The number e

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What is e*?*

What is $\mathbf{e}$*?*

Is this just a number

$$2.718281828459045\cdots$$

or something more?

The first hint of e

e *in financial matters*

How much time will it take for a sum of money to double if invested at 20% interest rate compounded annually?

Mesopotamia ; around 1700BC

How will you solve this?

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$$(1.2)^t = 2.$$

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Babylonians did not know logarithms.

Nevertheless, they could calculate t=3.7870 remarkably well.

A question on compound interest

If a principal amount P is invested at an interest rate of $r\%$ compounded annually, then what will be the amount after t years?

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If a principal amount P is invested at an interest rate of $r\%$ compounded annually, then what will be the amount after t years?

$$P\left(1+\frac{r}{100}\right)^t.$$

Let us take $P = 1$, $r = 100$ and $t = 1$. Then the amount after 1 year is 2.

If the amount is compounded twice, each time at the interest rate of 50%, the amount obtained at the end of 1 year will be

$$S_2 = \left(1 + \frac{1}{2}\right)^2 = 2.25.$$

What happens if compounded three times, with the interest rate of $100/3\%$ each time?

$$S_3 = \left(1 + \frac{1}{3}\right)^3 = 2.370.$$

If the interest is compounded n times with interest rate of $100/n\%$ each time?

$$S_n = \left(1 + \frac{1}{n}\right)^n.$$

n	10	100	1000	10,000	1,00,000
S_n	2.5937	2.7048	2.7169	2.7181	2.7182

Observations:

- The amount obtained increases as n increases.
- But the difference between S_n's decreases as n becomes large and large.

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- The amount obtained increases as n increases.
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What will be the amount if compounded at every instant in this manner?

This will be the amount

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

And this limit is what we call the number e.

The number e

$$\mathrm{e} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n}.$$

Does the limit exist?

How to prove it exists?

Let's see...

Binomial theorem:

$$(1+x)^n = 1 + \binom{n}{1}x + \cdots + x^n.$$

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} \\ &= 2 + \sum_{k=2}^{n}\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right).\end{aligned}$$

Now for a fixed k

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \to 1 \text{ as } n \to \infty.$$

Definition of e

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\sum_{k=0}^{\infty}\frac{1}{k!}.$$

This limit is denoted by the constant e.

$\sum_{k=0}^{\infty}\frac{1}{k!}$ converges;

$$\frac{1}{k!}\leq\frac{1}{2^{k-1}}\quad k\geq 2.$$

Logarithm and John Napier

Publicly propounded Logarithms in 1614.

To ease trigonometric computations.

John Napier ('1550-1617)

Observation: For a fixed number a,

$$a^m a^n = a^{m+n}.$$

So the problem of multiplication and division can be reduced to that of addition and subtraction.

$$y = \text{Naplog}\, x \text{ if } x = 10^7(1 - 10^{-7})^y.$$

Henry Briggs introduced the base 10 logarithms in 1617.

Squaring a hyperbola

The problem of finding *area* of a planar shape is called the *quadrature* or *squaring*.

The problem of squaring different conic sections dates back to around 200BC.

Greeks, in particular, Archemedes calculated the areas of parabola and circle through his method of exhaustion.

But the area of a hyperbola could not be computed.

Coming to the early seventeenth century

Pierre de Fermat

• Computed the area of the region bounded by the curves $y = x^n$, $n = 1, 2, \ldots$ and the vertical lines $x = 0$ and $x = a$.

Pierre de Fermat (1601-1665)

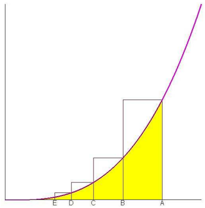
Area of the region bounded by $y = x^n$, $x = 0$ and $x = a$.

The total area of the rectangles=

$$a(1-r)\cdot a^n + ar(1-r)\cdot(ar)^n + \cdots + ar^k(1-r)\cdot(ar^k)^n + \cdots$$
$$= a^{n+1}\frac{1-r}{1-r^{n+1}}$$
$$= \frac{a^{n+1}}{1+r+r^2+\cdots+r^n}.$$

This is true for all $0 < r < 1$. So as $r \to 1$, we get the required area.

The required area is $\frac{a^{n+1}}{n+1}$.

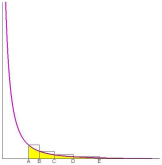


Area under the curve $y = x^3$

$(a = 4,\ r = 3/4)$.
A-$(a, 0)$, B-$(ar, 0)$, C-$(ar^2, 0)$...

Similarly, the area of the region bounded by the curves $y = x^{-n}$, $x \geq a$ $(a > 0)$ can be computed provided $n \neq 1$.

The required area is $\frac{a^{-n+1}}{-n+1}$.



Area under the curve $y = 1/x^2$

$(a = 1,\ r = 4/3)$.
A-$(a, 0)$, B-$(ar, 0)$,
C-$(ar^2, 0)$...

What is the problem with $n = 1$?

The curve when $n = 1$ is the *rectangular hyperbola*.

