1. Let S and T be subset of real numbers $\mathbb{R}$. Suppose $f : S \to T$

$$y = f(x) = \begin{cases} x & \text{if} \quad 0 \le x \le 1 \\ 2 - x & \text{if} \quad 1 \le x \le 3 \\ x - 4 & \text{if} \quad 3 \le x \le 4. \end{cases}$$

   (a) Find the domain S, range T of the function f and Draw its graph.
   (b) Graph each of the following on the respective domains:-

   (i) $3 + f(x-2)$, (ii) $1 - f(x+2)$, (iii) $f(4x)$, (iv) $f(\frac{x}{4})$, (v) $f(-x)$, (vi) $f(1-2x)$.

   (c) Find the domain D of the following:

   (i) $\sqrt{f(x)}$, (ii) $\frac{1}{f(x)}$, (iii) $\sqrt{2f(x+3)}$,

2. Express each of the following functions in terms of $f(t)$, where f is defined as above with x replaced by t. (Your answer should be of the form $y = Af(B(t - C)) + D$)

   (a) Naina is attending ICTS SWMS-2034 and she tries without success to find an open canteen for a midnight snack. Leaving the campus at time t = 0, she walks East at the constant speed of 3 km per hour for an hour, then return past the campus, walks the same distance to the West, and finally return to ICTS campus, hungry and despondent (always walking at the same speed). Let y be Naina's distance from ICTS campus at time t (with East taken as the positive direction).
   (b) Madurai never seemed to sleep, so its stock exchange decided to be open for 24 hours trading. The stock market's MSE index on Monday (time t = 0) is at 3400. On Tuesday it goes up 30 points, then on Wednesday and Thursday it drops 30 points each day, and finally it closes out the week at 3400.
   (c) On the 5th through 9th day, the number of people staying in ICTS campus is at 70, 90, 70, 50, and 70, respectively.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 5x + 6$. Find the domain of the function

$$z = \frac{1}{f(x) - 2} + \sqrt{f(x)}$$

4. *Extra Credit:*[1] Let A and B be two non-empty sets. Define what is meant by a function $f : A \to B$ and when is it called one-one and onto. Provide examples of f that are (and are not) one-one and (or) onto when A, B are finite or countable or uncountable.

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SWMS 2019, ICTS Bangalore, May 23, 2019

A road to the infinities: Some topics in set theory

Sujata Ghosh
ISI Chennai
sujata@isichennai.res.in

Acknowledgement: An ESSLLI course of Benedikt Löwe and Grzegorz Plebanek

Two protagonists in set theory

- Cardinal numbers
- Ordinal numbers

Two protagonists in set theory

- Cardinal numbers

  Measuring the size of infinity and comparing the sizes of infinite sets.

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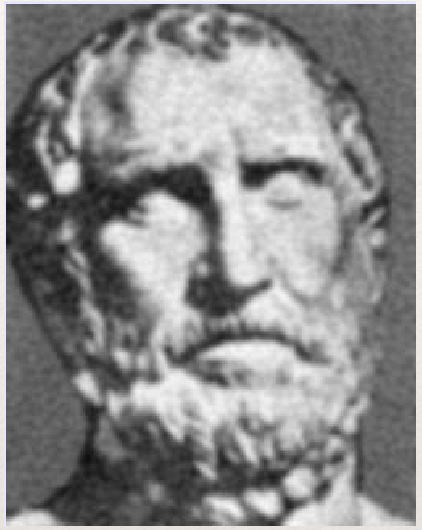
Until the 19th century, infinity has been considered to be a rather problematic concept

Achilles and the tortoise

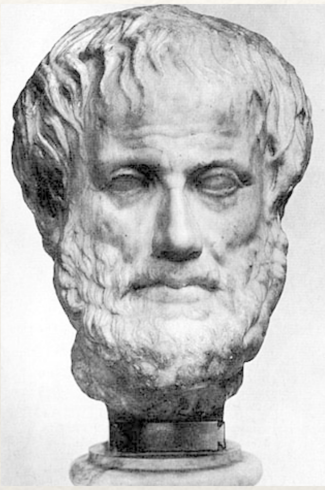


Zeno of Elea, c. 490 B.C. - c. 430 B.C.

