SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction

So far we have seen how to apply a variety of techniques to ordinary differential equations, some linear and some nonlinear. However in all cases so far the quest was to find a single unknown function. Applications are rarely this kind to us. Usually there are several unknown functions that must be determined simultaneously. We all have been told at some time or the other that masses move according to Newton's Law

$$F = ma$$

i.e. the force acting on a body is mass times acceleration. So if the position of a body m_1 is given by $x_1(t)$, then the acceleration of m_1, which is given by $x_1''(t)$, is equal to the total force F divided by m_1 or

$$x_1''(t) = \frac{F}{m}.$$

But what if the force acting on m_1 was actually another mass m_2, say the two were colliding into one another? Then there are two unknowns we'd like to know: the position of each mass as a function of time. That is *two* unknown functions. We could write Newton's Law for each mass, but that would be a differential equation for each mass or in other words *two* differential equations.

If you thought that situation was hairy, what would happen if we had three masses colliding with each other? How about four or even more? Hopefully I've convinced you that it is not unimaginable that we may need to solve for several ordinary differential equations simulataneously for several unknown functions. In these notes I'll stick to solving for only

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two unknown functions, but the methods do generalize and I recommend you try and work out how. Don't worry I'll give you plenty of hints.

2. Linear systems of differential equations

It's always a good idea to start simple in mathematics. Consider the following system of equations

$$\frac{dx}{dt} = 2x, \tag{1}$$

$$\frac{dy}{dt} = y. \tag{2}$$

By system I mean a set of relations between our unknowns. The above system is an example of a 2×2 system: two equations with two unknown functions x and y. Your goal is to find these two functions. We sometimes refer to the first equation as the equation for x and the second equation as the equation for y.

Remark 2.1. You could say that up to now you have been considering 1×1 systems since there was one equation for the one unknown function.

The nice thing about the above system is that the equation for x does not depend on y and vice-versa. This is nice because we could consider the two equations independently, in fact we could solve the first equation to obtain $x(t) = \alpha e^{2t}$ without even considering the second equation. Of course we could also solve the second equation to obtain $y(t) = \beta e^{t}$ without worrying about the first equation. Such a system of equations are called *Uncoupled*.

Definition 2.1. *A system of equations is* **uncoupled** *if each equation can be solved independently of the others. In other words, y does not appear in the equation for x and x does not appear in the equation for y.*

Uncoupled systems are the easiest kinds of systems to solve. The above example was also a linear differential equation. This is because the unknowns x and y appear linearly on the right-hand side of the equations which is to say there are no powers of x or y (even a term that looked like xy would make the equation nonlinear). For now we stick to solving linear equations. We expect they will be easier than nonlinear ones. Later on we will see how we can use the information we gather here to attack nonlinear equations.

As nice as uncoupled equations are, not all linear systems of differential equations are uncoupled. Here is an example of a *coupled* system

$$\frac{dx}{dt} = 5x + 4y, \tag{3}$$

$$\frac{dy}{dt} = -3x - 2y. \tag{4}$$

The way we solve coupled equations is to translate them into uncoupled equations. I'll show you two different ways. The first method is based on a substitution and the second one is based on an educated guess. Lets do the substitution method first.

Adding (3) and (4) we get

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + 2y,$$

or in other words

$$\frac{d}{dt}(x+y) = 2(x+y).$$

This suggests we make a substitution $u = x + y$ resulting in

$$\frac{du}{dt} = 2u.$$

This is a much easier equation to solve. In fact it's the same one we saw earlier in our example for an uncoupled equation. The equation for u can be solved to obtain

$$u = \alpha e^{2t} = x + y,$$

for some constant α and I've explicitly shown what u is in terms of x and y.

We could also multiply (3) by 3 and add that to 4 times (4). This results in

$$3\frac{dx}{dt} + 4\frac{dy}{dt} = 3x + 4y,$$

suggesting a substitution $v = 3x + 4y$ and hence

$$\frac{dv}{dt} = v.$$

Again this is an easy equation to solve: $v = \beta e^t = 3x + 4y$. So it seems we have two relations between x and y now

$$x + y = \alpha e^{2t},$$
$$3x + 4y = \beta e^t.$$

The above system of equations consists of two linear equations (no longer differential!). We can easily solve for the unknowns to obtain

$$x = 4\alpha e^{2t} - \beta e^t, \quad y = -3\alpha e^{2t} + \beta e^t.$$

Summarizing what we did:

(1) We began with the system (3) and (4).
(2) We make the susbtitution $u = x + y$ and $v = 3x + 4y$.
(3) Differentiate u and v with respect to t and substitute what $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are from (3) and (4).
(4) Solve the resulting equations for u and v. In particular they will be uncoupled.
(5) Once you obtain u and v in terms of t, solve for x and y from the definition of u and v.

The success of this method (the reason why it enabled us to solve the system of equations) is because we made the right guess for u and v. For different problems you will need different substitutions which result in decoupled equations. That means we need to know how to make the correct guesses. But it also raises an important question:

Q: Can we always go from a coupled system to an uncoupled system?

The answer is given by taking a detour and learning some linear algebra.

3. Some linear algebra

First of all, lets introduce some notation. The reason for introducing this notation is because we (by that I mean me) want to emphasize the fact that the constant coefficient linear system (3-4) is really an equation of the form

$$\frac{du}{dt} = Cu.$$

Those were the kind of equations we ended up with when we solved using substitutions. To this end I'm going to rewrite (3-4) as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{5}$$

Since the above equation is supposed to exactly the same as (3-4), we know how to interpret the rather odd looking product (parentheses mean products remember!).

Definition 3.1 (Vector)**.** *The term*

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

is called a vector. Think of it as the usual arrow with some length and direction but with components x and y along two coordinate axes, i.e. $x\hat{i} + y\hat{j}$.

We can stretch vectors by multiplying them by a constant scalar

$$\alpha\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}.$$

We can also add two vectors

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}.$$

The zero vector is the vector which has zero in both components

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Finally two vectors are equal if and only if their respective components are equal.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

is true if and only if $x_1 = x_2$ and $y_1 = y_2$.

Definition 3.2 (Matrix)**.** *A collection of elements* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *which takes a vector and gives out another vector is called a matrix. You can think of a matrix as a function which works on vectors and outputs vectors.*

1×1 matrix: Also known as a scalar. Mutliplying a scalar with a vector results in the vector being stretched.

2×2 matrix: Typically the kind of matrix we will mean when thinking of these things. Mutliplying a matrix with a vector results in the vector usually being stretched *and* rotated.

Action of a matrix on a vector: The way in which a matrix acts on a vector is a very special rule. The reason why it works this way is probably best explained in a class on linear algebra but I will base my motivation on the fact that our new notation (5) is really just another way of writing (3-4). If you think of (3-4) as really

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x+4y \\ -3x+2y \end{pmatrix},$$

then this suggests that

$$\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x+4y \\ -3x+2y \end{pmatrix},$$

and this defines what we mean by multiplying a matrix with a vector. (The vector on the left hand side $\begin{pmatrix} x \\ y \end{pmatrix}$ was mutliplied by the matrix to obtain the vector on the right-hand side. In went a vector, out came another.)

In general

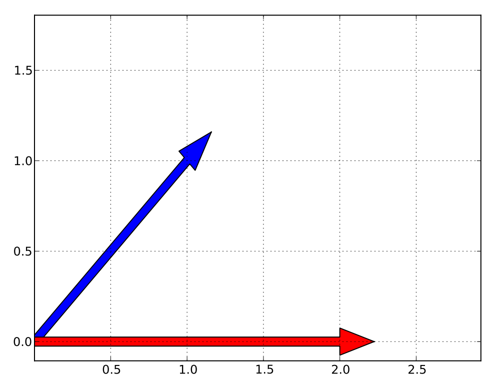
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

Let's do an example of matrix vector multiplication (there will be plenty of this stuff going on, so we had better get the hang of it). This example will also explain why matrices in general can rotate and stretch a vector.

Consider

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1+1\cdot 1 \\ 1\cdot 1+1\cdot(-1) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

You should really draw the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ with coordinate axes. A vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a line segment from the origin to the point $(1, 1)$ in the $x-y$ plane. Drawing the input vector as well as the output vector should convince you that the initial vector is getting rotated and stretched. If you're feeling lazy about drawing vectors look below.



Product of two matrices: It is possible to multiply two matrices to obtain a third matrix. The way we do this is to generalize the action of a matrix on a vector. Consider two matrices

$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ for which we want to find the product. The product of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{AB}$.

