A journey with infinite series

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Finite sums

We know how to compute

$$a_1 + a_2$$

$$a_1 + a_2 + a_3$$

$$a_1 + a_2 + \cdots + a_n$$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

But what if

$$1 + 2 + \cdots + n + \cdots ?$$

$$1 + 2^2 + \cdots + n^2 + \cdots ?$$

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots ?$$

or

$$a_1 + a_2 + \cdots + a_n + \cdots ?$$

The number of summands become infinite?

Real sequence

A function x from the set of natural numbers $\mathbb{N}$ to the set of real numbers $\mathbb{R}$.

$$\begin{array}{cccccc} x & : & \mathbb{N} & \to & \mathbb{R} \\ & & n & \to & x_n \end{array}$$

Denoted by $(x_n)/\langle x_n \rangle/\{x_n\}_{n=1}^{\infty}$.

We shall denote it by (x_n).

Note: The order of the numbers x_n is important.

Infinite series

Given a real sequence (x_n),

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \cdots + x_n + \cdots$$

is called the infinite series.

$$1 + 2 + \cdots$$

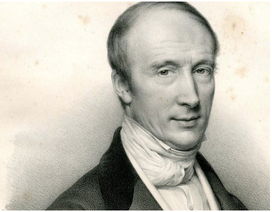
$$1 + 2^2 + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \cdots$$

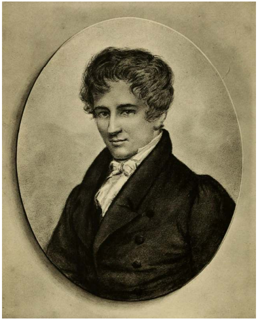
are all infinite series.

What to make of this infinite sum?



Augustin-Louis Cauchy (1789-1857)

Modern formal notions on infinite series from the works of Cauchy and Abel.



Niels Henrik Abel (1802-29)

Convergent series

$$s_n = x_1 + \cdots + x_n,\ \ n \in \mathbb{N} \text{ (sequence of partial sums).}$$

If (s_n) converges to a real number s, then the series $\sum\limits_{n=1}^{\infty} x_n$ is convergent and its sum is s.

We write $\sum\limits_{n=1}^{\infty} x_n = s$.

Divergent series

If the series is not convergent, then it is divergent.

The simplest example: Geometric series

$a, r \in \mathbb{R}$,

$$a + ar + ar^2 + \cdots + ar^n + \cdots$$

Finite sum:

$$a + ar + ar^2 + \cdots + ar^n = a\frac{1 - r^{n+1}}{1 - r}.$$

Convergent if and only if $|r| < 1$, and has sum $\frac{a}{1-r}$.

Going back to the ancient times

One of the earliest examples of summing finite geometric series: *Euclid Elements (300BC)*

Perfect number

A positive integer m is called a perfect number if it is the sum of all its divisors except itself, i.e.,

$$m = \sum_{\substack{d|m \\ d \neq m}} d.$$

If $2^n - 1$ is a prime, then $2^{n-1}(2^n - 1)$ is perfect.

$2^{n-1}(2^n-1) = 1+2+\cdots+2^{n-1}+(2^n-1)(1+2+\cdots+2^{n-2})$.

This involves summing the finite geometric series.

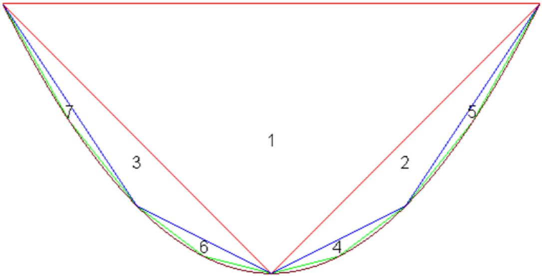
Euler: If m is an even perfect number, then it is of the form $2^{n-1}(2^n - 1)$ for some n.

One of the oldest unsolved problems: Are there any odd perfect numbers?.

Infinite geometric series in the antiquity

Archimedes (287-212 BC) used "infinite" geometric series to find the area of a parabolic segment using the *Method of Exhaustion*.