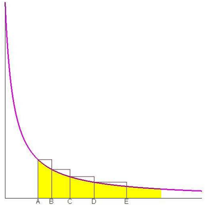
A look at the hyperbola

Let $A(t)$ be the area of the region bounded by the curves $xy = 1$, $x = 0$ and $x = t$.

Let's look at the area of each rectangle shown in the figure:

width=$r^k(r-1)$.
Height=$\frac{1}{r^k}$.
Area=$r-1$
(constant for each rectangle).

This happens for all $r > 1$ even if $r \to 1$.



Area under the curve $y = 1/x$

($r = 4/3$.)
A-$(1,0)$, B-$(r,0)$, C-$(r^2,0)$...

The appearance of logarithm

Observation

As t increases geometrically, $A(t)$ increases arithmetically. In other words,

$$A(st) = A(s) + A(t).$$

Saint Vincent

This is the characteristic property of *logarithmic function*.

A. A. de Sarasa

Hence $A(t) = \log_b t$ for some positive number b.

$A(t)$ is independent of the base of the logarithm.

The base b should be unique.

Natural logarithm and e

$$b \text{ is the unique number } t \text{ such that } \int_1^t \frac{1}{x}\mathrm{d}x = 1.$$

This base is e and the corresponding logarithm is called the natural logarithm.

The term *natural logarithm* was introduced by *Nicholas Mercator* in 1668.

The series for e and the work of Newton (1671)

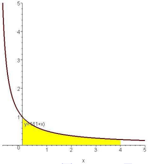
$$\frac{1}{1+x} = 1 - x + x^2 - \cdots .$$

On integration

$$\int_0^x \frac{\mathrm{d}t}{1+t} = \int_0^x \left(1 - t + t^2 - \cdots\right) \mathrm{d}t$$

Now $\int\limits_1^x \frac{1}{t}\mathrm{d}t = \log_b x$. So

$$\int\limits_0^x \frac{1}{1+t}\mathrm{d}t = \log_b(1+x).$$



Hence

$$\log_b(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots .$$

Newton's method of series inversion (Newton 1671):

Let y denote the inverse function of $\log_b(1+x)$, i.e., $y = \log_b(1+x) \Leftrightarrow b^y - 1 = x$.

$$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \tag{1}$$

Solve for y in terms of x. Equivalently put

$$x = a_0 + a_1 y + a_2 y^2 + \cdots$$

and substitute this value in the right-hand side of (1),

$$y = (a_0 + a_1 y + \cdots) - \frac{(a_0 + a_1 y + \cdots)^2}{2} + \frac{(a_0 + a_1 y + \cdots)^3}{3} - \cdots .$$

Compare the coefficients of powers of y on both sides.

Comparing the constant term:

$$0 = a_0 - \frac{a_0^2}{2} + \frac{a_0^3}{3} - \cdots = \log_b(1 + a_0)$$

which implies $a_0 = 0$.

Comparing the coefficient of y,

$$1 = a_1.$$

Comparing the coefficient of y^2,

$$0 = a_2 - \frac{a_1^2}{2} = a_2 - \frac{1}{2}.$$

Comparing the coefficient of y^3,

$$0 = a_3 - \frac{2a_2 a_1}{2} + \frac{a_1^3}{3} = a_3 - \frac{1}{2} + \frac{1}{3} = a_3 - \frac{1}{6}.$$

Newton found some initial terms and inferred the remaining coefficients by analogy.

The number e

$$b^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots .$$

For $y = 1$ this is the series for e.

Hence $b = \mathrm{e}$.

$$\mathrm{e}^y = 1 + y + \frac{y^2}{2!} + \cdots + \frac{y^n}{n!} + \cdots$$

Derivative of an exponential function

First let's understand the function e^x, x a real number.

b a positive real number. What is b^x?

- $x = n$, a nonnegative integer: $b^n = b \cdots b$(n times), and $b^0 = 1$.
- $x = -n$, n a natural number: $b^{-n} = \frac{1}{b} \cdots \frac{1}{b}$($n$ times).
- $x = \frac{m}{n}$, a rational number, $n > 0$: $b^{m/n} = \sqrt[n]{b^m}$.
- $x = w.d_1 d_2 \cdots$ an irrational number:

$$x_n = w.d_1 \cdots d_n, \quad b^x = \lim_{n \to \infty} b^{x_n}.$$

Derivative of b^x

$$\frac{\mathrm{d}}{\mathrm{d}x} b^x = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = b^x \lim_{h \to 0} \frac{b^h - 1}{h} = k b^x,$$

where $k = \lim_{h \to 0} \frac{b^h - 1}{h}$ is a constant independent of x but dependent on b.

The “natural ”exponent e

Derivative of any exponential function is a constant multiple of the function itself.

The best case: $k = 1$.

$$k = 1 \Leftrightarrow b = \mathrm{e}.$$

How do we see this?

Claim: $\lim_{h \to 0} \frac{b^h - 1}{h} = 1$ iff $b = \mathrm{e}$.