We know how to multiply the matrix A with $\begin{pmatrix} p \\ q \end{pmatrix}$. The resulting vector is the first column of the matrix $\mathbf{AB}$. Similarly, the vector obtained by multiplying $\mathbf{A}$ with $\begin{pmatrix} r \\ s \end{pmatrix}$ is the second column of the matrix $\mathbf{AB}$. Hence we get

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{pmatrix} ap + br & cp + dr \\ aq + bs & cq + ds \end{pmatrix}.$$

A characteristic feature of matrix multiplication is that $\mathbf{AB}$ does not always equal $\mathbf{BA}$. Go ahead and try it for the matrices $\mathbf{A}$ and $\mathbf{B}$ defined above. The first column of $\mathbf{BA}$ is $\mathbf{B}$ acting on the first column of $\mathbf{A}$. Similarly for the second column.

We can also multiply a matrix with a scalar (a single number).

$$m\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ma & mb \\ mc & md \end{pmatrix}.$$

Identity matrix: A rather special matrix is the *identity* matrix usually denoted by $\mathbf{I}$. It is defined as

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The identity matrix acts on vectors by neither stretching nor rotating them. It leaves vectors unchanged. This also means multiplying a matrix with $\mathbf{I}$ leaves the matrix unchanged.

Inverse of a matrix: We've seen that matices can be multiplied with vectors as well as with themselves. Suppose we want to find a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ such that

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

In this case, it would be great if we could somehow divide by matrices to find the unknown vector. Dividing by matrices is not defined. However we can multiply both sides of the equation above by another matrix such that the resulting product matrix on the left-hand side is the identity. Think of this matrix as a kind of reciprocal matrix. This reciprocal matrix is called the *inverse*. Multiply both sides of the equation by the matrix $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. This results in

$$\begin{aligned}
& \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \\
\Rightarrow \quad & \begin{pmatrix} 2\cdot 1 + 1\cdot(-1) & 2\cdot 1 + (-1)\cdot 2 \\ (-1)\cdot 1 + 1\cdot 1 & (-1)\cdot 1 + 1\cdot 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2\cdot 2 + (-1)\cdot 0 \\ (-1)\cdot 2 + 1\cdot 0 \end{pmatrix}, \\
\Rightarrow \quad & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \\
\Rightarrow \quad & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.
\end{aligned}$$

Thus $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ is the inverse of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Here is a general formula to find inverses. Consider $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then it's inverse

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The number $\dfrac{1}{ad - bc}$ is a scalar which multiplies every entry of the matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Check that $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ is the inverse of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Also find the inverse of $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$.

3.1. **Eigenvalues and Eigenvectors.** It is possible that multiplying a matrix with a special vector results in the vector only getting stretched (or compressed) but *not* rotated. Consider the following examples

$$\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \cdot 1 + 4 \cdot (-1) \\ (-3) \cdot 1 + (-2) \cdot (-1) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Here the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ was not rotated by multiplying with the matrix. Incidentally it was also not stretched. This is not typical as the next example shows.

$$\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \cdot 4 + 4 \cdot (-3) \\ (-3) \cdot 4 + (-2) \cdot (-3) \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

These special vectors are called *eigenvectors* of the matrix $\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$. In the first case the vector was stretched by the scalar 1 and in the second case it was stretched by 2. The numbers 1 and 2 are called *eigenvalues* of the matrix $\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$.

In general, if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \lambda \begin{pmatrix} p \\ q \end{pmatrix},$$

then λ is called the eigenvalue and $\begin{pmatrix} p \\ q \end{pmatrix}$ is the associated eigenvector. In the next section we will see why eigenvalues and eigenvectors are important for solving differential equations. But first let's see how we can compute them.

In a course on Linear Algebra you will learn how to compute eigenvalues and eigenvectors in all generality. Since in these notes we will always concern ourselves with 2×2 systems, I'll restrict the discussion to finding these quantities for 2×2 matrices. Each 2×2 matrix has two eigenvalues. Let λ_1 and λ_2 be the two eigenvalues of $\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$. Then the sum of the eigenvalues is the sum of the terms along main diagonal (the diagonal that starts from top left and goes to bottom right, here it consists of the elements 5 and -2). Hence

$$\lambda_1 + \lambda_2 = \text{ Sum along main diagonal } = 5 + (-2) = 3.$$

To solve for the eigenvalues we would like another relation. The second relation between the eigenvalues is the following. The product of the eigenvalues is equal to the difference of the product along the main diagonal and the product along the other diagonal. In other words

$$\begin{aligned}\lambda_1\lambda_2 &= \text{ Product along main diagonal } - \text{ Product along other diagonal },\\ &= 5(-2) - 4(-3),\\ &= -10 + 12,\\ &= 2.\end{aligned}$$

With the two relations between the eigenvalues we can readily solve to obtain either. Solving for λ_2 using the first relation

$$\lambda_1 + \lambda_2 = 3 \Rightarrow \lambda_2 = 3 - \lambda_1,$$

and substituting in to the second

$$\lambda_1\lambda_2 = 2 \Rightarrow \lambda_1(3 - \lambda_1) = 2 \Rightarrow \lambda_1^2 - 3\lambda_1 + 2 = 0.$$

This quadratic equation can be solved to get $\lambda_1 = 2, 1$. Since λ_1 satisfies a quadratic equation there are two roots. In either case we can solve for $\lambda_2 = 3 - \lambda_1$. However this does not imply that there are four eigenvalues since $\lambda_2 = 1, 2$ here. In fact the eigenvalues of the matrix $\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$ are just 2 and 1. Hence solving the quadratic equation for λ_1 tells you what both eigenvalues are.

Let us now compute corresponding eigenvectors. Consider the eigenvalue $\lambda = 2$ then

$$\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = 2\begin{pmatrix} p \\ q \end{pmatrix},$$

by the definition of the eigenvector. Multiplying out the matrix and vector on the left-hand side we get

$$\begin{pmatrix} 5p + 4q \\ -3p - 2q \end{pmatrix} = \begin{pmatrix} 2p \\ 2q \end{pmatrix}.$$

We noted earlier that two vectors were equal if and only if their components were equal. This implies

$$5p + 4q = 2p, \quad -3p - 2q = 2q,$$

simplifying which we get

$$3p + 4q = 0, \quad -3p - 4q = 0.$$

Note a very important comment here, both equations are really the same relation! That means we have two unknowns but only one equation and so we cannot solve for both p and q. One of them will remain undetermined. In other words we cannot find the eigenvector uniquely but only up to a constant. Let us solve for p in terms of q. Hence

$$3p + 4q = 0 \Rightarrow p = -\frac{4}{3}q.$$

Hence our eigenvector is given by

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\frac{4}{3}q \\ q \end{pmatrix}.$$

As you can see this eigenvector contains an unknown constant q. However, the nice thing is we don't need to find what this constant is. In fact all vectors of this form get stretched by 2 when they are multiplied the matrix $\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$ as you can see below

$$\begin{aligned} \begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} -\frac{4}{3}q \\ q \end{pmatrix} &= \begin{pmatrix} 5 \cdot (-\frac{4}{3}q) + 4 \cdot q \\ (-3) \cdot (-\frac{4}{3}q) + (-2) \cdot q \end{pmatrix}, \\ &= \begin{pmatrix} -\frac{8}{3}q \\ 2q \end{pmatrix}, \\ &= 2 \begin{pmatrix} -\frac{4}{3}q \\ q \end{pmatrix}. \end{aligned}$$

This means we can take the vector $\begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix}$ and multiply it by any non-zero number q and it will remain an eigenvector. In fact we could take any non-zero scalar multiple of $\begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix}$ and call that the eigenvector too. Let us pick $q = -3$ to obtain

$$\begin{pmatrix} -\frac{4}{3}q \\ q \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix},$$

and claim this is the eigenvector associated with $\lambda = 2$ with the understanding that eigenvectors are only determined up to a constant.

We should probably find the eigenvector associated with the eigenvalue $\lambda = 1$ for the matrix $\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$. Lets do that now.

$$\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = 1 \begin{pmatrix} p \\ q \end{pmatrix},$$

implies

$$\begin{pmatrix} 5p + 4q \\ -3p - 2q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix},$$

which upon equating components we get

$$5p + 4q = p, \quad -3p - 2q = q,$$

or in other words

$$p + q = 0.$$

Again note that both equations led to the same condition between p and q. Solving for p in terms of q we get $p = -q$. Hence the eigenvector is given by

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -q \\ q \end{pmatrix},$$

where once again we see the unknown constant q. We are free to choose any value of q to fix our eigenvector. I'm going to choose $q = -1$ to obtain $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are the eigenvector corresponding to $\lambda = 1$.

It would be good practice to do some more examples. We'll learn about different situations that are possible along the way.

Example 1: Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.