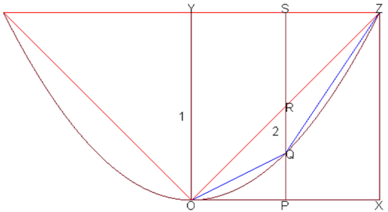
$$\text{Area of parabolic segment enclosed by } y = x^2 \text{ and } y = 1$$
$$= \Delta_1 + (\Delta_2 + \Delta_3) + (\Delta_4 + \Delta_5 + \Delta_6 + \Delta_7) + \cdots$$



Let's compute ourselves

Note $QR = \frac{1}{4}PS$.
Now

$$\begin{aligned}\Delta_2 &= \Delta(OQR) + \Delta(ZQR)\\ &= \frac{1}{2}(OP + SZ)QR\\ &= \frac{1}{2}YZ \cdot QR\\ &= \frac{1}{8}YZ \cdot PS\\ &= \frac{1}{8}\Delta_1.\end{aligned}$$



Similarly, $\Delta_3 = \frac{1}{8}\Delta_1$.

Therefore,

$$(\Delta_2 + \Delta_3) = \frac{1}{4}\Delta_1.$$

In the same way

$$\sum_{i=4}^{7} \Delta_i = \frac{1}{4}(\Delta_2 + \Delta_3) = \frac{1}{4^2}\Delta_1.$$

$$\cdots\cdots$$

Hence the area of the parabolic segment equals

$$\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right)\Delta_1 = \frac{4}{3}\Delta_1.$$

Infinity and infinite series: going further back

Zeno's paradox of the tortoise and Achilles (Zeno (490BC)

There is a race between Achilles, the legandry Greek warrior, and the tortoise. Achilles runs at the speed of $10ms^{-1}$ while the tortoise runs at a speed of $1ms^{-1}$. Achilles gives the tortoise a head start of $10m$. According to Zeno, Achilles will never be able to overtake the slow tortoise.

The resolution of the paradox lies in the convergence of the infinite series

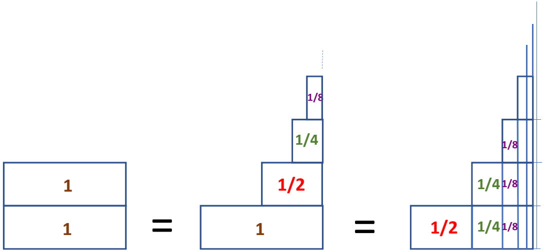
$$10 + 1 + \frac{1}{10} + \cdots$$

Series other than the geometric series

Liber calculationum, Richard Suiseth (or Swineshead, known as the Calculator), 1350 showed

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots = 2.$$

Oresme, 1350 Summed the above series through geometric methods.



Oresme's Summation

How will you find the sum of this series?

Series for trigonometric functions

Madhava (fifteenth century), Gregory, Leibniz, Newton (seventeenth century)

Gave the series for $\tan^{-1} x, \sin x, \cos x$:

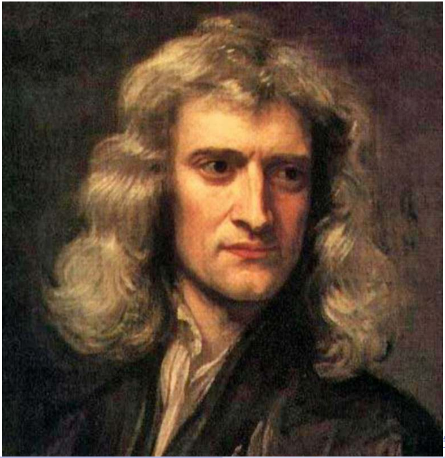
$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

As a special case, the first series for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

Newton’s calculus

Issac Newton (1642-1727) based calculus on the manipulation of infinite series.



Power series

Geometric series, Madhava series : *Express a function $f(x)$ in the form* $\sum_{n=1}^{\infty} a_n x^n$, $a_n \in \mathbb{R}$.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Useful to sum numerical series:

$$\frac{4}{5} = 1 - \frac{1}{4} + \frac{1}{4^2} - \cdots ,$$

$\vdots$

Important to find other series by methods of integration and inversion:

Let's illustrate it with a simple example:

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots .$$

On integration

$$\int_0^x \frac{\mathrm{d}t}{1+t} = \int_0^x \left(1 - t + t^2 - \cdots\right) \mathrm{d}t$$

Hence

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots .$$

Newton's method of series inversion (Newton 1671):

Let y denote the inverse function of $\log(1+x)$, i.e.,
$y = \log(1+x) \Leftrightarrow \mathrm{e}^y - 1 = x$.

$$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \tag{1}$$

Solve for y in terms of x. Equivalently put

$$x = a_0 + a_1 y + a_2 y^2 + \cdots$$

and substitute this value in the right-hand side of (1),

$$y = (a_0 + a_1 y + \cdots) - \frac{(a_0 + a_1 y + \cdots)^2}{2} + \frac{(a_0 + a_1 y + \cdots)^3}{3} - \cdots .$$

Compare the coefficients of powers of y on both sides.