For small enough h, $\frac{b^h-1}{h}$ is close enough to 1, i.e., for small enough h,

$$\frac{b^h - 1}{h} \approx 1 \implies b \approx (1+h)^{1/h}.$$

Hence

$$\lim_{h \to 0} (1+h)^{1/h} = b.$$

This gives

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = b.$$

We know the left hand limit is e.

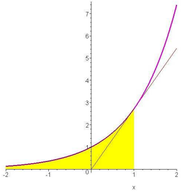
Hence $b = \mathrm{e}$.

The exponential function e^x

e^x is the only function (upto a constant multiple) that is equal to its derivative and hence also equal to its definite integral.

$$f'(x) = f(x) \Leftrightarrow f(x) = ce^x$$

for some constant c.



The notation e for the natural exponent

Euler introduced the symbol **e** for the natural exponent.

He used e in a letter to Christian Goldbach in 1731.

Leonhard Euler (1707-83)

Is e an integer?

$$\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

Certainly $\mathrm{e} > 2$.

$$\begin{aligned}
&\frac{1}{2!} + \frac{1}{3!} + \cdots + \\
&= \frac{1}{2}\left(1 + \frac{1}{3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{3 \cdot 4 \cdots n} + \cdots\right) \\
&< \frac{1}{2}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-2}} + \cdots\right) \\
&= \frac{1}{2}\frac{1}{1 - 1/3} = \frac{3}{4} \text{ (Sum of geom. series)}
\end{aligned}$$

***e* not an integer**

$$2 < \mathrm{e} < 2 + \frac{3}{4} = 2.75.$$

e rational or irrational ?

If possible, let $\mathrm{e} = \frac{p}{q}$, where p and q are coprime.

$$\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{q!} + \cdots$$

Multiplying both sides by $q!$

$$\begin{aligned} & p \cdot 1 \cdot 2 \cdots (q-1) \\ & = 2q! + (3 \cdots q) + (4 \cdots q) + \cdots + q + 1 \\ & + \left(\frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots \right). \end{aligned}$$

Since e is not an integer, $q \geq 2$.

Hence $q + 1 \geq 3$. We have

$$\begin{aligned} & \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots \\ & \leq \frac{1}{3} + \frac{1}{3^2} + \cdots \\ & = \frac{1}{3}\frac{3}{2} = \frac{1}{2} \text{ (Sum of geom. series)} \end{aligned}$$

Now the left hand side $q! \cdot \mathrm{e}$ is an integer but right hand side is not.

e is irrational between 2 and 3

Euler's proof of the irrationality of e

Euler established the irrationality of e by giving the *continued fraction* of e.

$$\mathrm{e} = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cdots}}}}}}}}}.$$

This goes indefinitely.

Continued fraction of any rational number is finite.

e is a transcendental number.

e is not a root of any polynomial that has integer coefficients.

The exponential function on complex numbers

x a real number,

$$e^{\imath x} = \cos x + \imath \sin x.$$

$$e^{x+\imath y} = e^{x} \cos y + \imath e^{x} \sin y.$$

Euler's identity

$$e^{\imath \pi} + 1 = 0.$$

This involves five fundamental math constants e, $\imath$, π, 1 and 0.

Occurrences of e

- A stick of length L is broken into n equal parts. What is the value of n for which the product of the lengths is maximum?

  Here n is either $\lfloor L/\mathrm{e} \rfloor$ or $\lfloor L/\mathrm{e} \rfloor + 1$.

  The value of x where the function $f(x) = x^{1/x}$ attains its maximum value is e.

- The expression $x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}$ as the number of exponents grows to infinity, tends to a limit if

$$\mathrm{e}^{-\mathrm{e}} \leq x \leq \mathrm{e}^{1/\mathrm{e}}.$$

- *Problem of misplaced envelopes*: If n letters are to be placed in n addressed envelopes, what is the probability that every letter will be placed in a wrong envelope?

$$\frac{1}{n!}\left(n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n\right) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

As $n \to \infty$, this goes to e^{-1}.

- *Real from complex*:

$$\imath^\imath = \mathrm{e}^{-\pi/2+2\imath k\pi},$$

$k = 0, \pm 1, \ldots$. The principal value of $\imath^\imath$ is $\mathrm{e}^{-\pi/2}$.

$e^{\imath\pi}$ Paradox

$$e^{\imath\pi} = -1.$$

Squaring

$$e^{2\imath\pi} = 1.$$

Multiplying both sides by e

$$e^{2\imath\pi+1} = e.$$

Replacing e by $e^{2\imath\pi+1}$ on the left hand side

$$e^{(2\imath\pi+1)^2} = e.$$

Since $(2\imath\pi + 1)^2 = 1 + 4\imath\pi - 4\pi^2$,

$$e \cdot e^{4\imath\pi} \cdot e^{-4\pi^2} = e.$$

This gives

$$e^{-4\pi^2} = 1 \implies -4\pi^2 = 0 \implies \pi = 0.$$