Let λ_1 and λ_2 be the eigenvalues of the matrix. Then

$$\lambda_1 + \lambda_2 = -2 - 2 = -4.$$

Also

$$\lambda_1 \lambda_2 = (-2)(-2) - (1)(1) = 3.$$

Solving the first equation for λ_2 and substituting in to the second we obtain

$$\lambda_1(4 - \lambda_1) = 3 \Rightarrow \lambda_1^2 + 4\lambda_1 + 3 = 0.$$

The roots of this quadratic equation are $-1, -3$ and hence we have found the eigenvalues to -1 and -3.

The eigenvector corresponding to $\lambda = -1$ is found as follows.

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = -1 \begin{pmatrix} p \\ q \end{pmatrix}.$$

Multiplying out the matrix and vector we obtain

$$\begin{pmatrix} -2p + q \\ p - 2q \end{pmatrix} = \begin{pmatrix} -p \\ -q \end{pmatrix},$$

which states

$$-2p + q = -p, \quad p - 2q = -q,$$

or

$$p = q.$$

Again note we only have one relation between p and q. The eigenvector is given as $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} q \\ q \end{pmatrix}$ and so we can again choose any q. I make the convenient choice of $q = 1$ to obtain eigenvector associated with $\lambda = -1$ as $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Similarly we can find the eigenvector for $\lambda = -3$.

$$\begin{aligned} & \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = -3 \begin{pmatrix} p \\ q \end{pmatrix}, \\ \Rightarrow \quad & \begin{pmatrix} -2p + q \\ p - 2q \end{pmatrix} = \begin{pmatrix} -3p \\ -3q \end{pmatrix}, \\ \Rightarrow \quad & -2p + q = -3p, \quad p - 2q = -3q, \\ \Rightarrow \quad & p = -q, \end{aligned}$$

which with the choice $q = -1$ we get the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Remember when we have the undetermined constant, we can choose any value (except 0). They are all equivalently the eigenvector.

The preceeding examples showed that each eigenvalue had an eigenvector. This is always true if the eigenvalues are not equal. When the eigenvalues are equal (usually called repeated eigenvalues) then they may have the same eigenvector or they may not.

Example 2: Find the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$.

Eigenvalues

$$\lambda_1 + \lambda_2 = 1 + 3 = 4,$$

and

$$\lambda_1 \lambda_2 = (1)(3) - (1)(-1) = 4.$$

Solving for λ_2 using the first equation and substituting in second we get

$$\lambda_1(4 - \lambda_1) = 4,$$

solving which we get the eigenvalues are $2, 2$. They are repeated eigenvalues since the quadratic for λ_1 has repeated roots.

Eigenvectors

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = 2 \begin{pmatrix} p \\ q \end{pmatrix}.$$

Multiplying out we get

$$\begin{pmatrix} p - q \\ p + 3q \end{pmatrix} = \begin{pmatrix} 2p \\ 2q \end{pmatrix}.$$

Setting components on either side equal we get

$$p = -q$$

and hence the eigenvector is

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -q \\ q \end{pmatrix}.$$

We take $q = -1$ to obtain $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as the eigenvector.

Note in this case there is only one eigenvector since if we repeat the calculation for the "other" eigenvalue $\lambda = 2$ we will obtain the same eigenvector.

Important comment: This will be the case when you have repeated eigenvalues for a non-diagonal matrix. A diagonal matrix is one which only has entries along the main diagonal. If you start with a non-diagonal matrix and obtain repeated eigenvalues, then the eigenvalues will share the eigenvector.

Example 3: Find eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. The given matrix is a diagonal matrix. Finding eigenvalues and eigenvectors is easiest for these matrices, we can simply read them off. The eigenvalues of a diagonal matrix are the elements along the diagonal. Hence $\lambda = 2, 1$. The eigenvector associated with the eigenvalue $\lambda = 2$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

and that associated with $\lambda = 1$ is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. These are always the eigenvectors of a diagonal matrix. For a diagonal matrix, the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is associated with the first element along the diagonal (here 2) and the eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is associated with the second element along the diagonal (here 1), in other words

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = d \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In the case that $a = d$, we have repeated eigenvalues but each eigenvalue has it's own eigenvector. Diagonal matrices have two unique eigenvectors. Recall the identity matrix, what are it's eigenvalues and eigenvectors?

Example 4: Find eigenvalues and eigenvectors of $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$.

Eigenvalues

$$\lambda_1 + \lambda_2 = 2 + 2 = 4,$$

and

$$\lambda_1 \lambda_2 = (2)(2) - (-1)(1) = 5.$$

Solving for λ_2 using the first equation and substituting it in the second we obtain

$$\lambda_1(4 - \lambda_1) = 5 \Rightarrow \lambda_1^2 - 4\lambda_1 + 5 = 0,$$

which has the roots $2 + i$ and $2 - i$. Hence the eigenvalues are $2 \pm i$. Note that if a matrix of real numbers has complex eigenvalues, they must be complex conjugates. If they are not, you have done a mistake in your calculations.

Eigenvectors

Consider the eigenvalue $2 + i$. We have

$$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = (2 + i) \begin{pmatrix} p \\ q \end{pmatrix}.$$

Multiplying out the matrix and vector we get

$$\begin{pmatrix} 2p - q \\ p + 2q \end{pmatrix} = \begin{pmatrix} 2p + ip \\ 2q + iq \end{pmatrix}.$$

Equating components from either side we get

$$2p - q = 2p + ip, \quad p + 2q = 2q + iq,$$

which is simply the one equation

$$p = iq.$$

Again we have a choice regarding q, I take $q = 1$ and hence the eigenvector associated with $\lambda = 2 + i$ is $\begin{pmatrix} i \\ 1 \end{pmatrix}$. Just as for the eigenvalues, the eigenvectors of a matrix of real numbers come in complex conjugate pairs. Hence the eigenvector associated with $\lambda = 2-i$ is $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ (the complex conjugate of the first eigenvector).

4. Solving linear systems of differential equations and the Phase Plane

In this section we see how eigenvalues and eigenvectors can be used to obtain solutions to linear systems of differential equations. Let's go back to the question that started all this. We want to solve the following system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{6}$$

We've had a lot of luck with making good guesses, so we'll continue this tradition. Assume that the solution has the following form

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

that is we are assuming that the solution is proportional to one of the eigenvectors of the matrix. Plugging this into the equation we get

$$\begin{aligned}
\frac{d}{dt}c_1(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} &= \begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix} c_1(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\
\frac{dc_1}{dt}\begin{pmatrix} 1 \\ -1 \end{pmatrix} &= c_1(t)\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ the scalar can come out in front,} \\
\frac{dc_1}{dt}\begin{pmatrix} 1 \\ -1 \end{pmatrix} &= c_1(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ using the fact that we chose an eigenvector with eigenvalue 1,} \\
\left(\frac{dc_1}{dt} - c_1(t)\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}$$

and the only way the last line is true is if

$$\frac{dc_1}{dt} = c_1(t),$$

or in other words if $c_1(t) = \alpha e^t$, α being any real number. This implies our solution must have the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now you might ask, why did I chose this eigenvector and not the other one, $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$? You'd be right to question. In fact I can repeat all of the above for the choice

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_2(t)\begin{pmatrix} 4 \\ -3 \end{pmatrix},$$

which would lead to

$$\frac{dc_2}{dt} = 2c_2(t),$$

(recall that the other eigenvalue was 2) which implies $c_2(t) = \beta e^{2t}$ and hence we have the solution of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \beta e^{2t} \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

Now this is all well and good, but why should the solution be proportional to either of the eigenvectors? That seems to be assuming a bit too much. Let's try something a bit more general; try the following form for the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = a_1(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + a_2(t) \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

Plugging this in to the equation (6) we get

$$\begin{aligned}
\frac{d}{dt}\left[a_1(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + a_2(t)\begin{pmatrix} 4 \\ -3 \end{pmatrix}\right] &= \begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}\left[a_1(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + a_2(t)\begin{pmatrix} 4 \\ -3 \end{pmatrix}\right], \\
\frac{da_1}{dt}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{da_2}{dt}\begin{pmatrix} 4 \\ -3 \end{pmatrix} &= a_1(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2a_2(t)\begin{pmatrix} 4 \\ -3 \end{pmatrix}, \\
\left[\frac{da_1}{dt} - a_1(t)\right]\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \left[\frac{da_2}{dt} - 2a_2(t)\right]\begin{pmatrix} 4 \\ -3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (7)
\end{aligned}$$

Our experience suggests that the terms in the square brackets should be zero. This would result in the differential equations we obtained earlier. At this point we cannot equate components on either side. That doesn't lead us anywhere. If you don't believe me, go ahead and try. I'll wait.