Comparing the constant term:

$$0 = a_0 - \frac{a_0^2}{2} + \frac{a_0^3}{3} - \cdots = \log(1 + a_0)$$

which implies $a_0 = 0$.

Comparing the coefficient of y,

$$1 = a_1.$$

Comparing the coefficient of y^2,

$$0 = a_2 - \frac{a_1^2}{2} = a_2 - \frac{1}{2}.$$

Comparing the coefficient of y^3,

$$0 = a_3 - \frac{2a_2 a_1}{2} + \frac{a_1^3}{3} = a_3 - \frac{1}{2} + \frac{1}{3} = a_3 - \frac{1}{6}.$$

Newton found some initial terms and found the remaining coefficients by analogy.

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots .$$

Nowadays we use different methods to find series expansions for given functions.

Summation of series

Most of the series studied were of the form

$$f(x) = \sum_{n=1}^{\infty} a_n x^n,$$

where f is a known function.

Not difficult to sum the series.

Converse problem

Given a series. To find its sum.

This is a difficult problem.

Examples:

- Richard Suiseth, Oresme (1350)

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots = 2.$$

- Mengoli(1650)

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots .$$

Easy to sum.

Note

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Hence

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}$$
$$= 1 - \frac{1}{n+1}.$$

This gives

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

The First hard example

Basel Problem

Sum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots .$$

Unsuccessfully tried by Mengoli;

Unsuccessfully tried by Jacobi and Johann Bernoulli.

They instead summed the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$, and gave some trivial results for the original problem.

Proved by Euler in 1734.

Leonhard Euler (1707-83): Fundamental contributions in the study of infinite series

The first proof of Euler

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}.$$

$$\frac{\sin\sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n-1}}{(2n-1)!}.$$

Roots of $\frac{\sin\sqrt{x}}{\sqrt{x}}$ are $x_1 = \pi^2, x_2 = (2\pi)^2, x_3 = (3\pi)^2, \ldots$.

Let

$$1 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

have roots $u_1, \ldots, u_n$.

Then

$$\begin{aligned} 1 + a_1 x + \cdots + a_n x^n &= (u_1 - x) \cdots (u_n - x) \\ &= \left(1 - \frac{x}{u_1}\right) \cdots \left(1 - \frac{x}{u_n}\right). \end{aligned}$$

This gives

$$\frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_n} = -a_1.$$

Assuming this is also true for “infinite polynomials”,we get

$$\frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \cdots = -\text{coeeficient of } x = \frac{1}{3!}.$$

Hence

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

This is a remarkable proof but lacks rigour.

A number of proofs were given by Euler and many mathematicians afterwards.

An elementary rigorous proof

Let

$$A_n = \int_0^{\pi/2} \cos^{2n} x \mathrm{d}x \text{ and } B_n = \int_0^{\pi/2} x^2 \cos^{2n} x \mathrm{d}x \text{ for all } n \geq 0.$$

$$\begin{aligned} A_n &= \int_0^{\pi/2} \cos x \cos^{2n-1} x \mathrm{d}x \\ &= (2n-1) \int_0^{\pi/2} \sin^2 x \cos^{2(n-1)} x \mathrm{d}x \text{ (Integrating by parts)} \\ &= (2n-1)(A_{n-1} - A_n) \text{ (Using } \sin^2 x = 1 - \cos^2 x). \end{aligned}$$

Since

$$\frac{A_n}{2n-1} = A_{n-1} - A_n,$$

$$\frac{A_n}{2n-1} = \frac{A_{n-1}}{2n}. \tag{2}$$

Also

$$\int_0^{\pi/2} \sin^2 x \cos^{2n} x \mathrm{d}x = \frac{A_{n+1}}{2n+1} = \frac{A_n}{2(n+1)}. \tag{3}$$