The argument says that it is impossible for [Achilles] to overtake the tortoise when pursuing it. For in fact it is necessary that what is to overtake [something], before overtaking [it], first reach the limit from which what is fleeing set forth. In [the time in] which what is pursuing arrives at this, what is fleeing will advance a certain interval ... And in the time again in which what is pursuing will traverse this [interval] which what is fleeing advanced, in this time again what is fleeing will traverse some amount ...

Simplicius, On Aristotle's Physics 6, 1014.10

Aristotle and the Actual Infinite



Aristotle, c. 384 B.C. - c. 322 B.C.

For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different.

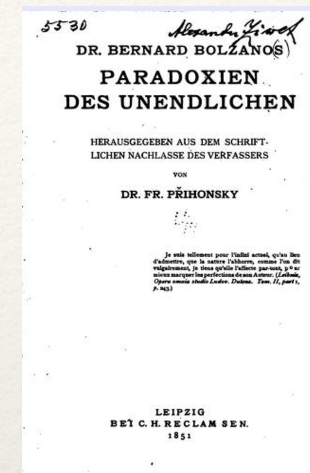
Aristotle, Physica, III.6

For the fact that the process of dividing never comes to an end ensures that this activity exists potentially, but not that the infinite exists separately.

Aristotle, Metaphysica, IX.6

Paradoxien des Unendlichen

Bernard Bolzano, 1781 - 1848


DR. BERNARD BOLZANO(S)