Do you believe me now? So it seems our previous argument will have to be modified. Assume for now that the terms in the square brackets are not zero. Then they must be some function of t. I'll rewrite the last line above as

$$f(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + g(t)\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If we take a step back and see what this line is really saying. We might realise, after staring at it for some time, that this equation says the addition of two vectors must be the zero vector. Intuitively, vectors represent a sort of instruction to walk in a direction for some distance. The zero vector would be interpreted as not moving. Then paraphrasing what the equation is saying, if I were given two instructions (to walk in two directions for some distance in each), is it possible that I end up not moving at all? This is possible if I walked some distance in one direction and then walked back the same distance in the opposite direction. But that would mean that one vector would be the negative of the other. In other words the vectors would have to be proportional to one another. However, that is clearly not the case for us. The other possibility is that I walk zero distance in one direction and then zero distance in the other. In this case, the functions f and g have to be zero.

To put all this in a more mathematical context, two vectors $\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$ and $\begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$ are linearly independent if and only if, the only numbers d_1 and d_2 such that

$$d_1\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + d_2\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

are $d_1 = d_2 = 0$. For two dimensions, this will be the case if $\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$ is not proportional (in other words, is not a scalar multiple) of $\begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$.

Going back to (7), the vectors are not proportional to one another. Hence the only way they can add up to zero, is if the terms in the square brackets are zero. This implies

$$\frac{da_1}{dt} = a_1, \quad \frac{da_2}{dt} = 2a_2,$$

both of which are easily solved to obtain

$$a_1 = \alpha e^t, \quad a_2 = \beta e^{2t}.$$

Hence we have the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta e^{2t} \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

Lastly, it should be mentioned that when you have linearly independent vectors, every solution is given as a combination of those vectors. In other words, the above represents every solution to the differential equation.

What all of this means is that we have a very straight-forward way of solving linear systems we've been considering so far. Find the eigenvectors and eigenvalues of the given matrix. Then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha e^{\lambda_1 t} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix},$$

is the general solution to the equation. If we are given initial conditions, say

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

where C_1 and C_2 are known numbers given in the problem, then evaluating the general solution at $t = 0$ gives two equations for the α and β which can be solved.

4.1. **Real unequal eigenvalues.** In the two next examples we will see how we can solve systems when we have two real, unequal eigenvalues. We will also see how we can analyse the solution for a range of initial conditions without having to explicitly solve each time. A graphical representation will hopefully provide a lot more intuition about how the solution behaves.

Example 1: Solve

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Eigenvalues

For the given matrix,

$$\lambda_1 + \lambda_2 = 1 + 1 = 2$$

and

$$\lambda_1 \lambda_2 = (1)(1) - (4)(1) = -3.$$

Solving for λ_1 in the usual way we obtain the eigenvalues are 3 and -1 (you should check this).

Eigenvectors

For the eigenvalue $\lambda = 3$, we have

$$\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = 3 \begin{pmatrix} p \\ q \end{pmatrix},$$

which results in

$$p + q = 3p, \quad 4p + q = 3q,$$

or in other words

$$p = \frac{q}{2}.$$

Hence the eigenvector corresponding to this eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ where we have taken $q = 2$.

For the eigenvalue $\lambda = -1$, we have

$$\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = -1 \begin{pmatrix} p \\ q \end{pmatrix},$$

which leads to

$$p = -\frac{q}{2}$$

and so the eigenvector is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Note we have found two eigenvalues with unique eigenvectors. Also the eigenvectors are not proportional to one another and so they are linearly independent. Hence the solution to equation is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The behavior of the solutions is easily analysed by plotting the solution x versus y.

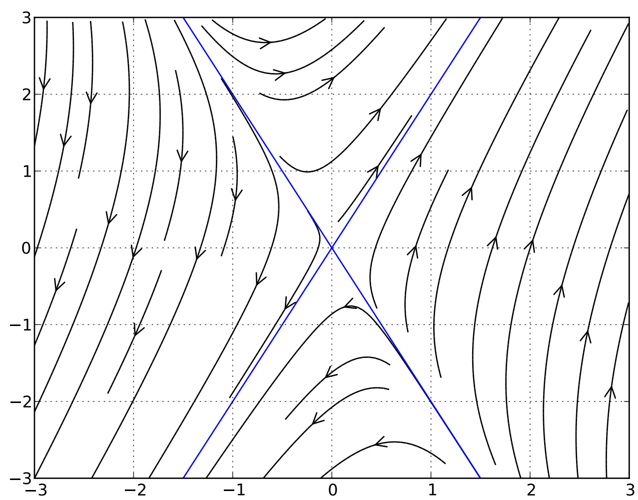


FIGURE 1. Phase-plane of the solution to (7). The blue lines represent the eigenvectors. Arrows indicate the direction the solution proceed with time.

When plotting by hand it is best to start with the eigenvectors. Draw the eigenvectors as lines that pass through the origin with slope q/p. In this case, the eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a line through the origin with slope -2. Similarly for the other eigenvector. Once the eigenvectors are drawn, the arrows indicate whether the solution goes toward the origin or away from it. If the eigenvalue is positive, then the arrows point away from origin. If eigenvalue is negative, arrows point toward the origin. The other curves are easy draw once we know the direction along the eigenvectors. The origin is an equilibrium point of the equation. If we start at the origin, the solution remains there forever. However, moving ever so little away from origin can push us further away unless we move along the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (the direction corresponding to the negative eigenvalue). Such an equilibrium is called a *saddle point*.

Example 2: Solve

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{8}$$

Eigenvalues

We begin as usual

$$\lambda_1 + \lambda_2 = -3 - 2 = -5,$$

and

$$\lambda_1\lambda_2 = (-3)(-2) - \sqrt{2}\sqrt{2} = 4.$$

This implies

$$\lambda_1(-5 - \lambda_1) = 4,$$

which simplifies to

$$\lambda_1^2 + 5\lambda_1 + 4 = 0,$$

which has roots -4 and -1. Hence the eigenvalues are -4 and -1.

Eigenvectors

The eigenvector corresponding to $\lambda = -4$ satisfies

$$\begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = -4\begin{pmatrix} p \\ q \end{pmatrix}.$$

Solving for p in terms of q we obtain $p = -\sqrt{2}q$. This leads to the eigenvector $\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$ with the choice $q = 1$.

The eigenvector corresponding to $\lambda = -1$ satisfies

$$\begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = -1\begin{pmatrix} p \\ q \end{pmatrix}.$$

Solving again for p in terms of q we see $p = \frac{q}{\sqrt{2}}$. On choosing $q = \sqrt{2}$, we obtain the eigenvector is $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$.

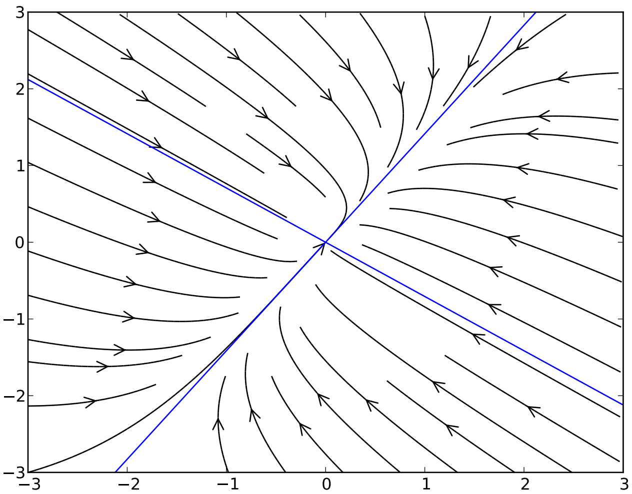


FIGURE 2. Blue lines indicate the eigenvectors. Arrows indicate the evolution of solution in time.

We have two distinct eigenvalues and hence their eigenvectors are linearly independent. Thus our solution is given as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha e^{-4t} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} + \beta e^{-t} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

When both eigenvalues have the same sign (both positive or both negative), we call such a point a *node*. The figure below depicts the solutions in the neighborhood of the equilibrium point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Here since both eigenvalues are negative, the solution always moves towards the equilibrium point. Thus the origin is a stable equilibrium.

4.2. **Complex conjugate eigenvalues.** We now consider the case of complex conjugate eigenvalues. Since the eigenvalues are complex conjugates, they cannot be equal unless they are real. Recall that eigenvectors of distinct eigenvalues are linearly independent. Hence our solution is essentially of the same form. However since the eigenvectors will be complex valued, we will have to do a little more work to get real valued solutions. We'll see what to do in the example below.

