What are Minimal surfaces?

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Topology of compact oriented surfaces – genus

Sphere, torus, double torus are usual names you may have heard.

Sphere and ellipsoids are abundant in nature – for example a ball and an egg. They are topologically "same" – in the sense if they were made out of plasticine, you could have deformed one to the other.

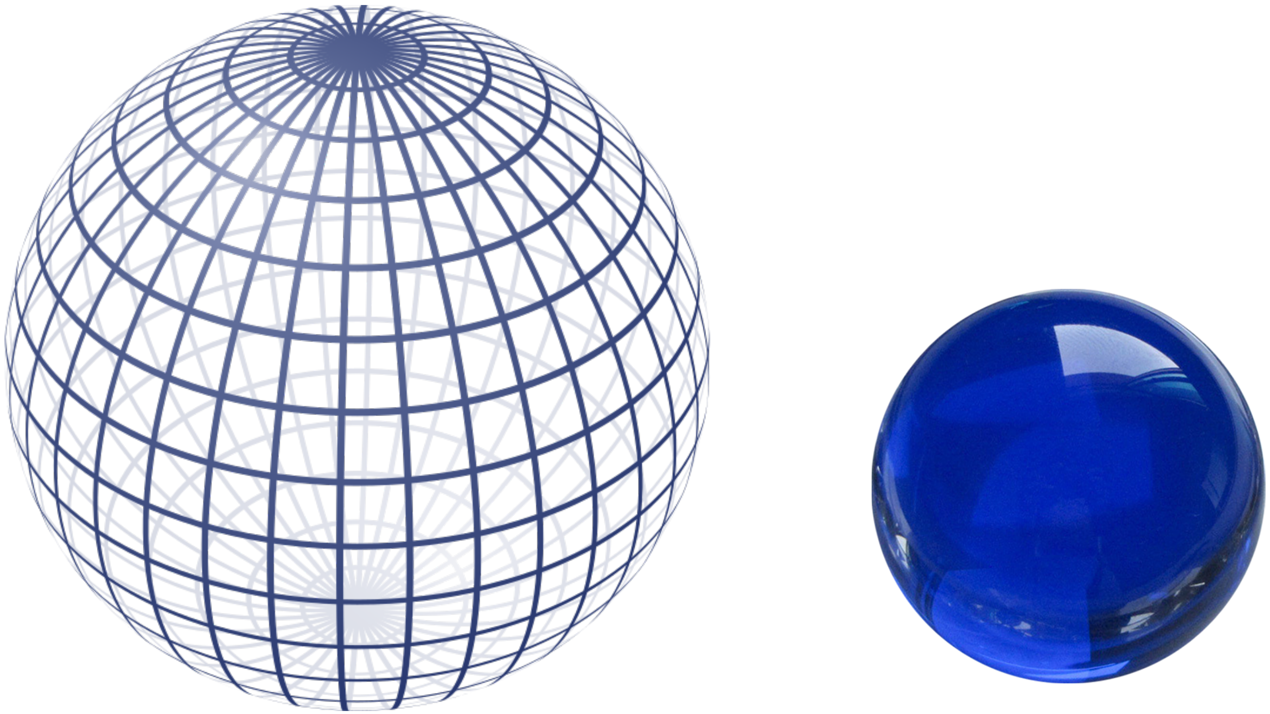

Figure 1: Sphere and a glass ball

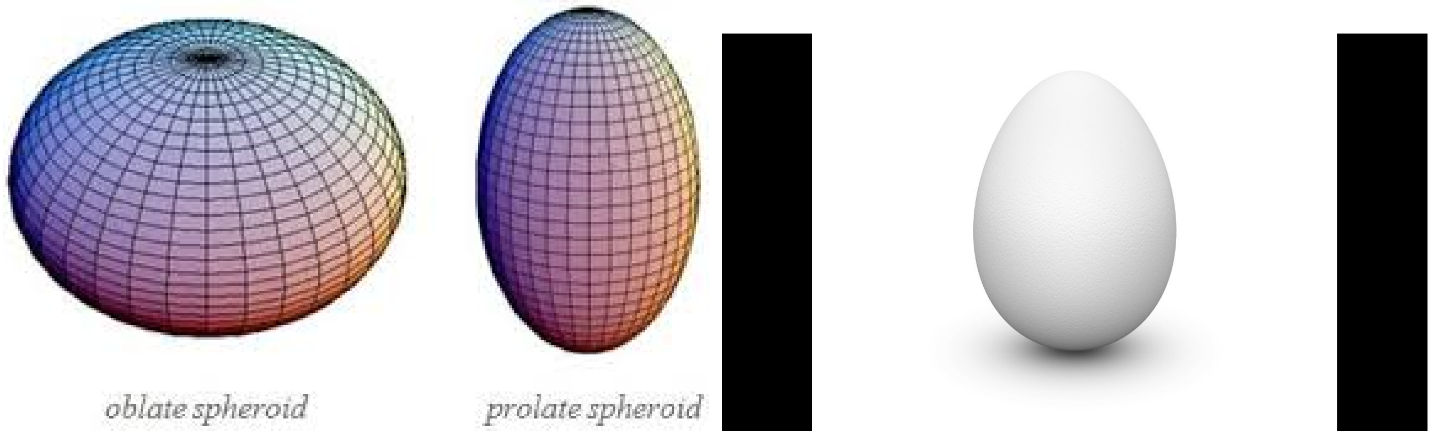

Figure 2: Ellipsoids and an egg

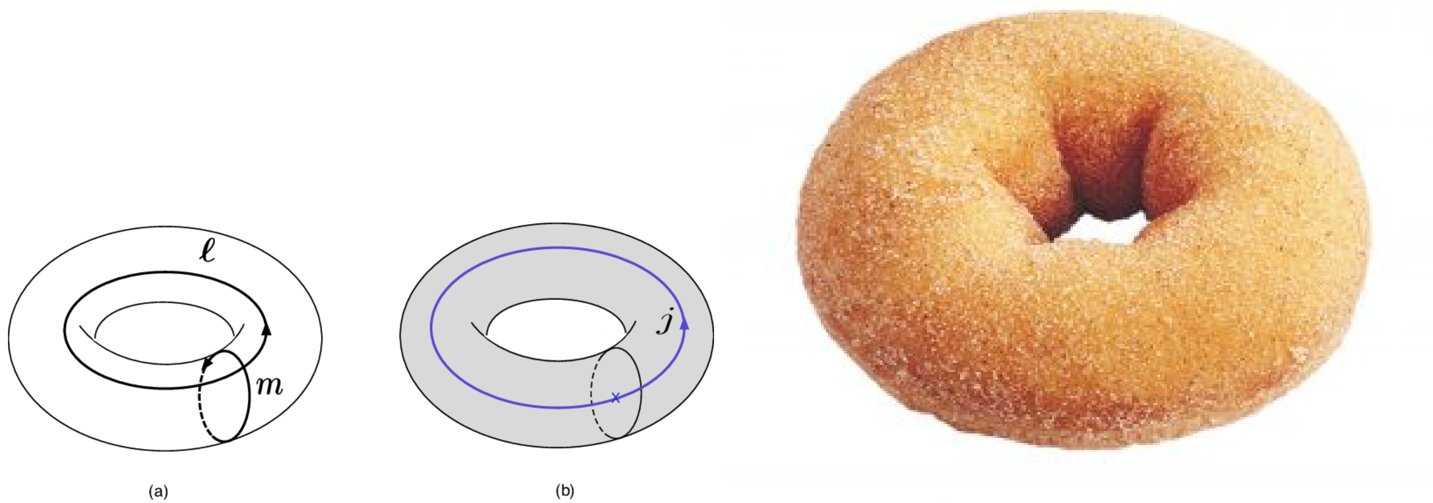

Figure 3: Tori and donut

Figure 4: Triple torus and surface of a pretzel

Handle addition to the sphere –genus