PARADOXIEN
DES UNENDLICHEN

HERAUSGEGEBEN AUS DEM SCHRIFT-
LICHEN NACHLASSE DES VERFASSERS

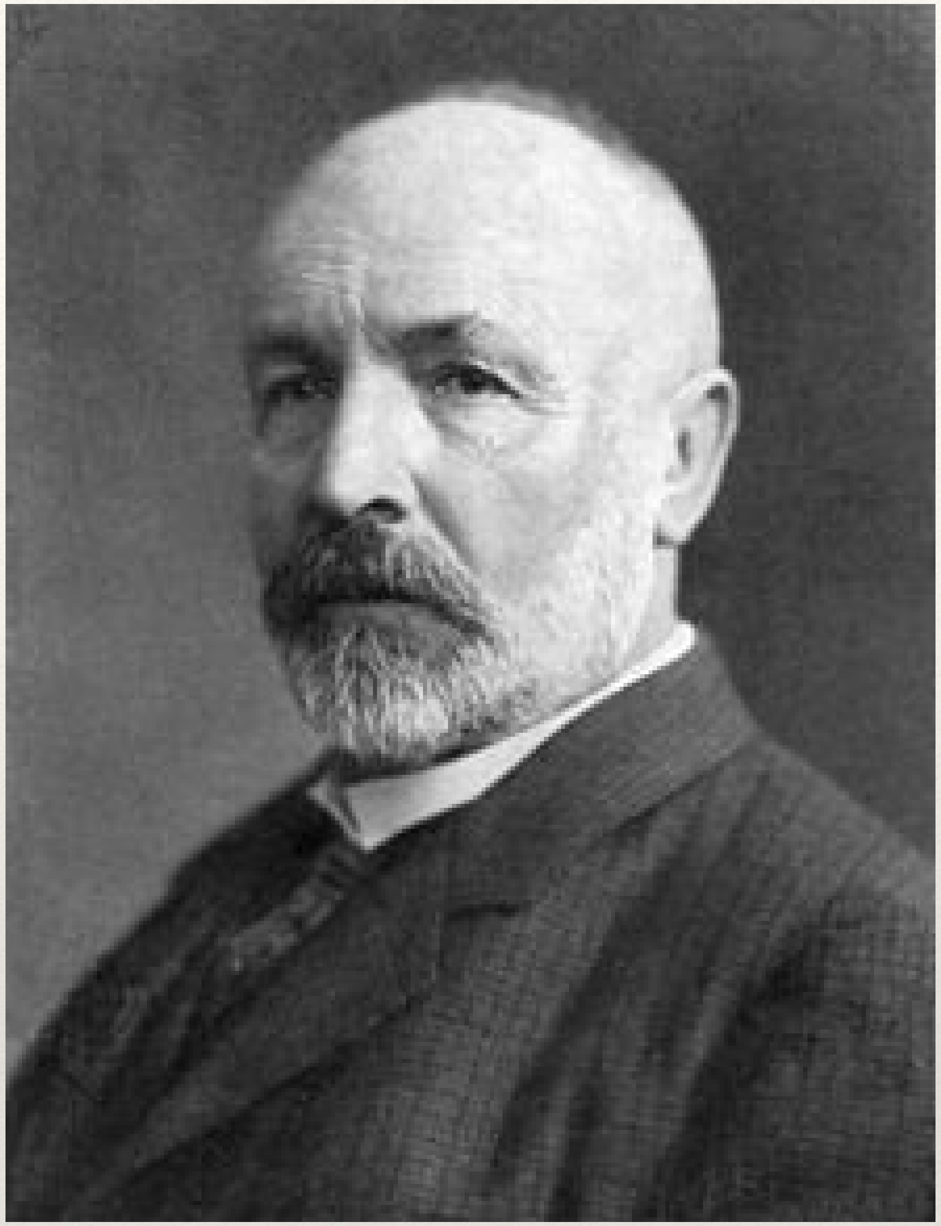
VON

DR. FR. PŘIHONSKY

Je suis tellement pour l'infini actuel, qu'au lieu d'admettre, que la nature l'abhorre, comme l'on dit vulgairement, je tiens qu'elle l'affecte par-tout, pour mieux marquer les perfections de son Auteur. (*Leibniz, Opera omnia studio Ludov. Dutens. Tom. II, part 1, p. 243.*)

LEIPZIG
BEI C. H. RECLAM SEN.
1851

- 18 Not every magnitude that is a sum of infinitely many finite magnitudes is itself infinite.
- 20 A remarkable relationship between two infinite sets: it is possible to pair each object of the first set to one of the second such that every object in the two sets has a unique partner.
- 21 And this situation can occur even if one of the sets is a proper subset of the other.



Georg Cantor, 1845 - 1918

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  Measuring the size of infinity and comparing the sizes of infinite sets.

- Ordinal numbers

  Counting beyond infinity and providing the means of exhausting infinite sets.

- ## Cardinal numbers

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Cantor, Georg (1872). Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen, *Mathematische Annalen*, 5, 123–132.

The dual nature of set theory

- Historically, set theory started as a field of mathematics: the study of infinite sets and their relationships.

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- Historically, set theory started as a field of mathematics: the study of infinite sets and their relationships.
- In subsequent years, set theory developed into more than that: the standard foundations for mathematics.

Ernst Zermelo

1871-1953

Abraham Fraenkel

1891-1965

Thoralf Skolem

1887-1963

John von Neumann

1903-1957

Some axioms of Zermelo-Fraenkel (ZF)

- **Union:** For any sets X and Y there is a set, denoted $X \cup Y$, containing all elements of X and all elements of Y.
- **Separation:** If A is a set and φ is some property then there is a set $\{x \in A : \varphi(x)\}$.
- **Power set:** For every set X there is a set $\{A : A \subseteq X\}$ (denoted $P(X)$).
- **Infinity:** There is an infinite set.

