…):*

$$c_s = \left(\frac{-\langle \tau_{jk} \tilde{S}_{jk} \rangle}{2\Delta^2 \langle |\tilde{S}| \tilde{S}_{ij} \tilde{S}_{ij} \rangle} \right)^{1/2}$$

An example result from atmospheric turbulence…:

Measure "empirical" Smagorinsky coefficient for atmospheric surface layer as function of height and stability (thermal forcing or damping):

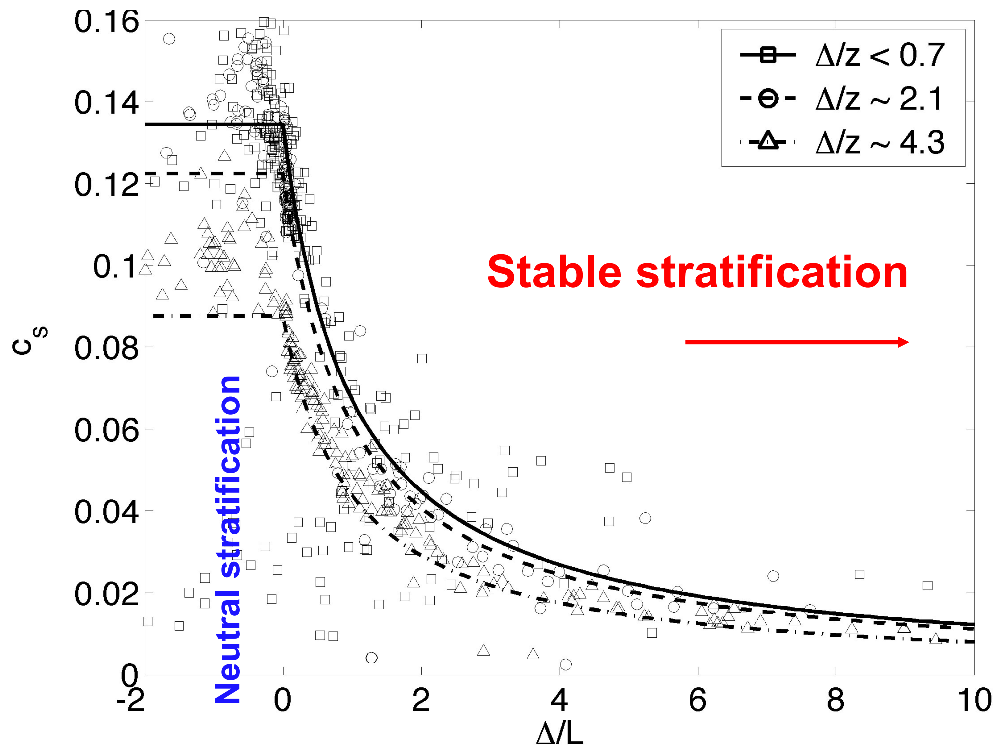
HATS - 2000
(with NCAR researchers: Horst, Sullivan)
Kettleman City (Central Valley, CA)



$$c_s = \left(\frac{-\langle \tau_{jk} \tilde{S}_{jk} \rangle}{2\Delta^2 \langle |\tilde{S}| \tilde{S}_{ij} \tilde{S}_{ij} \rangle} \right)^{1/2}$$

Example result: **effect of atmospheric stability on coefficient from sonic anemometer measurements in atmospheric surface layer (Kleissl et al., J. Atmos. Sci. 2003)**

$$c_s = c_s(\mathbf{x}, t)$$



How to avoid "tuning" and case-by-case adjustments of model coefficient in LES?