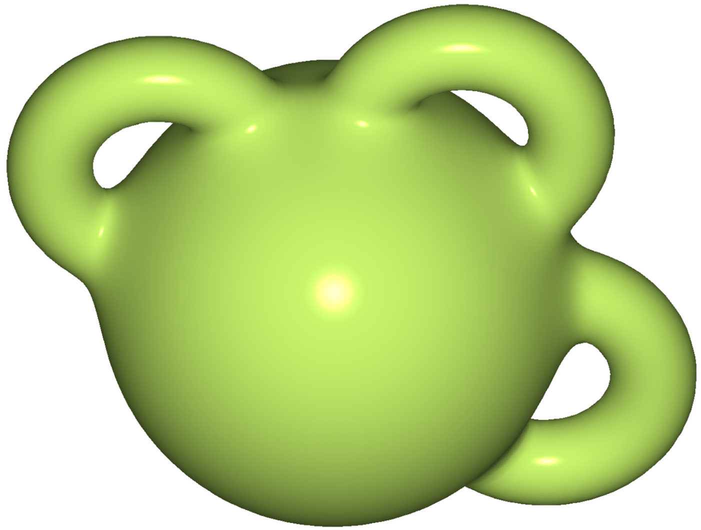

Figure 5: Triple torus

In fact as the above figure suggests one can construct more and more complicated surfaces by adding handles to a sphere – the number of handles added is the genus.

Genus classifies compact oriented surfaces topologically

Sphere (genus g =0) is not topologically equivalent to torus (genus g =1) and none of them are topologically equivalent to double torus (genus g =2) and so on.

One cannot continuously deform one into the other if the genus-es are different.

✦ **Euler's formula:**

Divide the surface up into faces (without holes) with a certain number of vertices and edges. Let no. of vertices = V, no. of faces = F, no. of edges = E, no. of genus-es = g.