Again

$$
\begin{aligned}
A_n &= \int_0^{\pi/2} 1 \cdot \cos^{2n} x \mathrm{d}x \\
&= 2n \int_0^{\pi/2} x \sin x \cos^{2n-1} x \mathrm{d}x \\
&= n \int_0^{\pi/2} x^2 \left((2n-1) \sin^2 x \cos^{2(n-1)} x - \cos^{2n} x \right) \mathrm{d}x \\
&= n(2n-1) B_{n-1} - 2n^2 B_n.
\end{aligned}
$$

Dividing both sides by $n^2 A_n,$

$$\begin{aligned} \frac{1}{n^2} &= \frac{(2n-1)B_{n-1}}{nA_n} - \frac{2B_n}{A_n} \\ &= \frac{2B_{n-1}}{A_{n-1}} - \frac{2B_n}{A_n} \text{ (Using (2))} \end{aligned} \tag{4}$$

From (4)

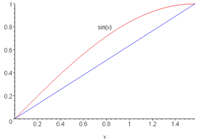
$$\sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=1}^{n} \left(\frac{2B_{k-1}}{A_{k-1}} - \frac{2B_k}{A_k} \right) = \frac{2B_0}{A_0} - \frac{2B_n}{A_n}.$$

$$A_0 = \int_0^{\pi/2} \mathrm{d}x = \frac{\pi}{2} \text{ and } B_0 = \int_0^{\pi/2} x^2 \mathrm{d}x = \frac{\pi^3}{24}.$$

Now we estimate $\frac{B_n}{A_n}$.

For $0 \leq x \leq \frac{\pi}{2}$, $\frac{\pi}{2} x \leq \sin x$.

Using this and (3)



$$B_n \leq \frac{4}{\pi^2} \int_0^{\pi/2} \sin^2 x \cos^{2n} x \mathrm{d}x = \frac{4A_n}{2\pi^2(n+1)}.$$

Hence

$$\frac{2B_n}{A_n} \leq \frac{4}{\pi^2(n+1)} \to 0 \text{ as } n \to \infty.$$

Thus we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2B_0}{A_0} = \frac{\pi^2}{6}.$$

The first criterian on convergence of series

The *alternating series*

$$a_1 - a_2 + a_3 - \cdots + (-1)^{n-1} a_n + \cdots ,$$

$a_n \geq 0$, $a_n \geq a_{n+1}$ is convergent if and only if

$$\lim_{n\to\infty} a_n = 0.$$

Gottfried Wilhelm Leibniz (1646-1716)

The first divergent series

Oresme (1350)

Oresme proved the divergence of the *harmonic series*.

$$1+\frac{1}{2}+\frac{1}{3}+\cdots .$$

His proof same as the modern one:

$$\begin{aligned}1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots\\ >1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\cdots\\ =1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots .\end{aligned}$$

Divergent series: different views

- Euler (1746): "I believe that every series should be assigned a certain value. However, to account for all the difficulties that have been pointed, this value should not be denoted by the name sum, ..."
- Abel (1828): "Divergent series are the invention of the devil and it is shameful to base on them any demonstration whatsoever".

G. H. Hardy's excellent book *Divergent Series* explains the rights and wrongs; the good the bad and the ugly.

Sum of divergent series

Granti series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Leibniz, Euler: Sum is $\frac{1}{2}$.

Sum of divergent series

Granti series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Leibniz, Euler: Sum is $\frac{1}{2}$.

Look the sequence of partial sums.

$$s_1 = 1, s_2 = 0, s_3 = 1, \ldots, s_{2n-1} = 1, s_{2n} = 0, \ldots$$

Clearly this is divergent.

But the average of the sequence of partial sums is $\frac{1}{2}$.

We can take the 'sum' to be $\frac{1}{2}$.

Another view

This is the series

$$1 - x + x^2 - x^3 + \cdots \tag{5}$$

when $x = 1$.

Sum of (5) is $\frac{1}{1+x}$ (for $|x| < 1$).

Thus the sum should be $\frac{1}{1+1} = \frac{1}{2}$.

One more example

What should be the sum of the series

$$1 - 2 + 3 - 4 + \cdots ?$$

One more example

What should be the sum of the series

$$1 - 2 + 3 - 4 + \cdots ?$$

We consider the power series

$$1 - 2x + 3x^2 - 4x^3 + \cdots .$$

This represents the function $f(x) = \frac{1}{(1+x)^2}$ for $|x| < 1$.