Example: Solve

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Eigenvalues

$$\lambda_1 + \lambda_2 = -1/2 - 1/2 = -1,$$

and

$$\lambda_1 \lambda_2 = (-1/2)(-1/2) - (1)(-1) = 5/4.$$

Solving for λ_2 in terms of λ_1 using the first equation and substituting in to the second one we get

$$\lambda_1(-1-\lambda_1) = 5/4 \Rightarrow \lambda_1^2 + \lambda_1 + 5/4 = 0,$$

which has roots $-1/2 \pm i$. Hence the eigenvalues are $-1/2 \pm i$. Notice that our eigenvalues are complex conjugates.

Eigenvectors

Recall that we need only find the eigenvector associated with one of the eigenvalues. The other eigenvector is the complex conjugate of the first. Consider $\lambda = -1/2 - i$. The eigenvector equation is

$$\begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = (-1/2 - i) \begin{pmatrix} p \\ q \end{pmatrix}.$$

This implies

$$-\frac{p}{2} + q = -\frac{p}{2} - ip, \quad -p - \frac{q}{2} = -\frac{q}{2} - iq.$$

Both of these equations lead to the condition $p = iq$. Hence the associated eigenvector is $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ on taking $q = -i$.

This implies the eigenvector associated with $\lambda = -1/2 + i$ is given by the complex conjugate of the first eigenvector, *i.e.* $\begin{pmatrix} 1 \\ i \end{pmatrix}$.

The solution is then given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha\vec{u} + \beta\vec{v},$$

where $\vec{u} = e^{(-1/2-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\vec{v} = e^{(-1/2+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$. However, this does not lead to real-valued solutions for real α, β. To remedy this we will take specific combinations of the individual vectors $\vec{u}$ and $\vec{v}$. Consider the first vector. Using Euler's identity we can write

$$\begin{aligned} \vec{u} &= e^{(-1/2-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \\ &= e^{-t/2}\left[\cos t - i\sin t\right]\begin{pmatrix} 1 \\ -i \end{pmatrix}, \\ &= e^{-t/2}\left[\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} - i\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right], \end{aligned}$$

Similarly,

$$\vec{v} = e^{-t/2}\left[\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right].$$

As we did in the case of second order linear differential equations, we will a new set of vectors to form our fundamental solutions. In particular we let

$$\vec{x}_1 = \frac{\vec{u}+\vec{v}}{2} = e^{-t/2}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \quad \vec{x}_2 = \frac{\vec{u}-\vec{v}}{2i} = e^{-t/2}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

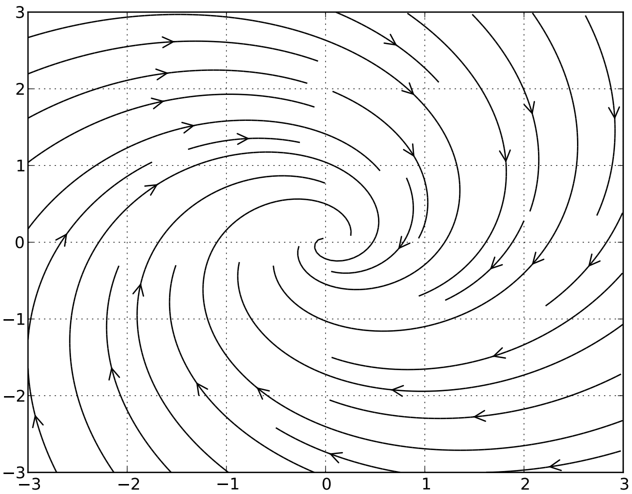


FIGURE 3. Stable spiral due to complex conjugate eigenvalues

Hence the solution to the differential equation is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha e^{-t/2} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + \beta e^{-t/2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Note we could have also taken the real and imaginary part of either $\vec{u}$ or $\vec{v}$ to form our new fundamental solutions. Finally we plot the phase plane for this situation. In this case, we obtain spirals either converging to the origin (if real part of eigenvalue is negative, as is in this case) called *sinks/stable spirals* or diverging from origin (if real part of eigenvalue is positive) called *sources/unstable spirals*. If the eigenvalue is purely imaginary, then the phase plane consists of concentric circles around the origin. The figure shows the situation for this example.

It should be fairly evident from the solution above, why the arrows all point toward the origin: the exponential term $e^{-t/2}$ decays to zero as $t \to \infty$. When we have complex conjugate eigenvalues, the solution will always rotate around the equilibrium point (here the origin). The question remains what the sense of rotation is. Is it clockwise or counter-clockwise? It is fairly easy to resolve this question. Pick any point (*i.e* an x_0 and y_0 value) in the $x - y$ plane that is near the origin. It does not matter which point you choose so long as you do not choose the origin itself. Now evaluate the right-hand side of the original differential equation at this point,

$$\begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -x_0/2 + y_0 \\ -x_0 - y_0/2 \end{pmatrix}.$$

Now draw a little arrow pointing in the direction of the above vector at the point (x_0, y_0) in the plane. This arrow should tell you the sense of rotation. If you cannot distinguish what the exact sense of rotation (clockwise or counter-clockwise), pick another point. Repeat the procedure and the two arrows should give you a better picture.

4.3. **Repeated eigenvalues.** We'll now look at the case of repeated eigenvalues. But first, since we always consider systems with real coefficients and so complex eigenvalues come in conjugate pairs, repeated eigenvalues must be real (because if a number is equal to its conjugate, then the number is real). Secondly, recall that distinct eigenvalues cannot share eigenvectors. For 2×2 systems, this means distinct eigenvalues will always have linearly independent vectors. Hence if we have repeated eigenvalues, we may or may not have two linearly independent eigenvectors. This could be a problem. We will split the discussion in two: when repeated eigenvalues have distinct eigenvectors and when repeated eigenvalues share an eigenvector (in this case we need to come up with anothe vector somehow). You might guess the second situation involves more work. If you did guess that, you'd be right.

Consider the situation when the eigenvalues of a matrix are repeated (say λ, λ) and we have two distinct eigenvectors for each eigenvalue. It turns out that in this case the matrix must be of the following form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

This is quite fortunate because the matrix is a diagonal matrix and we already know what the eigenvectors of a diagonal matrix are (see example 4 in 3.1). You may notice that the matrix is really just the identity matrix multiplied by λ. All of this is very good news because it makes recognizing this situation (distinct eigenvectors with repeated eigenvalues) quite trivial. Said another way, if you have a diagonal matrix with the same element along the diagonal, then the eigenvalues are repeated (they are the diagonal elements of the matrix) and the eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Having resolved the first case, we consider now the situation when a 2×2 matrix has repeated eigenvalues but only one eigenvector (example 2 of 3.1). Let us consider a concrete example

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{9}$$

The first step is the usual one: we need to find the eigenvalues and eigenvectors of the matrix. The matrix in this example is precisely the same one considered in example 2 of section 3.1. Hence I'll just state the results. The matrix has repeated eigenvalues $2, 2$ which have the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. This gives us one vector and solution of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{10}$$

. To find the other we take a hint from our method for dealing with repeated eigenvalues of second-order constant coefficient differential equations. In that case, we showed that te^t was linearly independent of e^t. Following this line of thought we will guess a form for the solutions of (9) as

$$\begin{pmatrix} x \\ y \end{pmatrix} = te^{2t}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + e^{2t}\begin{pmatrix} p_2 \\ q_2 \end{pmatrix}. \tag{11}$$

Plugging this in the equation (9) we get

$$2te^{2t}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + e^{2t}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + 2e^{2t}\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = te^{2t}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} p_2 \\ q_2 \end{pmatrix},$$

which can be simplified to get

$$te^{2t}\left[\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} - 2\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}\right] + e^{2t}\left[\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} - 2\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} - \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}\right] = 0.$$

The above statement should be true for all t. Further, e^{2t} and te^{2t} are linearly independent functions. This means there is no linear combination of these functions which results in zero, except for the trivial combination, *i.e.* zero times e^{2t} plus zero times te^{2t}. This implies each term in the square brackets must be zero. This leads to

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = 2\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \Rightarrow \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

since this is just the equation for the eigenvector for the given matrix with eigenvalue 2 and we have already computed this in example 2 of section 3.1. The second square bracket set to zero results in

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = 2\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} + \begin{pmatrix} p_1 \\ q_1 \end{pmatrix},$$

which can be solved since we know $\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$. The vector $\begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$ is known as the generalized eigenvector of the matrix. We'll see that the generalized eigenvector is linearly independent of the original eigenvector.