Then Euler's formula for compact oriented surface of genus g is

$\mathbf{V - E + F = 2 - 2g}$.

Euler's formula for the sphere

Let us test this formula for the sphere.

For example triangulation of sphere:

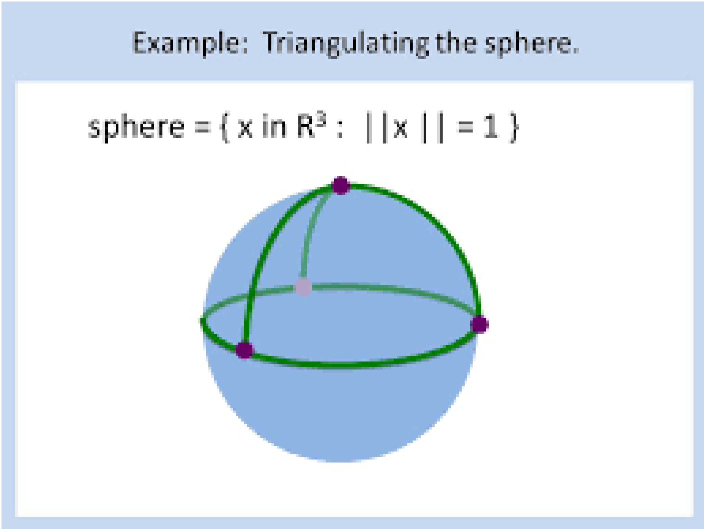

Figure 6: Triangulation of the sphere

It is easy to check that $V - E + F = 2$ and hence confirm genus $g = 0$

Polygonal shapes topologically equivalent to the sphere:

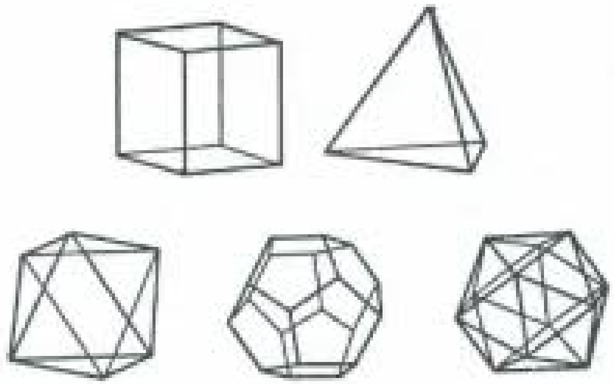

Fig. 1

Figure 7: Polyhedrons

It is easy to check that $V - E + F = 2$ and hence confirm genus $g = 0$

Euler formula for the torus

The torus can be obtained from a rectangular sheet of paper with opposite edges identified.

