Early Ramsey-type Theorems in Combinatorial Number Theory

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Prologue:

- Early Ramsey-type theorems include the theorem of Ramsey; the other results being those of Schur, van der Waerden and Hilbert, which share the credit of preceding the result of Ramsey in the class of Ramsey-type theorems.

- Origins of some of the famous recent developments in mathematics can be traced back to these results.

Planning of this lecture:

- The general philosophy of this theme.
- The pigeonhole principle
  (of which the classical Ramsey theorem is a generalization).
- Some of these theorems.
  Interrelations and generalizations.

General philosophy:

Existence of regular substructures in general combinatorial structures is the phenomena which can be said to characterize the subject of Ramsey theory.

Most often, we shall come across results saying that if a large structure is divided into finitely many parts, at least one of the parts will retain certain regularity properties of the original structure.

In some results in Ramsey theory, ‘large’ substructures will be seen to have certain regularities.

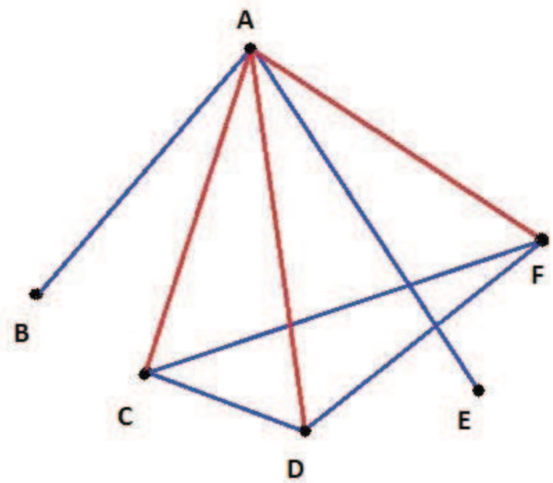
Theodore Motzkin: ”Complete disorder is impossible.”

The pigeonhole principle:

The pigeonhole principle: If $n + 1$ objects are put in n pigeonholes, then there will be a pigeonhole containing more than one object.

A well-known example: If you have a party of at least 6 people, you can guarantee that there will be a group of 3 people who all know each other, or a group of 3 people who all do not know each other.

Proof by graphs:



(joining by red line when two people know each other)

Theorems of Ramsey and Schur:

Ramsey's Theorem (1930):

Let $k, r, l(\geq k)$ be positive integers. Then there exists a positive integer $n = n(k, r, l)$ such that for any r-colouring of the k-subsets of the set $[n]$, there is an l-subset of $[n]$ all of whose k- subsets are of the same colour.

Schur's Theorem (1916):

For any r-colouring of $\mathbf{Z}^+$, $\exists$ a monochromatic subset $\{x, y, z\}$ of $\mathbf{Z}^+$ such that $x + y = z$. (The situation is described by saying that the equation $x + y = z$ has a monochromatic solution.)

Deduction of Schur's Theorem from Ramsey's Theorem:

Let $N = n(2, r, 3)$, where $n(k, r, l)$ is as defined in Ramsey's Theorem above.

An r-colouring $\chi : [N] \to [r]$ induces an r-colouring χ^* of the collection of 2-element subsets of $[N]$:

$$\chi^*(\{i, j\}) = \chi(|i - j|), i \neq j \in [N].$$

By definition, $\exists$ a 3-element subset $\{a, b, c\}$ of $[N]$ with $a < b < c$ such that $\chi^*(\{a, b\}) = \chi^*(\{b, c\}) = \chi^*(\{c, a\})$, that is, $\chi(b - a) = \chi(c - b) = \chi(c - a)$.