The Dynamic Model

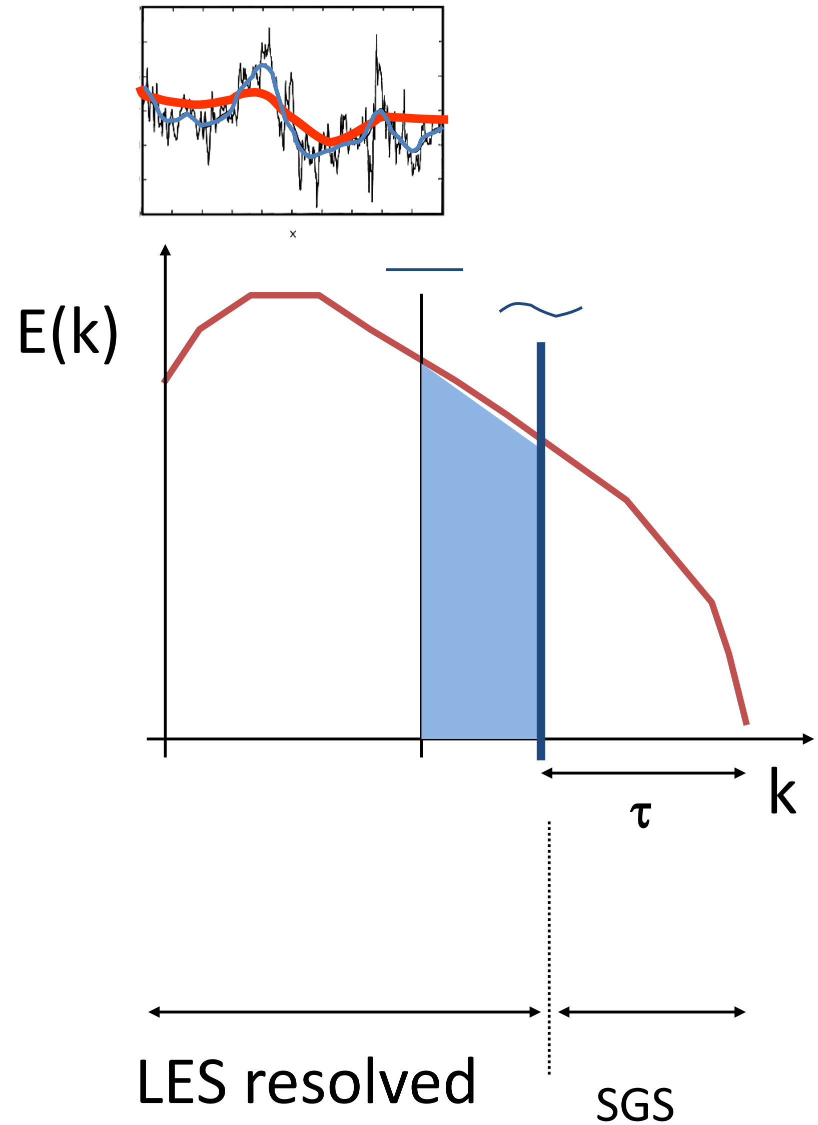
(Germano et al. Physics of Fluids, 1991)

Germano identity and dynamic model

(Germano et al. 1991):

Exact ("rare" in turbulence):

$$\overline{\widetilde{u_i u_j}} - \bar{\tilde{u}}_i \bar{\tilde{u}}_j = \overline{\widetilde{u_i u_j}} \qquad\qquad - \bar{\tilde{u}}_i \bar{\tilde{u}}_j$$