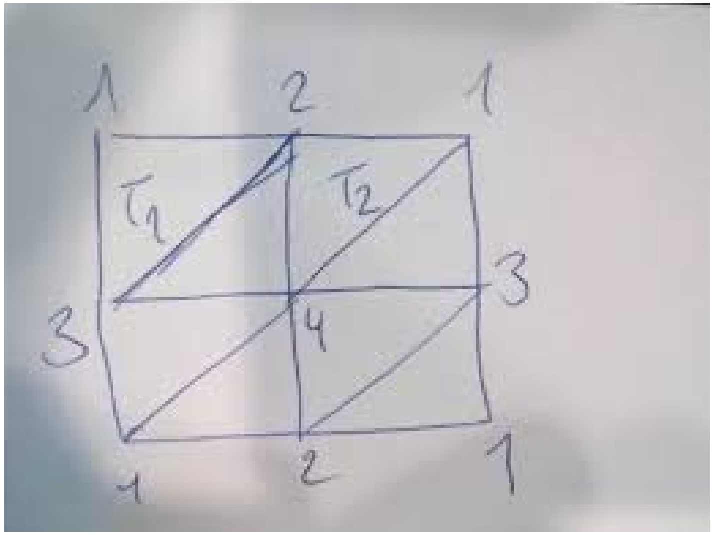

Figure 8: Triangulation of the torus

It is easy to check that $V - E + F = 0$ confirming genus $g = 1$.

A planar figure

A triangle or any connected planar figure $V - E + F = 1$

(Please check)

An example of a non-oriented surface with a boundary

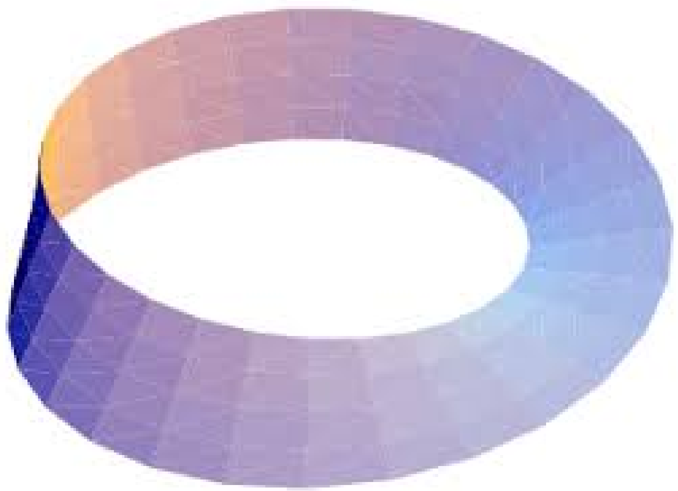

Figure 9: Mobius strip

Geometry of surfaces

When we talk of geometry, we talk of lengths of curves on surfaces, area of the surface, Gaussian curvature, mean curvature etc. In this sense, sphere and ellipsoids (though topologically equivalent) are very different. Sphere for example is totally symmetric while the ellipsoid is not!

Parametrized surfaces in $\mathbb{R}^3$

Let $\mathbf{X} = \mathbf{X}(u,v)$ is a surface in $\mathbb{R}^3$ parametrized by two parameters u, v, i.e. $\mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v))$ *locally*.

For example, the helicoid: $z = tan^{-1}(y/x)$ can be locally parametrized in many ways:

(1) $\mathbf{X}(u,v) = (u, v, tan^{-1}(v/u))$

(2) $\mathbf{X}(r,\theta) = (rcos\theta, rsin\theta, \theta)$

(3) Now we introduce a novel parametrization:

Let $\zeta = u + iv$, $\mathbf{X}(u,v) = \mathbf{X}(\zeta,\bar{\zeta}) = (x(\zeta,\bar{\zeta}), y(\zeta,\bar{\zeta}), z(\zeta,\bar{\zeta}))$ where

$x(\zeta,\bar{\zeta}) = -\frac{1}{2}\mathrm{Im}(\zeta + \frac{1}{\zeta})$

$y(\zeta,\bar{\zeta}) = \frac{1}{2}\mathrm{Re}(\zeta - \frac{1}{\zeta})$

$z(\zeta,\bar{\zeta}) = -\frac{\pi}{2} + \mathrm{Im}\ln(\zeta)$

This is actually a helicoid!!