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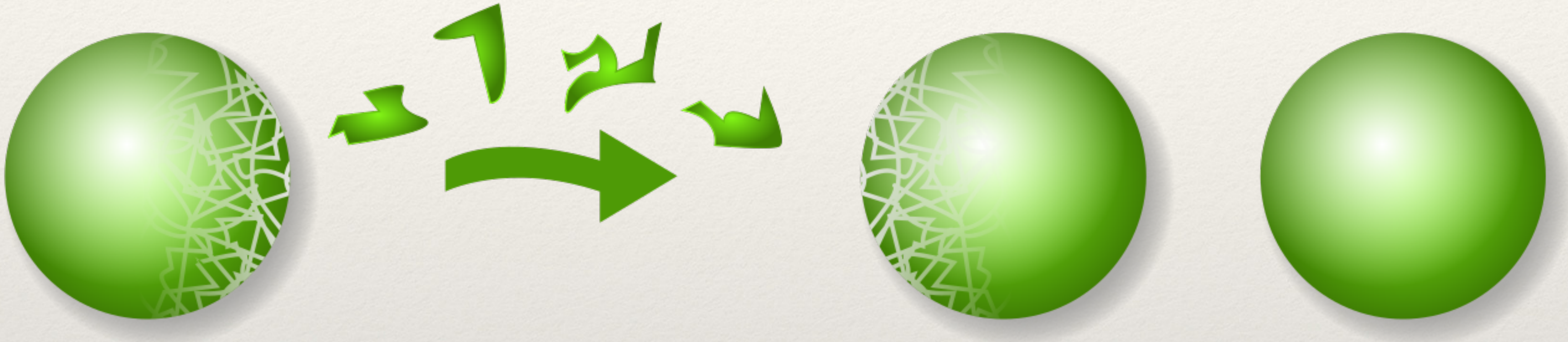
Axiom of Choice (AC)

ZFC = ZF + AC

For every family $\mathcal{A}$ of nonempty sets there is a choice function f, such that $f(A) \in A$ for every $A \in \mathcal{A}$.

Banach-Tarski Paradox



The ball of radius 1 (in $\mathbb{R}^3$) can be, by AC, decomposed into 5 pieces. Using those sets one can, using rotations and translations, form two balls of radius 1.

Measuring the infinite ……

Many infinite sets in mathematics

- the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots, 2018, 2019, \ldots\}$
- the set of rational numbers (all quotients), $\mathbb{Q} = \{0, 1, 2, 2/3, 7/8, \ldots.\}$
- the set of all reals, $\mathbb{R}$ (including $\mathbb{Q}$ and many others)

The natural numbers

- $\mathbb{N}$ is the 'smallest' infinite set
- It is infinite but 'countable'
- Theoretically, we can imagine naming all its elements

Countable Set

We say that a set A is **countable** if we can write

$$A = \{a_1, a_2, \ldots, a_n, \ldots\}$$

where $a_1, a_2, \ldots$ are all distinct.

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Aleph Zero

The infinity represented by $\mathbb{N}$ is denoted by $\aleph_0$; we write

$$|\mathbb{N}| = \aleph_0.$$

Having introduced $\aleph_0$, we can write $|A| = \aleph_0$ instead of saying that A has as many elements as $\mathbb{N}$.

Why Aleph ?   Why Aleph indexed with zero ?