But first let's solve the generalized eigenvector problem. We want to solve

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = 2\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which can be rewritten as

$$\begin{pmatrix} p_2 - q_2 \\ p_2 + 3q_2 \end{pmatrix} = \begin{pmatrix} 2p_2 + 1 \\ 2q_2 - 1 \end{pmatrix}.$$

Equating components on either side of the equality we get

$$p_2 + q_2 = -1,$$

for both components. Let $p_2 = k$ and so $q_2 = -1 - k$. Then

$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} k \\ -1-k \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} k \\ -k \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The constant k is arbitrary and can be safely set to zero. You may choose to keep it along for the ride and eventually notice that it merely redefines some constants in the final answer. The general solution to this differential equation is then given as a linear combination of (10) and (11)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta\left[te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right].$$

The solution in this case is called an *improper node* which evidently is also unstable as solutions move away from the equilibrium point. We call the term proportional to α the

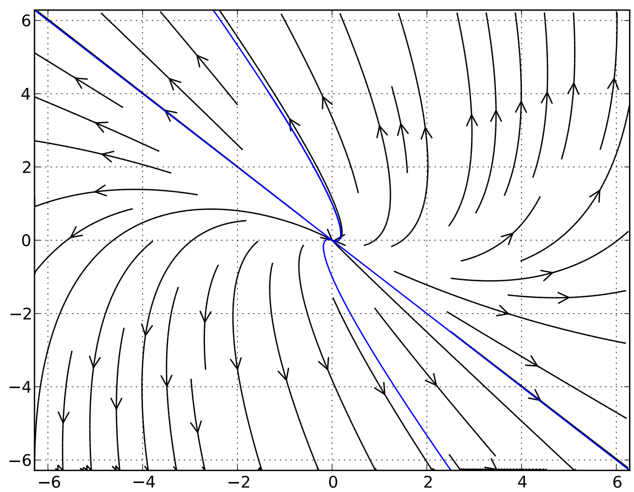


Figure 4. Improper node phase portrait. The blue lines are the linearly independent solutions. Arrows indicate the direction of time from $-\infty$ to ∞.

first linearly independent solution whereas the term proportional to β as the second linearly independent solution.

Let us see how we can draw the phase plane portrait for this case. As usual the idea is to draw the linearly independent solutions. Consider the first linearly independent solution $e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. We first draw a line representing the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Note this is a line with slope -1 (the slope of a vector is the y-component divided by the x-component) that passes through the origin. We draw arrows that point away from the origin since the exponential term has a positive coefficient. Hence solutions move away from the origin as $t \to \infty$.

We now rewrite the second linearly independent solution as

$$\beta e^{2t}\left[t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] = \beta e^{2t}\begin{pmatrix} t \\ -1-t \end{pmatrix}.$$

Hence the second linearly independet solution looks like an exponential times a vector that also depends on t. The direction of this solution is given by the direction of the vector, which we evidently see depends on t. Hence unlike the first independent solution, the direction of this solution changes with t (a bit like the real-valued independent solutions for the complex-conjugate eigenvalue case). You may notice that the vector is the parametric representation of a line with slope -1 that passes through the point $(0, -1)$. I have drawn this line as a dashed blue line in Figure (5). The red dot represents the point $(0, -1)$. If $\beta = -1$, then $\beta\begin{pmatrix} t \\ -1-t \end{pmatrix}$ is a line that passes through $(0, 1)$ with slope -1. This is the upper dashed blue line. The first step is to draw these lines, *i.e.* the lines represented by the vector $\beta\begin{pmatrix} t \\ -1-t \end{pmatrix}$. The next step is to note the direction along the line as t goes from $-\infty$ to

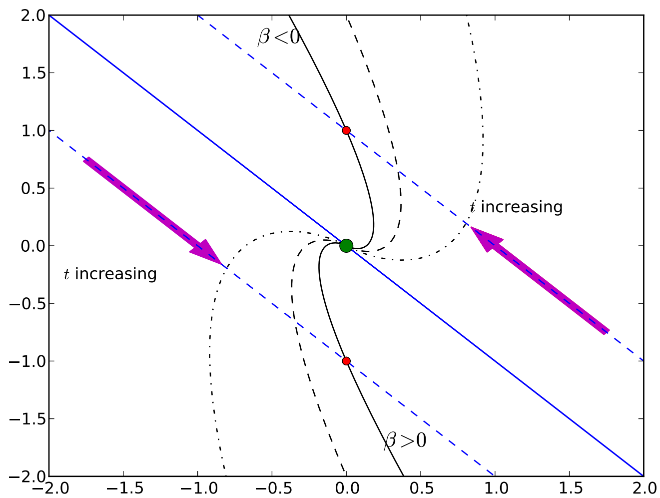


FIGURE 5. Second linearly independent solution. The various black lines depict secondary solution curves (the $x(t)$ versus the $y(t)$ in the second linearly independent solution) for various β values. The solid black line is $\beta = 1$.

∞. Consider first $\beta > 0$, then $\beta \begin{pmatrix} t \\ -1-t \end{pmatrix}$ has a negative x component and positive y component for $t < 0$. In particular, this means the for $t \to -\infty$, the line starts in the third quadrant. As $t \to \infty$, the vector $\beta \begin{pmatrix} t \\ -1-t \end{pmatrix}$ has positive x component and negative y component. Thus we end up in the fourth quadrant. Hence we know the direction of t along the lower dashed blue line. I have represented this direction with the purple arrow on the left. You should work out the case $\beta < 0$ for yourself and be convinced of the direction of the right purple arrow.

With the direction of the vector now resolved we turn out attention to the exponential term. When t is large and negative, the exponential term is practically zero. Hence this exponential term starts near the zero when $t \to -\infty$ and moves away from the origin as $t \to \infty$. The overall shape of this solution is that of an "S" reflected across the y axis, with half of it lying above the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and half it below. To draw the upper-half, we note that as $t \to -\infty$, we begin at the origin and must move away. However as we move away from the origin we must also move from right to left as dictated by the arrow on the right side. This results in the general shape shown in the figure. Similarly for the lower half, we begin near the origin (due to the exponential term being small for large negative t) and must move away while also moving left to right (as dictated by the arrow). This leads to the shape which completes the "S" shape. In the case of a negative repeated eigenvalue, one would obtain a solution that looks like a normal "S" shape.

5. Nonlinear systems and linearizing

Typically applications lead to equations more complicated than those we've studied so far. Most applications lead to nonlinear equations. These equations will have the form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix},$$

where f and g are nonlinear equations of x and y. In these cases we cannot find explicit solutions. However we can still use techniques we've learned to analyze the solutions. The two main steps in the analysis of nonlinear systems are

(1) Equilibrium solutions

Definition 5.1. *A pair (x_0, y_0) is said to be an equilibrium solution if both*

$$f(x_0, y_0) = 0, \quad g(x_0, y_0) = 0.$$

Equilibrium solutions imply that if our initial condition were precisely $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ then the solution will forever remain at that point. They are important since they play a significant role in the dynamics and behavior of the overall solution, *i.e.* knowledge of the all equilibrium points will tell you how the general solution to the nonlinear system of differential equations behaves.

(2) Stability of equilibrium solution

As mentioned above, initial conditions exactly equal to the equilibrium solution lead to solutions that remain there forever. When investigating stability of equilibrium solutions we are interested in what happens when we take an initial condition that is close to the equilibrium point, but not equal to that point.

Recall that for linear systems, the eigenvalues and eigenvectors dictated what happened to initial conditions near the origin (the equilibrium point for linear systems is the origin. Can you show why?). The main problem is that eigenvalues and eigenvectors are only defined for linear systems. There is no sensible definition of an eigenvalue for a nonlinear system. Thus we somehow need to convert our nonlinear system to a linear system. In particular we want to convert a nonlinear system near an equilibrium point to an appropriate linear system. Then eigenvalue-eigenvector analysis will tell us the stability of the equilibrium solution.

It remains for us to suitably *linearize* a nonlinear system. You might remember from your Calculus courses that the linear approximation to a function $h(x)$ near a point $x = x_0$ was given by

$$L[h(x)]_{x_0} = h(x_0) + h'(x_0)(x - x_0).$$

Remember that this results in a function that is *linear* in x and is a good approximation to the original nonlinear function $h(x)$ when x is close to x_0. In fact

$$|h(x) - L[h(x)]_{x_0}| \leq C|x - x_0|^2.$$

In other words, the error in approximating a nonlinear function with a linear function is roughly equal to the square of the difference between x and x_0. This approximation is known as the tangent line approximation. Since we are concerned with approximating functions of two variables, $h(x, y)$, we need the generalization known as the tangent plane approximation. It is given by

$$L[h(x,y)]_{x_0,y_0} = h(x_0, y_0) + h_x(x_0, y_0)(x - x_0) + h_y(x_0, y_0)(y - y_0).$$

Again the error in this appoximation is bounded as follows

$$|h(x,y) - L[h(x,y)]_{x_0,y_0}| \le C_1|x-x_0|^2 + C_2|y-y_0|^2.$$

The function we wish to approximate by a linear one is $\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$. This is a function of two variables that gives out two numbers. We approximate, near some point (x_0, y_0), each component of the vector to obtain the following linear approximation

$$\begin{aligned}
L\left[\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}\right]_{x_0,y_0} &= \begin{pmatrix} L[f(x,y)]_{x_0,y_0} \\ L[g(x,y)]_{x_0,y_0} \end{pmatrix}, \\
&= \begin{pmatrix} f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ g(x_0,y_0) + g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0) \end{pmatrix}, \\
&= \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} + \begin{pmatrix} f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0) \end{pmatrix}, \\
&= \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} + \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix},
\end{aligned}$$

where in the last step I have used matrix-vector product notation to introduce the matrix

$$\begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix},$$

which is also known as the **Jacobian matrix at the point** (x_0, y_0). It is the derivative of a vector. Note the order of the partial derivatives and their location in the matrix. Changing the order leads to the wrong matrix and can result in the wrong conclusions about stability.