Hence the sum should be $f(1) = \frac{1}{4}$.

The series $1+2+3+\cdots$

Ramanujan (1913) gave its ‘sum’

Let s be the sum

$$s = 1 + 2 + 3 + \cdots \ . \tag{6}$$

We have

$$\frac{1}{4} = 1 - 2 + 3 - 4 + \cdots \ . \tag{7}$$

Subtract (7) from (6) to get

$$\begin{aligned} s - \frac{1}{4} &= 4 + 8 + 12 + \cdots \\ &= 4(1 + 2 + 3 + \cdots) = 4s. \end{aligned}$$

Thus

$$s = -\frac{1}{12}.$$

What do these ‘sums’ denote?

These sums are clearly not the limits of the sequences of partial sums.

Abel summability

Let (a_n) be a real sequence such that the power series $\sum\limits_{n=1}^{\infty} a_n x^n$ is convergent for every $|x| < 1$ to a function $f(x)$.

Suppose $\lim\limits_{x \to 1^-} f(x)$ exists and equals a. Then $\sum\limits_{n=1}^{\infty} a_n$ is Abel summable and has the Abel sum a.

The series $1 - 1 + 1 - 1 + \cdots$ and$1 - 2 + 3 - 4 + \cdots$ are Abel summable with Abel sums $\frac{1}{2}$ and $\frac{1}{4}$, respectively.

Cesaro summability

Let (a_n) be any real sequence, and put $s_n = a_1 + \cdots + a_n$. Define

$$H_n^{(k)} = \begin{cases} \frac{1}{n}\sum_{i=1}^{n} s_i & k = 1 \\ \frac{1}{n}\sum_{i=1}^{n} H_i^{(k-1)} & k > 1. \end{cases}$$

The series $\sum_{n=1}^{\infty} a_n$ is C_k-summable if $(H_n^{(k)})$ is convergent and its C_k-sum is $\lim_{n\to\infty} H_n(k)$..

The series $1 - 1 + 1 - 1 + \cdots$ is C_1-summable to $\frac{1}{2}$, and the series $1 + 2 + 3 + \cdots$ is C_2-summable to $-\frac{1}{12}$.

Riemann-ζ function

The series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent if and only if $s > 1$.

Define a function $\zeta : (1, \infty) \to \mathbb{C}$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

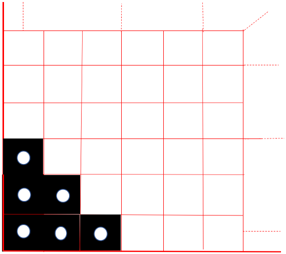
This function can be "nicely extended" to the complex plane $\mathbb{C} \setminus \{1\}$.

Studied by Euler in 1740 for real numbers, and by Riemann in 1859 for complex numbers.

- $\zeta(2) = \frac{\pi^2}{6}$.
- $\zeta(4) = \frac{\pi^4}{90}$.
- $\zeta(-1) = -\frac{1}{12}$.
- $\zeta(0) = -1$.

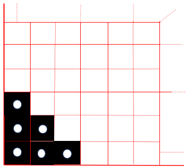
Let's end the talk with a puzzle

Pebbling problem: Consider the following infinite grid of squares with one corner.



Setup for the Pebbling Problem

- Consider an infinite grid of squares with one corner,i.e., the first quadrant.
- Six of the colors are colored, as shown in the figure.
- At the beginning, these six coloured squares are each occupied by a single pebble.



Setup for the Pebbling Problem

We are allowed to remove any pebble, provided we replace it by two new pebbles in the square directly above and in the square directly to the right. If either of these squares are already occupied, the pebble cannot be removed at that time.

Questions

(1) Is it possible, by these operations, to evacuate all six of the coloured squares?

(2) Suppose instead that only the corner square is occupied by a pebble. In this case, is it possible to evacuate the six coloured squares?

Hint for the puzzle

Use infinite series.

1/32	1/64	1/128	1/256	1/512	1/1024
1/16	1/32	1/64	1/128	1/256	1/512
1/8	1/16	1/32	1/64	1/128	1/256
1/4	1/8	1/16	1/32	1/64	1/128
1/2	1/4	1/8	1/16	1/32	1/64
1	1/2	1/4	1/8	1/16	1/32

Labeling for the Quarter-Checkerboard