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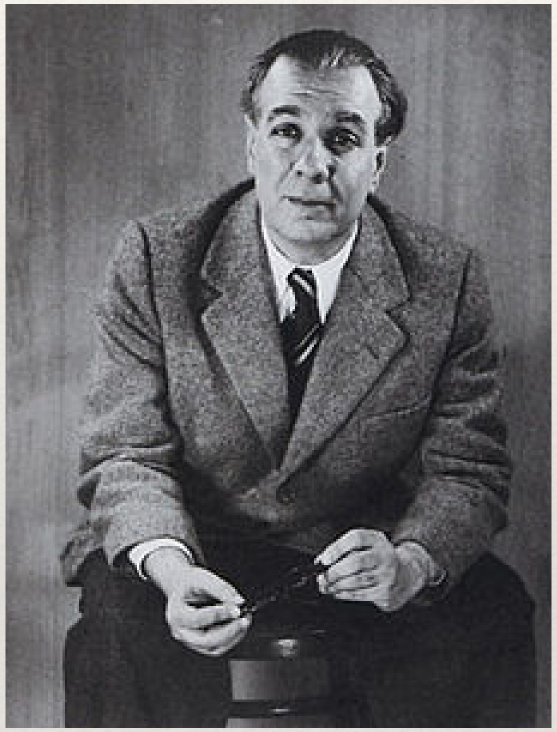
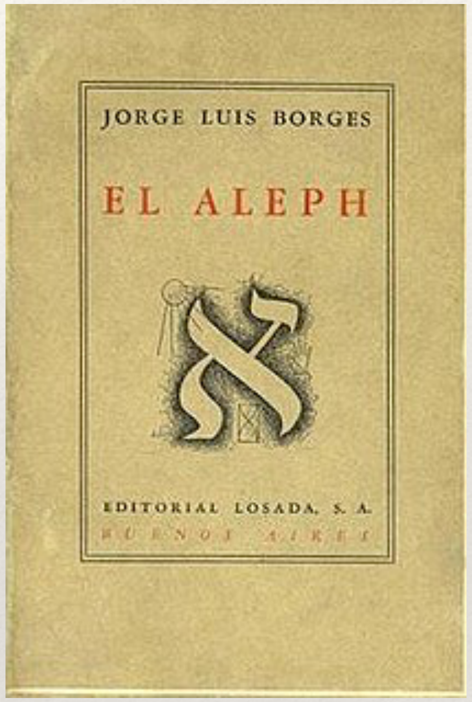


Jorges Luis Borges, 1899-1986



Hotel $\aleph_0$

You are the owner of a hotel having inifnitely many rooms (numbered 1,2,...). Therefore if one day you have infinitely many guests $g_1, g_2, \ldots, g_n, \ldots$ then you can provide accomodation for all of them. Late in the evening another guest arrives? No problem:

$$g_1 \to 2, \quad g_2 \to 3, \quad \ldots, g_n \to n+1.$$

You will have the room no 1 free for the late guest.
Next day you face another infinite group of tourists $h_1, h_2, \ldots, h_n, \ldots$. Still no problem:

$$g_1 \to 2, \quad g_2 \to 4, \ldots, g_n \to 2n \ldots.$$

This makes all the rooms with odd numbers free, and

$$h_1 \to 1, \quad h_2 \to 3, \ldots, h_n \to 2n-1 \ldots.$$

“Paradoxes of the infinite arise only when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited”

– *Galileo*

Paradoxes, or rather theorems !

$$\aleph_0 + 1 = \aleph_0$$

Paradoxes, or rather theorems !

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Properties of Countable Sets

- If A and B are countable then $A \cup B$ is countable.
- If A and B are countable then $A \times B$ is countable, too, where

$$A \times B = \{\langle a, b \rangle : a \in A, b \in B\}.$$

- If $A_1, A_2, \ldots$ are all countable then the set

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containing all elements of all those sets, is countable too.

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Theorem

The set $\mathbb{Q}$ of rational numbers is countable: $|\mathbb{Q}| = \aleph_0$

Theorem

The set $\mathbb{R}$ of all real numbers is not countable

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Proof

In fact we shall check that already the interval $[0,1]$ is not countable.
Suppose that we have managed to create a list $a_1, a_2, \ldots$ of all real numbers $x \in [0,1]$.

number	0.	1st	2nd	3rd	...	nth	...
a_1	0.	? x_1					
a_2	0.		? x_2				
a_3	0.			? x_3			
...	0.						
a_n	0.					? x_n	
...	0.						

The number $0.x_1x_2\ldots x_n\ldots$ is not on our list!

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The cardinality of $\mathbb{R}$ is called **continuum** and denoted by $\mathfrak{c}$:

$$|\mathbb{R}| = \mathfrak{c}.$$

Comparing arbitrary sets

- We say that two sets X and Y are **equinumerous** if there is a bijection $f : X \to Y$, that is one-to-one correspondence between all elements of X and all elements of Y.
- Equinumerous sets have the same cardinality: $|X| = |Y|$.
- Note that a set X is countable if it is equinumerous with $\mathbb{N}$.