Since $(b - a) + (c - b) = (c - a)$, we get a monochromatic solution of $x + y = z$.

van der Waerden's theorem:

Rightly finding its place among the 'pearls' that Kinchin presents in his 'Three pearls of Number theory', the theorem of van der Waerden we are going to state now, led to many interesting developments in combinatorics and number theory.

van der Waerden's theorem (1927): Given $k, r \in \mathbf{Z}^+$, there exists $W(k,r) \in \mathbf{Z}^+$ such that for any r-colouring of $\{1, 2, \cdots W(k,r)\}$, there is a monochromatic arithmetic progression (A.P.) of k terms.

Some generalizations of Schur's theorem:

Schur's Theorem: For any finite colouring of $\mathbf{Z}^+$, $\exists$ a monochromatic subset $\{x, y, z\}$ of $\mathbf{Z}^+$ satisfying the equation

$$x + y = z.$$

Special case of van der Waerden's theorem: For any finite colouring of $\mathbf{Z}^+$, $\exists$ a monochromatic subset $\{x, y, z\}$ of $\mathbf{Z}^+$ satisfying the equation

$$x + z = 2y.$$

Rado's theorem (abridged version). Given an equation

$$c_1x_1 + \cdots + c_nx_n = 0, \qquad c_i(\neq 0) \in \mathbf{Z},$$

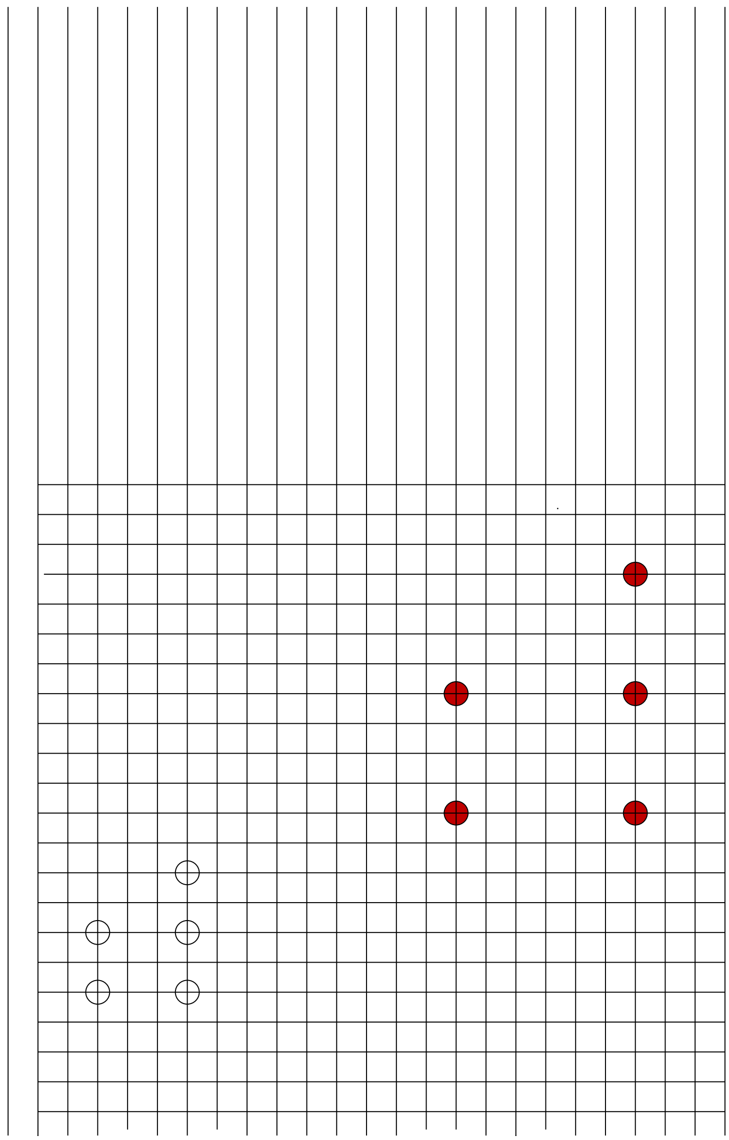
it has a monochromatic solution $(x_1, \cdots, x_n)$ (where x_i's may not be distinct) in $\mathbf{Z}^+$ with respect to any finite colouring if and only if some non-empty subset of $\{c_1, \cdots, c_n\}$ sums to zero.