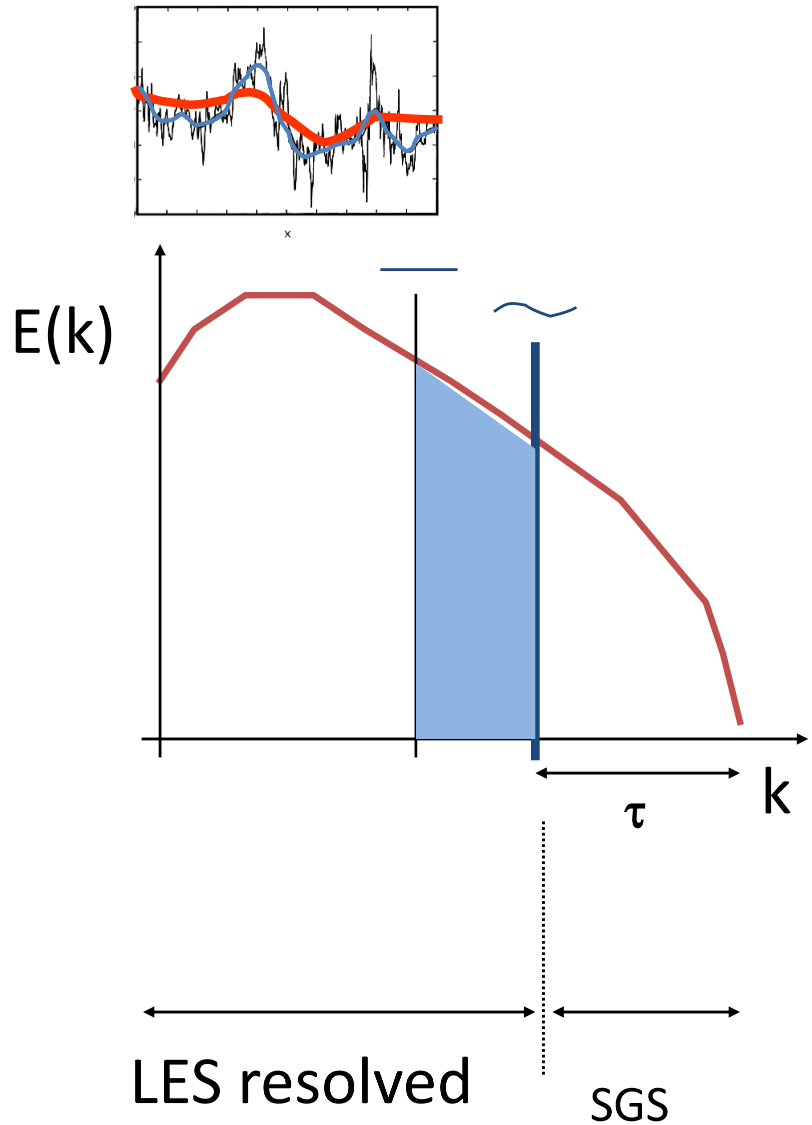


Germano identity and dynamic model

(Germano et al. 1991):

Exact ("rare" in turbulence):

$$\overline{\widetilde{u_i u_j}} - \bar{\tilde{u}}_i \bar{\tilde{u}}_j = \overline{\widetilde{u_i u_j}} \quad - \overline{\tilde{u}_i \tilde{u}_j} + \overline{\tilde{u}_i \tilde{u}_j} \quad - \bar{\tilde{u}}_i \bar{\tilde{u}}_j$$



Germano identity and dynamic model

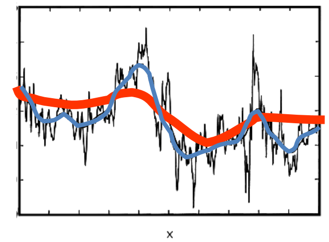
(Germano et al. 1991):

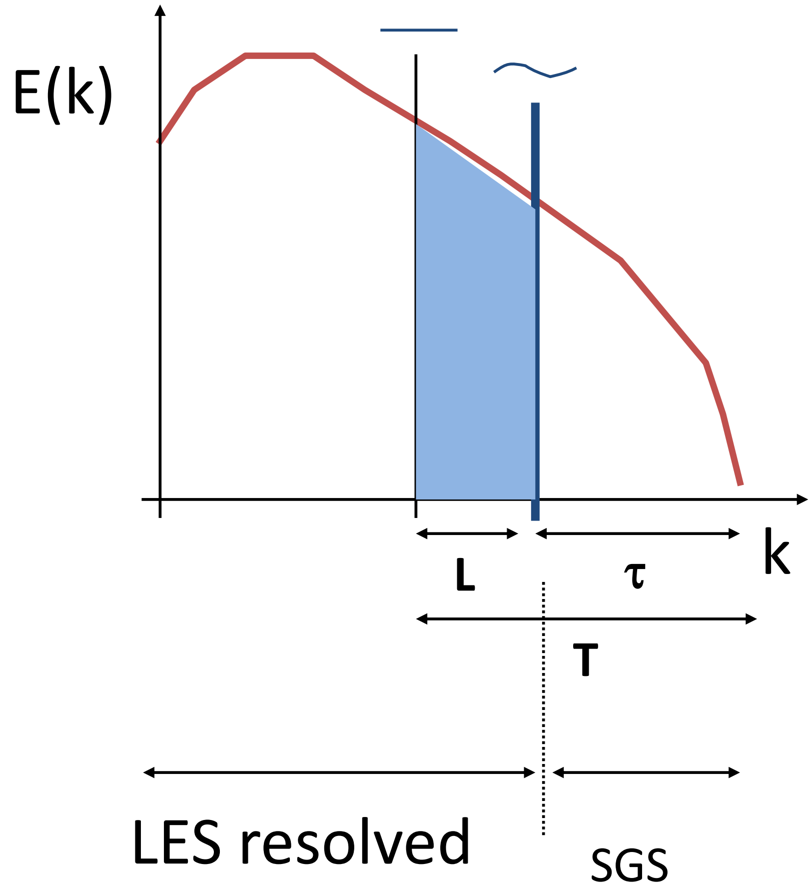
Exact ("rare" in turbulence):