This representation fails for $\zeta = 0$.

The helicoid

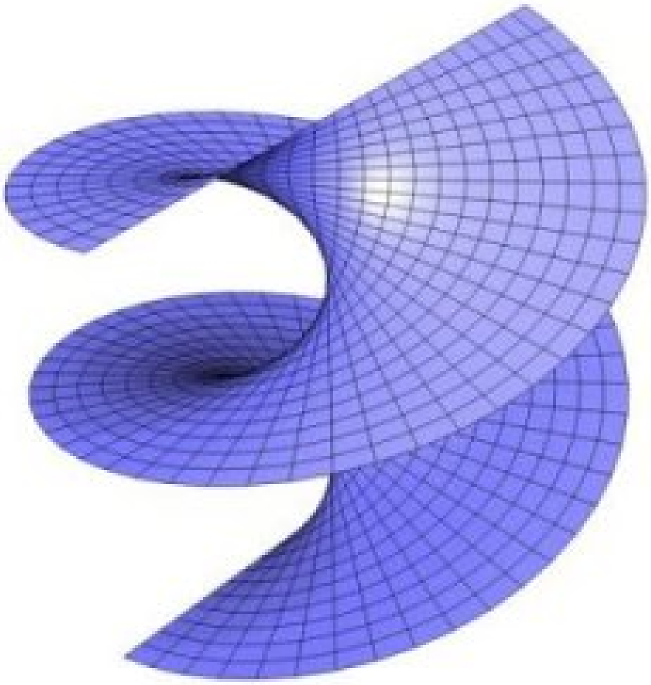

Figure 10: Helicoid: a minimal surface

Normal and tangent plane to a regular surface at a point on the surface

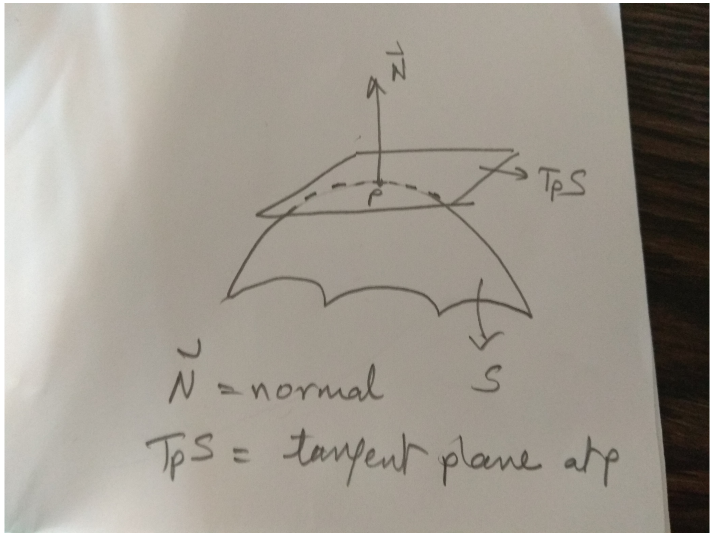

Figure 11: normal and tangent plane to a regular surface at p

✦ Suppose $\mathbf{X} = \mathbf{X}(u, v)$ is a regular surface S in $\mathbb{R}^3$ parametrized by two parameters u, v, i.e. $\mathbf{X}(u, v) = (x(u, v), y(u, v), z(u, v))$ *locally*.

Then $\mathbf{X}_u$ and $\mathbf{X}_v$ evaluated at p are linearly independent vectors in $\mathbb{R}^3$ and span the tangent plane T_pS at p.

Then, the unit normal $\mathbf{N}$ at p is given by $\frac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|}$, evaluated at p.