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Examples

- Every two nonempty intervals (a, b) and (c, d) on the real line are equinumerous and have cardinality $\mathfrak{c}$.
- **Theorem.** The plane $\mathbb{R} \times \mathbb{R}$ is equinumerous with $\mathbb{R}$.
- All the Euclidean spaces $\mathbb{R}^1, \mathbb{R}^2, \dots, \mathbb{R}^d, \dots$ have cardinality $\mathfrak{c}$.

Comparing arbitrary sets (contd.)

- $|X| \leq |Y|$ if there is a one-to-one function $f : X \to Y$, that is a bijection between X and some part of Y.
- $|X| < |Y|$ if $|X| \leq |Y|$ but $|X| \neq |Y|$.

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Cantor-Shröder-Bernstein Theorem

If $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$.

Power sets and their cardinalities

If X is any set we denote by $P(X)$ the power set of X, that is the family of all subsets of X.

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If $X = \{1, 2, \ldots, n\}$ then $P(X)$ has 2^n elements.

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If X is a set of cardinality κ then 2^κ denotes the cardinality of $P(X)$.

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If X is a set of cardinality κ then 2^κ denotes the cardinality of $P(X)$.

For a finite set X we have $2^{|X|} > |X|$ since $2^n > n$.

Cantor's theorem

For every set X the power set $P(X)$ has more elements than X; in other words

$$2^{\kappa} > \kappa,$$

for any cardinal number.

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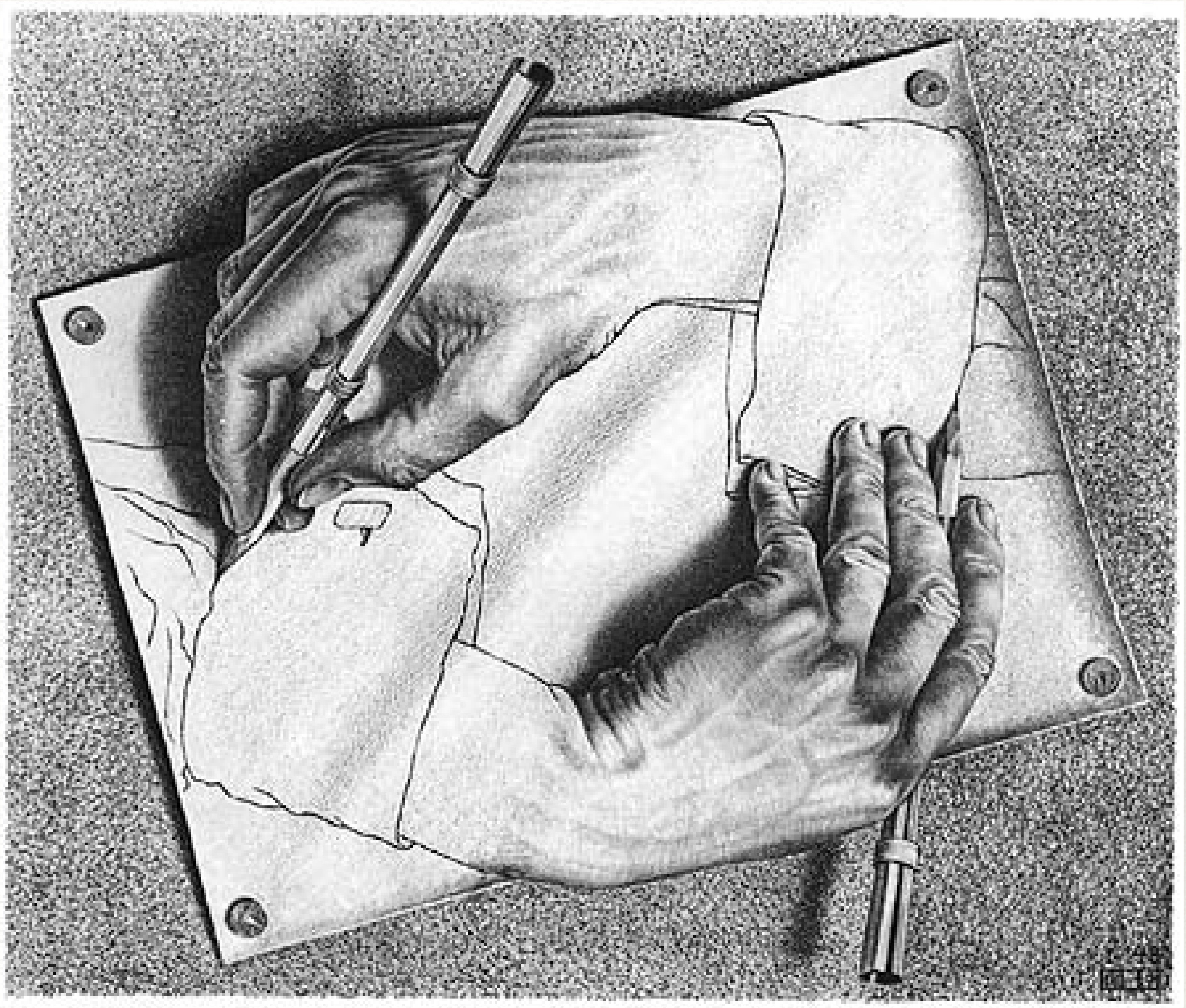
We have $|X| \leq |P(X)|$ since we can define one-to-one function $f : X \to P(X)$ by $f(x) = \{x\}$.
Suppose that $g : X \to P(X)$ is a bijection. Consider the set $A \subseteq X$, where