$$\underbrace{\overline{\widetilde{u_i u_j}} - \bar{\tilde{u}}_i \bar{\tilde{u}}_j}_{T_{ij}} = \underbrace{\overline{\widetilde{u_i u_j}} - \overline{\tilde{u}_i \tilde{u}_j}}_{\bar{\tau}_{ij}} + \underbrace{\overline{\tilde{u}_i \tilde{u}_j} - \bar{\tilde{u}}_i \bar{\tilde{u}}_j}_{L_{ij}}$$

$$T_{ij} = \bar{\tau}_{ij} + L_{ij}$$

$$L_{ij} - (T_{ij} - \bar{\tau}_{ij}) = 0$$





Germano identity and dynamic model
(Germano et al. 1991):

Exact ("rare" in turbulence):

$$\underbrace{\overline{\widetilde{u_i u_j}} - \bar{\tilde{u}}_i \bar{\tilde{u}}_j}_{T_{ij}} = \underbrace{\overline{\widetilde{u_i u_j}} - \overline{\tilde{u}_i \tilde{u}_j}}_{\bar{\tau}_{ij}} + \underbrace{\overline{\tilde{u}_i \tilde{u}_j} - \bar{\tilde{u}}_i \bar{\tilde{u}}_j}_{L_{ij}}$$

$$T_{ij} = \bar{\tau}_{ij} + L_{ij}$$

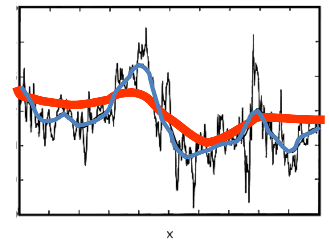
$$L_{ij} - (T_{ij} - \bar{\tau}_{ij}) = 0$$

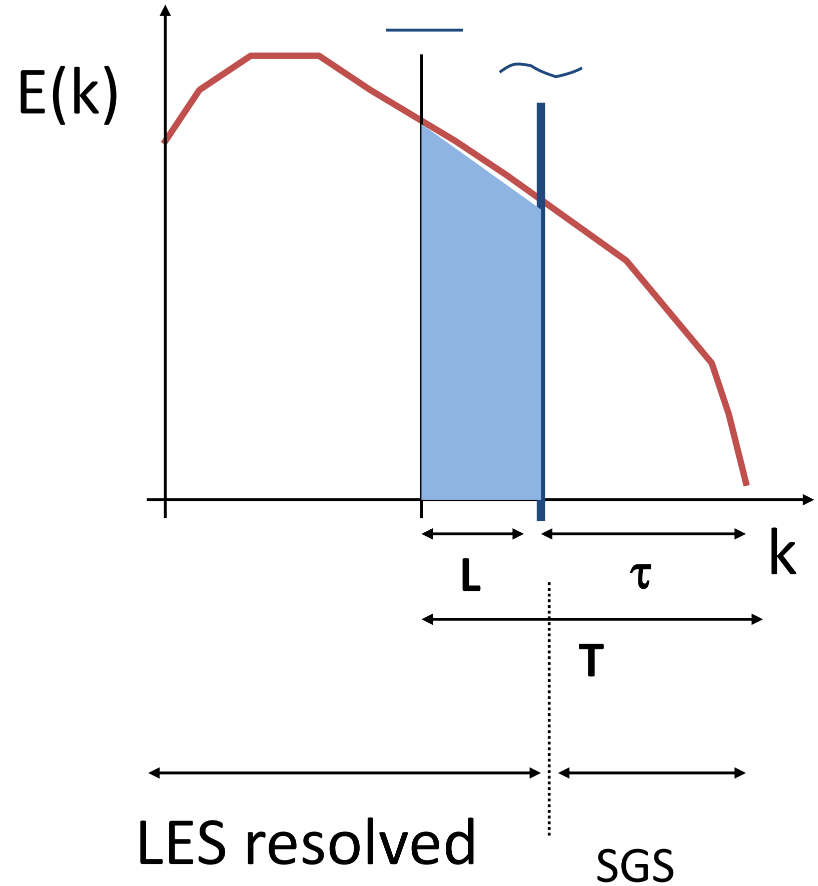
$$-2(c_s 2\Delta)^2 \,|\bar{\tilde{S}}|\,\bar{\tilde{S}}_{ij} \qquad -2(c_s\Delta)^2\, \overline{|\tilde{S}|\,\tilde{S}_{ij}}$$