Normal curvature at a point on the surface in a specified direction

- Recall that the curvature of a curve parametrised by s, namely,
  $\mathbf{l}(s) = (l_1(s), l_2(s), l_3(s))$
  in $\mathbb{R}^3$ is given by k where $|k| = |\mathbf{l}''(s)|$
- Normal curvature along a given direction: Take any direction $\mathbf{v}$ in the tangent plane to S at p. Take the plane containing the unit normal to the surface, $\mathbf{N}$, and $\mathbf{v}$. This plane intersects the surface S in a curve $\mathbf{l}$. It is called the normal section at p along $\mathbf{v}$. The curvature k of this curve is called the normal curvature k_n in direction $\mathbf{v}$.

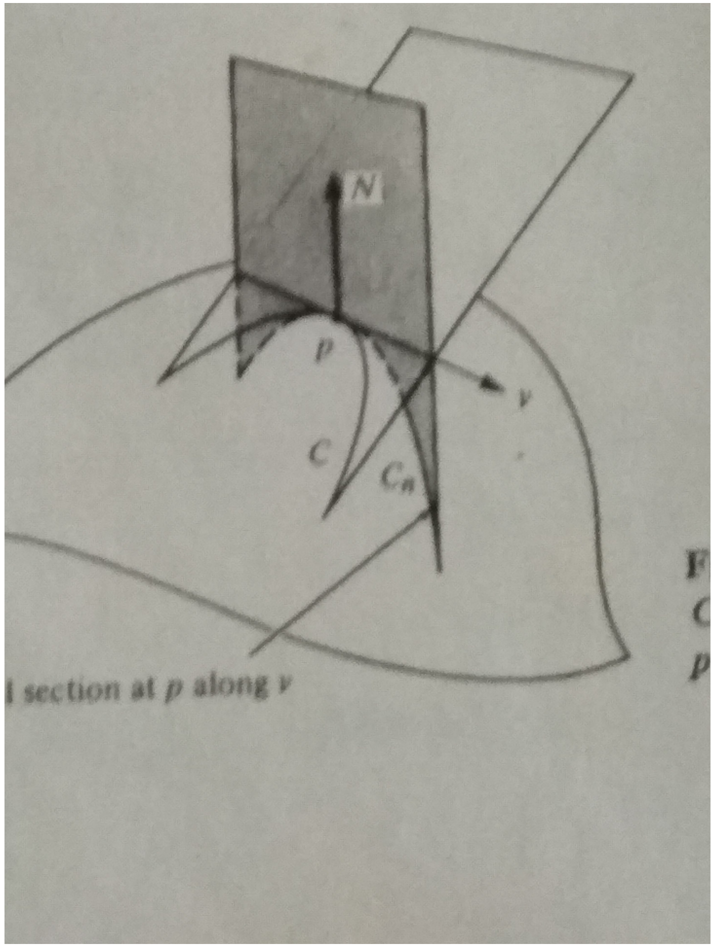

Figure 12: Normal section at p along $\mathbf{v}$

Principal curvatures of a surface at a point; Gaussian and Mean curvatures

Let the direction $\mathbf{v}$ vary and take all possible directions. One finds that k_n achieves a maximum and minimum – denoted by k_1 and k_2 – along two orthogonal directions $\mathbf{e_1}$ and $\mathbf{e_2}$. These are called principal curvatures and the two orthogonal directions are callled principal directions of curvature.