Folkman's theorem. Given any two positive integers r and k, there is a positive integer $n = n(r, k)$ such that if $[n]$ is r coloured, there are positive integers $a_1 < a_2 < \cdots < a_k$ satisfying $\sum_{1\leq i\leq k} a_i \leq n$ such the elements $\sum_{i\in I} a_i$ are identically coloured as I varies over different non-empty subsets of $\{1, 2, \cdots, k\}$.

Some generalizations of van der Waerden's theorem:

Grünwald's theorem. Let $d, r \in \mathbf{Z}^+$. Then given any finite set $S \subset (\mathbf{Z}^+)^d$, and an r-colouring of $(\mathbf{Z}^+)^d$, there exists a positive integer 'a' and a point 'v' in $(\mathbf{Z}^+)^d$ such that the set $aS + v$ is monochromatic.

Remark.
We note that when $d = 1$, putting $S = \{1, \cdots, k\}$, Grünwald's theorem implies van der Waerden's theorem.



(monochromatic translated homothety)

Hales-Jewett theorem: Let

$$C_t^n = \{x_1 x_2 \cdots x_n : x_i \in \{1, 2, \cdots t\}\}.$$

A combinatorial line in C_t^n is a set of t points in C_t^n ordered as $X_1, X_2, \cdots, X_t$ where

$$X_i = x_{i1} x_{i2} \cdots x_{in}$$

such that for j belonging to a nonempty subset I of $\{1, \cdots n\}$ we have $x_{sj} = s$ for $1 \leq s \leq t$ and $x_{1j} = \cdots = x_{tj} = c_j$ for some $c_j \in \{1, \cdots t\}$ for j belonging to the complement (possibly empty) of I in $\{1, \cdots n\}$.

Hales-Jewett theorem says that given any two positive integers r and t, there exists $n = HJ(r, t)$ such that if C_t^n is r-coloured then there exists a monochromatic combinatorial line.

Remark:

For $t = 3$ and $n = 5$, the following is an example of a combinatorial line in C_3^5:

$$\begin{array}{c} 12123 \\ 22223 \\ 32323 \end{array}$$

We see that the collection of words in C_3^5 are in one-to-one correspondence in the obvious way with a subset of the integers $1, 2, \cdots, 33333$ where the integers have their usual expression in decimal system, that is, with base 10. Thus, the three words in the above combinatorial line correspond to an arithmetic progression with common difference 10100.

Remarks:

The theorem of Hales and Jewett, revealing the combinatorial nature of van der Waerden's theorem would claim that this 'pearl of number theory' belonged to the ancient shore of combinatorics.

Perhaps nothing can better describe the role of this theorem as has been done in the following statement:

"the Hales-Jewett theorem strips van der Waerden's theorem of its unessential elements and reveals the heart of Ramsey theory".

–Graham, Rothschild and Spencer

However, on the other hand, van der Waerden's theorem was a prelude to a theme which essentially belongs to the realm of Number Theory.

The development of this theme, culminating in the theorem of Szemerédi, followed by the ergodic proof of Szemeredi's theorem due to Furstenberg and holding some yet unanswered questions to its bosom.

In the thirties, Erdős and Turan conjectured that for a subset of the set of positive integers, the property of possessing arithmetic progressions of arbitrary length actually depends on the 'size' of the set.

Erdős and Turan conjecture

Any subset of $\mathbf{Z}^+$ with positive upper natural density will have the property. For $A \in \mathbf{Z}^+$, *the upper natural density* $\bar{d}(A)$ of A is defined by

$$\bar{d}(A) = \limsup_{N \to \infty} \frac{|A \cap [N]|}{N}.$$

One can observe that in this regard, the situation is quite different in the case of Schur's theorem For, even though the set of odd integers and the set of even integers have the same upper natural density (namely, $\frac{1}{2}$), the set of odd integers do not possess any solution to the equation $x + y = z$.