With all this information about how to linearize a vector, we will now write down the linear system of equations near an equilibrium point. We begin with the nonlinear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}.$$

Let (x_0, y_0) be an equilibrium point. Now let

$$u(t) = x(t) - x_0, \quad v(t) = y(t) - y_0,$$

where we think of u and v as being small in absolute value. From the definition of u and v, we have

$$\frac{dx}{dt} = \frac{du}{dt}, \quad \frac{dy}{dt} = \frac{dv}{dt},$$

since x_0 and y_0 are constants (they are just numbers). If u and v are small enough, which means we consider points in the $x-y$ plane which are close to the point x_0, y_0 then we can replace the nonlinear functions f and g with their linear approximations. This means the equation satisfied by u and v is

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} + \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}, \\
&= \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} + \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.
\end{aligned}$$

But since we chose (x_0, y_0) to be equilibrium points, the first term on the right-hand side above is zero. Hence the linear equation near the equilibrium point is given by

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

Once we have the linearized system, we can use our eigenvalue-eigenvector analysis to understand the behavior near the equilibrium point (x_0, y_0). Nonlinear systems typically have more than one equilibrium point and so we repeat this analysis for each point, noting the stability properties of every equilibrium point. As we will see in the examples, this information is sufficient to understand the solution of the nonlinear problem for all initial conditions. In other words we will have solved the nonlinear problem.

Example: Undamped Pendulum

The equation of motion of an undamped pendulum is given by

$$\Theta'' + \omega^2 \sin\Theta = 0,$$

where $\Theta(t)$ is the angle between the rigid rod of the pendulum and the vertical axis and ω is some real positive constant related to the physical parameters of the problem (length of the pendulum, acceleration due to gravity, etc). We first convert this second order equation in to a first order system as follows. Let $x = \Theta$ and $y = \Theta'$. Then

$$\frac{dx}{dt} = \frac{d}{dt}\Theta = y,$$

and

$$\frac{dy}{dt} = \frac{d}{dt}\Theta' = \Theta'' = -\omega^2 \sin\Theta = -\omega^2 \sin x.$$

Thus we have

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -\omega^2 \sin x \end{pmatrix}.$$

I. *Equilibrium solutions*

The equilibrium solutions are given by those (x_0, y_0) such that

$$\begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here $f(x, y) = y$ and $g(x, y) = -\omega^2 \sin x$. Hence

$$y_0 = 0, \quad -\omega^2 \sin x_0 = 0,$$

implies

$$y_0 = 0, \quad x_0 = 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots$$

In other words our equilibrium solutions consist of

$$(x_0, y_0) = (0, 0), (\pi, 0), (-\pi, 0), (2\pi, 0), (-2\pi, 0), \ldots$$

II. *Stability of equilibrium solutions*

The linearized system near any of the equilibrium points is given by

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

$$= \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x_0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

For the equilibrium points $(x_0, y_0) = (\pi, 0), (-\pi, 0), (3\pi, 0), (-3\pi, 0), (5\pi, 0), (-5\pi, 0), \ldots$ we have the system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

since $\cos x_0$ at these points is -1. This system has eigenvalues

$$\omega, -\omega$$

with associated eigenvectors

$$\begin{pmatrix} 1 \\ \omega \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -\omega \end{pmatrix},$$

respectively (show this is indeed the case). Since ω is a real positive constant, we have two unequal eigenvalues of opposite signs. Each eigenvalue also has an eigenvector and so the general solution of the linear problem is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \alpha e^{\omega t}\begin{pmatrix} 1 \\ \omega \end{pmatrix} + \beta e^{-\omega t}\begin{pmatrix} 1 \\ -\omega \end{pmatrix}.$$

Evidently we have a situation of a **saddle point** at these equilibrium points since there is a stable and unstable direction, *i.e.* one positive eigenvalue which pushses the solution away from the origin in $u - v$ plane (or the equilibrium point x_0, y_0 in the $x - y$ plane) and another negative eigenvalue which takes the solution towards the equilibrium point.

For the equilibrium points $(x_0, y_0) = (0, 0), (2\pi, 0), (-2\pi, 0), \ldots$ we have the system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

since $\cos x_0$ at these points is 1. This leads to the eigenvalues

$$i\omega, \quad -i\omega,$$

with associated eigenvectors

$$\begin{pmatrix} 1 \\ i\omega \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -i\omega \end{pmatrix}.$$

Thus the general solution to this linear problem is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \alpha\begin{pmatrix} \cos(\omega t) \\ -\omega\sin(\omega t) \end{pmatrix} + \beta\begin{pmatrix} \sin(\omega t) \\ \omega\cos(\omega t) \end{pmatrix}.$$

This corresponds to a **clockwise rotating center**. It is a center due to the purely imaginary eigenvalue (and hence not a spiral). The direction of rotation of the solution around the equilibrium point is obtained as follows: the direction of rotation does not depend on the particular choice of initial conditions. All initial conditions rotate in same sense around the equilibrium point. The constants α and β are obtained from a particular choice of initial conditions. Since the sense of rotation is independent of initial conditions we may take any suitable α and β to obtain the direction of rotation. Choose $\alpha = 1$ and $\beta = 0$. Now consider two times $t = 0$ and $t = \pi/2$. The solution moves from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ -\omega \end{pmatrix}$ and since ω is positive, this means the solution is rotating in a clockwise sense. (Plot $(1, 0)$ and $(0, -\omega)$ to see why. You don't need to know the exact value of ω, only that it is positive.)

We summarize this information in the following figure.

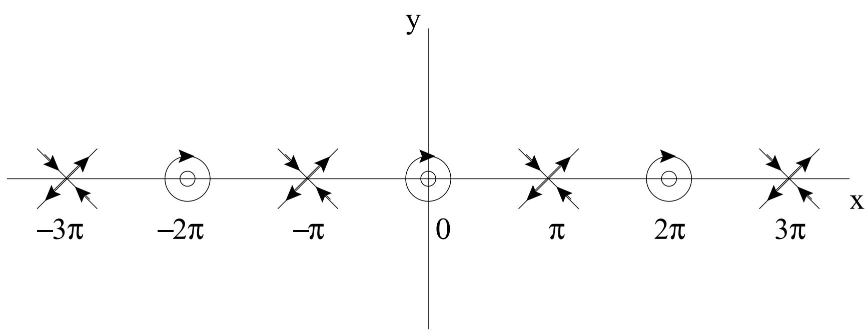


FIGURE 6. Equlibrium points and their stability.

Although the linear analysis results are insightful, their full power is not realized until we generalize our thinking of the above the figure. In particular, trajectories (the lines indicated by the arrows) cannot intersect except at equilibrium points, we can utilize the above figure to develop a full qualitative understanding of the solution to the original nonlinear problem. Specifically we can describe the behavior of the system far away from the equilibrium points. In the following figure, we develop the full nonlinear behavior by taking the phase-plane portrait above and generalizing it in the only way possible. This results in a dynamical picture which makes a great deal of

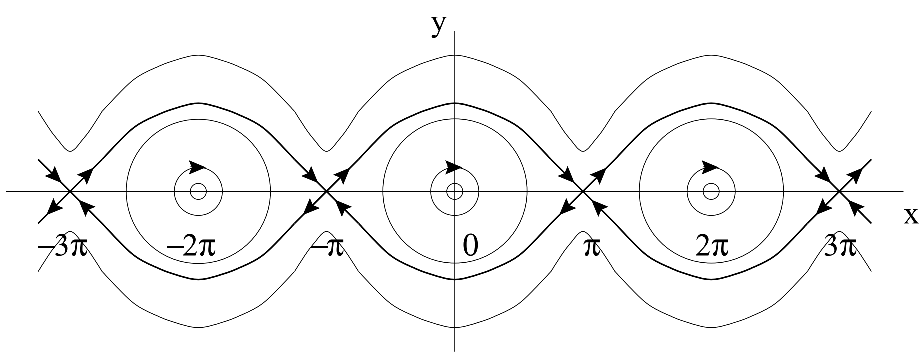


FIGURE 7. Full nonlinear behavior of pendulum

sense. Near the equilibrium points which are multiples of 2π the behavior is exactly as expected: oscillatory. This corresponds to the usual oscillation about the rest position of the pendulum. Near odd multiples of π (corresponding to the pendulum standing vertically up) we see that the solution is unstable; small disturbances make the solution move quickly away from the equilibrium point. The trajectory separating the oscillatory solutins (closed loops) from the trajectories above these is called the *separatrix*. This is denoted in bold lines in the figure. The separatrix connects the unstable direction of one saddle point to the stable direction of the neighboring saddle point. The behavior above the separatrix corresponds to the undamped pendulum swinging around it's support non-stop. As you can see, these trajectories (the wavy line at the top of the figure) never cut the x-axis. This means their y value (that is Θ') is never zero, which means that they have non-zero angular speed all the time or in other words they continuously rotate. Also, notice that their is a minimal y value (depending on x) that must be given as initial condition for the system to display this behavior. Simply put, you cannot expect the pendulum to complete full rotations if you do not give it enough of an initial kick. Since there is no damping in the model, the pendulum will continue to rotate about it's support for all time.