$$A = \{x \in X : x \notin g(x)\}.$$

Then A cannot be associated with any $x \in X$. If we suppose that $A = g(x_0)$ then we have a puzzle whether x_0 is in A or not:

- if $x_0 \in A$ then $x_0 \notin g(x_0) = A$;
- if $x_0 \notin A$ then $x_0 \in g(x_0) = A$,

a contradiction!. $\square$



‘Drawing hands’ by Maurits Cornelis Escher

What we have!

- $\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \ldots$;
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- there are **infinitely** many kinds of infinity;
- there is no set X which is the biggest one.

Theorem

$\mathfrak{c} = 2^{\aleph_0}$

Proof

Note that $2^{\aleph_0}$ (by the definition the cardinality of $P(\mathbb{N})$) is the cardinality of the set $\{0,1\}^{\mathbb{N}}$ of all infinite sequences of of 0's and 1's.

The function $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$, where

$$f(x_1, x_2, \ldots) = \sum_{n=1}^{\infty} \frac{2x_n}{3^n},$$

is one-to-one. It follows that $2^{\aleph_0} \leq \mathfrak{c}$.

Every $x \in [0,1]$ has a unique **infinite** binary expansion

$$x = (0, x_1 x_2 \ldots)_{(2)}.$$

This shows that $[0,1]$ admits one-to-one function into $\{0,1\}^{\mathbb{N}}$, and $\mathfrak{c} = |[0,1]| \leq 2^{\aleph_0}$. Finally $\mathfrak{c} = 2^{\aleph_0}$ by the Cantor-Bernstein theorem.

Now for something slightly different. Have you ever counted up to 1000?

$$1, 2, 3, \ldots, 1000.$$

It takes more than 16 minutes but surely we can do it.
We can also imagine ourselves counting up to $10^{10^{10}}$ though it will be really time-consuming.
Can we count beyond infinity? If so we need a new name:

Where are we going ?

Having ω (sometimes denoted ω_0) at hand we can continue:

$$0, 1, 2, \ldots, 2019, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega+\omega, \ldots$$

Let's do things a bit formally !