The first progress towards the Erdös-Turan conjecture was made by K. F. Roth (1953) who proved that any subset A of the set $\mathbf{Z}^+$ of positive integers with positive upper natural density will always contain a three-term arithmetic progression.

In Szemerédi (1969) improved Roth's result to that of A possessing a four-term arithmetic progression. Later in 1974, in a famous paper Szemerédi proved the general Erdös-Turan conjecture by combinatorial method.

In 1977, Furstenberg gave an ergodic theoretic proof of Szemerédi's theorem which opened up the subject of Ergodic Ramsey Theory.

The following conjecture of Erdős is still open.

If $A \subset \mathbf{Z}^+$ satisfies

$$\sum_{a \in A} \frac{1}{a} = \infty,$$

then A contains arithmetic progressions of arbitrary length.

A theorem of Hilbert

Let T be a continuous map of a topological space X into itself. Then a point $x \in X$ is called a recurrent point for T if for any neighbourhood V of x, $\exists\ n \geq 1$ with $T^n(x) \in V$.

Now, let X be a compact topological space and $T : X \to X$ a continuous map. Let $\mathcal{F}$ denote the family of nonempty closed subsets of X invariant under T.

Ordering by inclusion, we observe that because of compactness of X, by the finite intersection property, the intersection of a totally ordered chain in $\mathcal{F}$ belongs to $\mathcal{F}$. Hence, by Zorn's lemma, $\mathcal{F}$ has a minimal element Y_0.

We claim that each point of Y_0 is a recurrent point for T.

Let $y \in Y_0$ and $Y = \overline{\{T^n(y) : n \geq 1\}}$, the forward orbit closure of y.

Now, Y_0 being T invariant, $\{T^n(y) : n \geq 1\} \subset Y_0$ and therefore, Y_0 being closed, $\overline{\{T^n(y) : n \geq 1\}} \subset Y_0$.

But by definition, Y is nonempty and closed. Further, since T is continuous, Y is invariant under T.

Therefore, $Y \in \mathcal{F}$ and by minimality $Y = Y_0$. Hence $y \in Y$ which means that each neighbourhood of y contains $T^n(y)$ for some $n \geq 1$.

Thus in particular, we have:

For a continuous map T from a compact topological space X into itself, the set of recurrent points for T is nonempty.

For $a, b \in \Lambda$ the metric d defined below gives the discrete topology on Λ.

$$d(a,b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b. \end{cases}$$

The space $\Omega = \Lambda^{\mathbf{Z}^+}$ with the product topology is metrizable. If for $\omega, \omega' \in \Omega$ one defines

$$D(\omega, \omega') = \sum_{n=1}^{\infty} \frac{d\left(\omega(n), \omega'(n)\right)}{2^n},$$

then it is easy to verify that the metric D corresponds to the product topology on Ω. By Tychonoff's theorem, (Ω, D) is therefore compact.

Also the semigroup $\mathbf{Z}^+$ acts on the elements of Λ in the obvious way by shifting. That is by the action of $n \in \mathbf{Z}^+$, an element $\omega \in \Omega$ goes to $\omega' \in \Omega$ where $\omega'(m) = \omega(m+n)$. The map on Ω corresponding to $n = 1$ (which in fact determines the action of the semigroup $\mathbf{Z}^+$) will be called the shift mapand we shall denote it by σ. The map $\sigma : \Omega \to \Omega$ is continuous.

The space Ω endowed with the metric D and the $\mathbf{Z}^+$ action is a symbolic flow in the terminology of Dynamical Systems. In a symbolic flow, by saying that a point is recurrent one means that it is recurrent for the shift map. Λ is sometimes called the alphabet and a finite sequence of elements in Λ is called a word.