Example: Damped Pendulum

We can extend our model of the nonlinear pendulum to account for damping due to air friction by adding a term to the original equations. This leads to

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -\omega^2 \sin x - \gamma y \end{pmatrix}.$$

The overall theme is the same as before. Fist find equilibrium points and then study the stability/nature of the points using the corresponding linear system. With this information we try to draw the nonlinear phase portrait to get the qualitative properties of the full problem.

I. Equilibrium points

To find equilibrium points we set the right-hand side of the differential equation to zero. This leads to

$$\begin{pmatrix} y \\ -\omega^2 \sin x - \gamma y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and so by equating components on either side we get, $y = 0$ and $-\omega^2 \sin x - \gamma y = 0$. Hence we find the same equilibrium points as before,

$$y_0 = 0, \quad x_0 = 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots$$

II. Stability/Nature of equilibrium points

The linearized system near any equilibrium point is given by

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \\ &= \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x_0 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \end{aligned}$$

since $f(x, y) = y$ and $g(x, y) = -\omega^2 \sin x - \gamma y$ here. For the equilibrium points

$$(x_0, y_0) = (\pi, 0), (-\pi, 0), (3\pi, 0), (-3\pi, 0), \ldots$$

that is all odd multiples of π, we obtain the following system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

The system has the eigenvalues

$$\lambda = -\frac{\gamma}{2} \pm \sqrt{\omega^2 + \frac{\gamma^2}{4}}.$$

Hence we have one positive and one negative eigenvalue, or in other words we have a saddle point at these equilibrium points, just as before. Further the eigenvectors are given as $\begin{pmatrix} 1 \\ -\frac{\gamma}{2} + \sqrt{\omega^2 + \frac{\gamma^2}{4}} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -\frac{\gamma}{2} - \sqrt{\omega^2 + \frac{\gamma^2}{4}} \end{pmatrix}$, showing that you have one eigenvector with positive slope and one with negative slope. The qualitative behavior for the system at these equilibrium points is same as before.

For the equilibrium points

$$(x_0, y_0) = (0,0), (2\pi, 0), (-2\pi, 0), \ldots$$

we have the linearized system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

which has eigenvalues

$$-\frac{\gamma}{2} \pm i\sqrt{\omega^2 - \frac{\gamma^2}{4}}.$$

Hence depending on whether $\omega^2 > \gamma^2/4$, we get qualitatively different behavior at these points. The case when $\omega^2 > \gamma^2/4$ is called an underdamped pendulum and leads to a stable spiral equilibrium point (stable since the term $-\gamma/2$ will lead to exponential decay in the solution through $e^{\lambda t}$). The case when $\omega^2 = \gamma^2/4$ leads to a stable improper node (repeated roots) and is called the critically damped pendulum. Lastly, the case when $\omega^2 < \gamma^2/4$ is called the overdamped pendulum and leads to a stable node (both eigenvalues are negative in this case). You should work out the details and convince yourself that these are indeed the cases.

You may have noticed that whether or not the pendulum is over-, critically or underdamped, the points which are even multiples of π are always stable. Given a slight nudge away from these equilibrium points, the solution always returns to these points. The difference is whether the pendulum oscillates about the point, while getting closer (underdamped) or simply comes to rest monotonically (overdamped). Below is a picture of what happens in the underdamped case. Note in this case there is no separatrix since all solutions eventually come to rest at one of the equilibrium points (odd multiple of π means the pendulum is vertically upright whereas even multiple of π means the pendulum is vertically down). Of course, being a saddle point, the vertically upright positiion is unstable.

5.1. **Stability of equilibrium points to nonlinear systems.** So far we have discussed the stability of linearized systems. We used this to infer some details about the nonlinear system. It would be worthwhile to ask whether the conclusions we made about the linear system do indeed hold for the fully nonlinear system. Otherwise everything we did would have been for nothing! Thankfully in almost all cases the stability of the origin in the linearized system will be the same as the stability of equilibrium point in the nonlinear equation. In other words, if the origin ($u = v = 0$) has arrows pointing towards it, that is solutions progressively come closer to origin in time, then the equilibrium point of the nonlinear system will also have this property: solutions that start near the equilibrium will get closer. On the other hand, if the linearized system has an unstable origin, then the nonlinear system will have an unstable equilibrium.

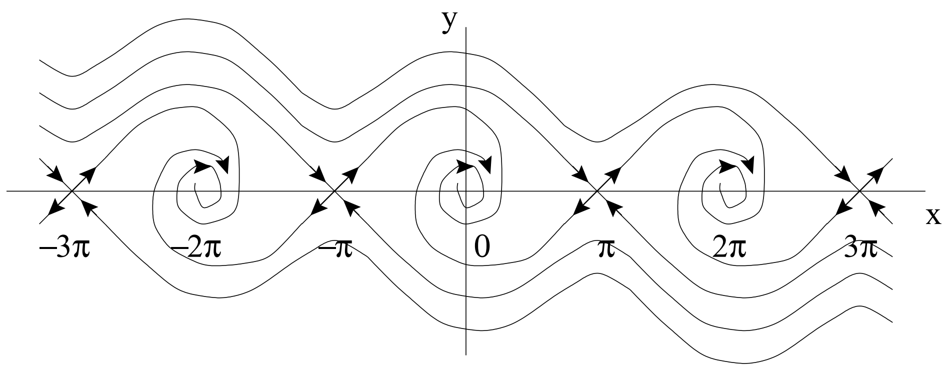


FIGURE 8. Underdamped pendulum. Note absence of separatrix.

The only two cases where the behavior of the linear system does not match that of the nonlinear system are the following two cases

(a) Equal eigenvalues

When the linear system has two real equal eigenvalues both of which are positive, then it may be a proper or improper node depending on whether we needed to find a generalized eigenvector. The nonlinear system in this case will still be unstable, but it need no longer be a node. The nonlinear system could be a unstable node or an unstable spiral point.

Similarly when the linear system has two real equal eigenvalues both of which are negative, then it may again be a proper or improper node. The nonlinear system will then either be a stable node or stable spiral.

(b) Purely imaginary complex conjugate eigenvalues

When the linear system has purely imaginary complex conjugate eigenvalues, it indicates a center type origin for the linearized system. The linear system is stable; near by solutions don't get closer to the origin, but they don't go further away either. The nonlinear system in this case could be a center or a spiral point. Further we have no information on the stability, the nonlinear system could be stable or unstable.

In all cases except the center for a linear system, the word Stable in the following table also means asymptotically stable. That is, in all cases except the center, stable equilibrium points actually bring nearby solutions back to the equilibrium point as time increases. In the case of a center for a linear system, solutions neither come nearer to the equilibrium point, nor go further away.

Table 1. Stability and Instability of linear and nonlinear systems

Eigenvalues	Linear System		Nonlinear System	
	Nature	Stability	Nature	Stability
$\lambda_1 > \lambda_2 > 0$	Node	Unstable	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Stable	Node	Stable
$\lambda_1 < 0 < \lambda_2$	Saddle	Unstable	Saddle	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or Improper Node	Unstable	Node or Spiral Point	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or Improper Node	Stable	Node or Spiral Point	Stable
$\lambda_1, \lambda_2 = a \pm bi$				
$a > 0$	Spiral Point	Unstable	Spiral Point	Unstable
$a < 0$	Spiral Point	Stable	Spiral Point	Stable
$\lambda_1 = ib$, $\lambda_2 = -ib$	Center	Stable	Center or Spiral Point	Indeterminate