Assumes scale-invariance:

$$\boxed{L_{ij} - c_s^2 M_{ij} = 0}$$

where $M_{ij} = 2\Delta^2\left(\overline{|\tilde{S}|\,\tilde{S}_{ij}} - 4\,|\bar{\tilde{S}}|\,\bar{\tilde{S}}_{ij}\right)$





Germano identity and dynamic model

(Germano et al. 1991):

$$L_{ij} - c_s^2 M_{ij} = 0$$

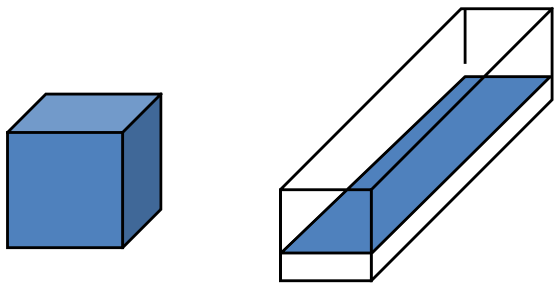
Over-determined system:
solve in "some average sense"
(minimize error, Lilly 1992):

$$\mathrm{E} = \left\langle \left(L_{ij} - c_s^2 M_{ij} \right)^2 \right\rangle$$

Minimized when:

$$c_s^2 = \frac{\left\langle L_{ij} M_{ij} \right\rangle}{\left\langle M_{ij} M_{ij} \right\rangle}$$

Averaging over regions of
statistical homogeneity
or fluid trajectories



Dynamic subgrid model: scale dependence + Lagrangian averaging

$$\tau^{sgs} = \widetilde{uu} - \tilde{u}\tilde{u} = -2\nu_T \tilde{S}$$

$$\nu_T = C_s^2 \Delta^2 |\tilde{S}|$$

Dynamic model:
Germano et al. PoF 1991

Major Features of the LASD Subgrid-Scale Model

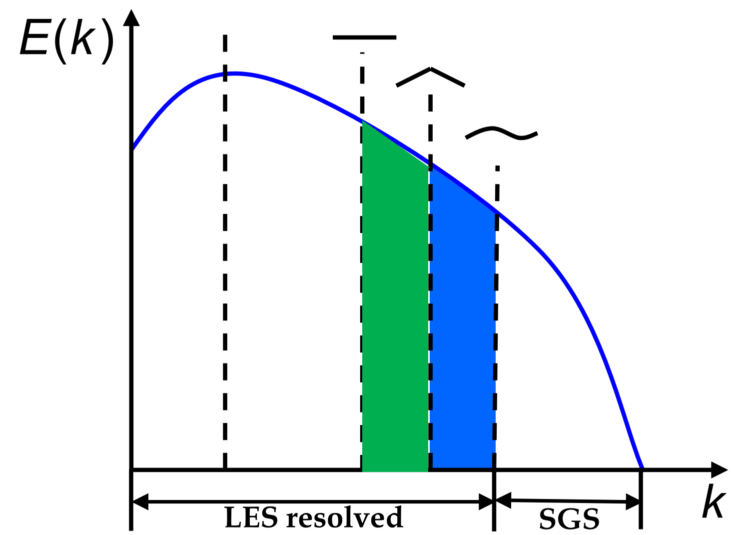
➢ **Scale-dependence:**

Porte-Agel, Meneveau & Parlange (JFM, 2000)

Assume that *Smagorinsky coefficient* is scale-dependent.

$$c^2_{S,\hat{\Delta}} = \gamma_c \; c^2_{S,\Delta}$$

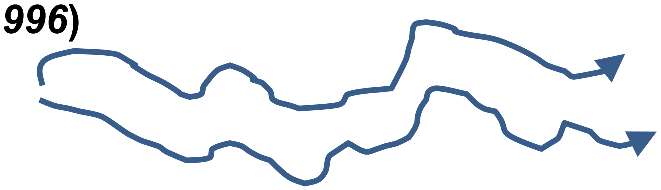
$$\gamma_c = c^2_{S,\hat{\Delta}} / c^2_{S,\Delta} = c^2_{S,\bar{\Delta}} / c^2_{S,\hat{\Delta}}$$



➢ **Lagrangian averaging:**

Averaging backward in time along particle trajectory.

Meneveau, Lund & Cabot (JFM, 1996)



$$\mathcal{L}_f = \int_{-\infty}^{t} f\left(z(t'),t'\right)\frac{1}{T}\exp\left(-\frac{t-t'}{T}\right)dt'$$

Example appliction of LES:

Using LASD SGS model