It is clear that two points $\omega, \omega' \in \Omega$ are close if they agree on a large block of numbers $(1, 2, \cdots, N)$. Therefore, in a symbolic flow, a sequence $\omega \in \Omega$ is recurrent if and only if every word occurring in ω occurs a second time. We further note that a most general recurrent point ω will look like

$$\begin{aligned} \omega &= [(a\omega^{(1)}a)\omega^{(2)}(a\omega^{(1)}a)]\omega^{(3)} \\ &\quad [(a\omega^{(1)}a)\omega^{(2)}(a\omega^{(1)}a)]\cdots \qquad * \end{aligned}$$

where $a = \omega(1) \in \Lambda$ and $\omega^{(i)}$'s are arbitrary words composed of elements of Λ.

Hilbert's theorem. Given a finite colouring on $\mathbf{Z}^+$ and a positive integer l, one can find l elements $m_1 \leq m_2 \leq \cdots \leq m_l$ in $\mathbf{Z}^+$ such that if $P(m_1, \cdots, m_l)$ denotes the set of sums $\sum_{i=1}^{l} c(i)m_i, \ \ c(i) = 0$ or 1, then infinitely many translates of $P(m_1, \cdots, m_l)$ are of the same colour.

Proof. Let $\chi : \mathbf{Z}^+ \to \{c_1, c_2, \cdots, c_q\}$ be a q-colouring on $\mathbf{Z}^+$. Let $\Lambda = \{1, 2, \cdots, q\}$ and Ω be defined as above. We consider the element $\xi \in \Omega$ where

$$\xi(n) = i \iff n \in \chi^{-1}(c_i), i = 1, 2 \cdots q.$$

Case I (ξ is a recurrent point)

Let ξ has the form given in (*) and

$$\begin{aligned} \omega_0 &= a \\ \omega_1 &= \omega_0\omega^{(1)}\omega_0 \\ \omega_2 &= \omega_1\omega^{(2)}\omega_1 \\ \cdots & \quad \cdots \\ \omega_n &= \omega_{n-1}\omega^{(n)}\omega_{n-1} \end{aligned}$$

Now, denoting the length of the word $\omega_n\omega^{(n+1)}$ by m_{n+1}, we see that if some symbol occurs at position p in w_n, then it occurs at positions p and $p+m_{n+1}$ in $\omega_{n+1} = \omega_n\omega^{(n+1)}\omega_n$. Thus the symbol a occurs at positions $1, 1+m_1, 1+m_2, 1+m_1+m_2$ and in general at positions belonging to $1+P(m_1, \cdots m_l)$ for any l. Since every finite configuration occurs infinitely often in ξ, it is now clear that $\chi^{-1}(a)$ contains infinitely many translates of $P(m_1, \cdots m_l)$.

Case II (ξ is not a recurrent point) In this case we consider the forward orbit closure X of ξ in Ω. The shift operator takes points of X to X and therefore there is a recurrent point say w for the shift operator σ in X. Let w be of the form given in (*). Therefore, there exists a sequence of positive integers $\{n_k\}$ such that $\sigma^{n_k}(\xi) \to w$. If a is the leading symbol in w, then arguing as before, a occurs at positions belonging to $1 + P(m_1, \cdots m_l)$. Choose k such that $\sigma^{n_k}(\xi)$ agrees with w for $(1+m_1+\cdots+m_l)$ terms. Then $\xi(n_k+p) = a$ whenever $p \in (1 + P(m_1, \cdots m_l))$. One can assume that $n_k \to \infty$. For, otherwise, a finite translate of ξ would be recurrent and one could then invoke the first case. Therefore we obtain that $1 + n_k + P(m_1, \cdots m_l) \in \chi^{-1}(a)$ for a sequence $\{n_k\}$ where $n_k \to \infty$ and this proves the theorem.