Linearly ordered sets

We say that a set X is **linearly ordered** by $<$ if for any $x, y, z \in X$

- $x \not< x$;
- $x < y$ and $y < z$ imply $x < z$;
- if $x \neq y$ then $x < y$ or $y < x$.

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Examples

The set $\mathbb{R}$ of reals is linearly ordered by the 'natural' order.
All words are linearly ordered by the lexicographic order.

Well-ordered sets

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Two well-ordered sets $(X,<)$ and $(Y,<)$ are **isomorphic** if there is a bijection $f: X \to Y$ such that

- $x_1 < x_2$ is equivalent to $f(x_1) < f(x_2)$;

for any $x_1, x_2 \in X$.

Initial segments

If X is well-ordered and $a \in X$ then the set $\{x \in X : x < a\}$ is called **the initial segment** of X given by a.

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Theorem

Let $(X, <)$ and $(Y, <)$ be two well-ordered sets. Then either

1. *X and Y are isomorphic, or*
2. *X is isomorphic to some initial segment of Y, or*
3. *Y is isomorphic to some initial segment of X.*

Ordinal numbers

An **ordinal number** is the order type of some well-ordered set.

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- 0 is the order type of the empty set;
- 1 is the order type of a set consisting of one element;
- $\omega = \omega_0$ is the order type of $\{0, 1, 2, \ldots\}$;

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ω_1 is the least order type of a well-ordered uncountable set.

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We have $\alpha < \omega_1$ whenever α is an order type of a countable set. We may think that $\omega_1 = \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \alpha, \ldots\}$ is the set of all order types of countable sets.

Ordinal and cardinal numbers

- An ordinal number α is a cardinal number if for every $\beta < \alpha$ we have $|\beta| < |\alpha|$.
- $0, 1, 2, \ldots$ are cardinal numbers.
- ω is a cardinal number (denoted $\aleph_0$).
- $\omega + 1$, $\omega + \omega$ are not cardinal numbers.
- ω_1 is the next cardinal number denoted as $\aleph_1$.
- ω_2 is the least order type of a set of cardinality $> \aleph_1$; $\aleph_2 = \omega_2$.
- We can define $\aleph_0 < \aleph_1 < \aleph_2 < \ldots$.
- Then $\aleph_\omega$ comes. **And so on . . .** Do you understand?[a]

[a] In mathematics, you don't understand things. You just get used to them. (John von Neumann)

Summing up!

- We have an exact list of cardinal numbers $\aleph_0 < \aleph_1 < \aleph_2 < \dots$.
- Before we defined another list $\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \dots$.
- We also considered $\mathfrak{c}$ — the cardinality of $\mathbb{R}$.

The Continuum Hypothesis

- $\aleph_1$ is the least cardinal greater than $\aleph_0$.
- $\mathfrak{c}$ is the cardinality of the real line $\mathbb{R}$.
- It is not obvious at all that there is **any** relation between $\aleph_1$ and $\mathfrak{c}$, as we do not know whether there is a cardinal that is equinumerous to $\mathbb{R}$ [redacted]
- If we assume that $\mathfrak{c}$ is a **cardinal** and not just a cardinality, then we know that $\mathfrak{c} \geq \aleph_1$ since cardinals are linearly ordered.
- Cantor conjectured (in 1877) that in fact $\mathfrak{c} = \aleph_1$. This statement is called **the Continuum Hypothesis** (CH).
- CH was the first problem on the famous Hilbert list (1900).
- In 1938, Kurt Gödel proved that there is a model of set theory in which CH holds. [redacted]
- In 1963, Paul Cohen proved that you cannot prove CH. In fact, for any $n \geq 1$, the statement $\mathfrak{c} = \aleph_n$ is consistent. With a few exceptions (e.g., $\aleph_\omega$), $\mathfrak{c}$ can be any $\aleph_\alpha$.

"No one shall expel us from the paradise that Cantor has created."

– David Hilbert

"The essence of mathematics lies entirely in its freedom."

– Georg Cantor