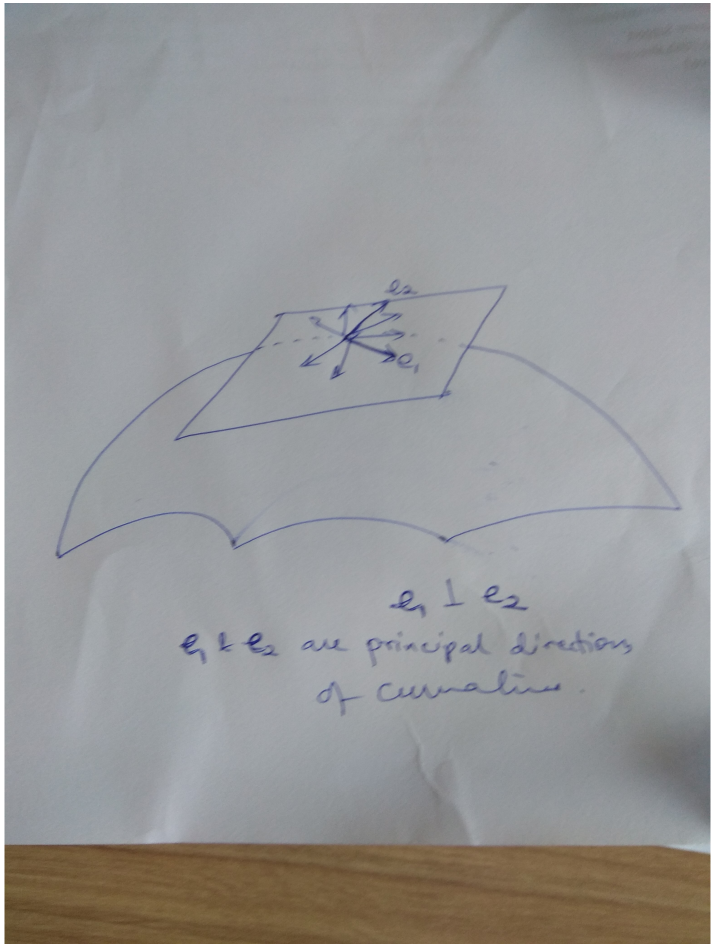

Figure 13: The two principal directions of curvature

The Gaussian curvature is given by $\mathbf{K} = \mathbf{k_1 k_2}$ and the Mean curvature is given by $\mathbf{H} = \mathbf{k_1} + \mathbf{k_2}$

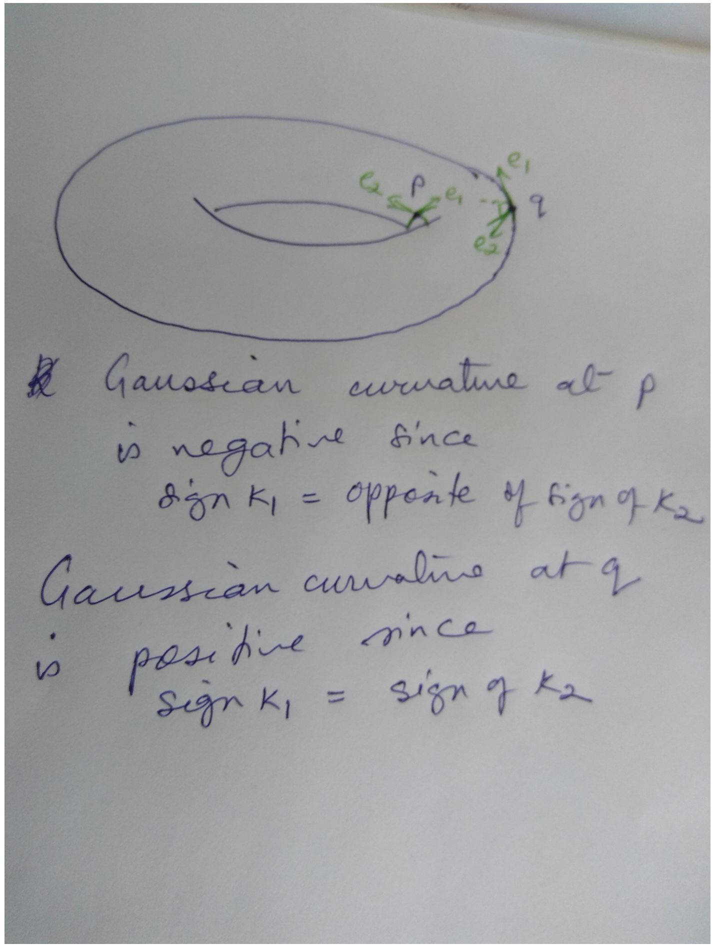

Figure 14: Positive and Negative Gaussian curvatures on a torus

Minimal Surfaces

✦ We will be talking about locally area minimizing surfaces called "Minimal Surfaces", which appear in nature as idealised soap-films.

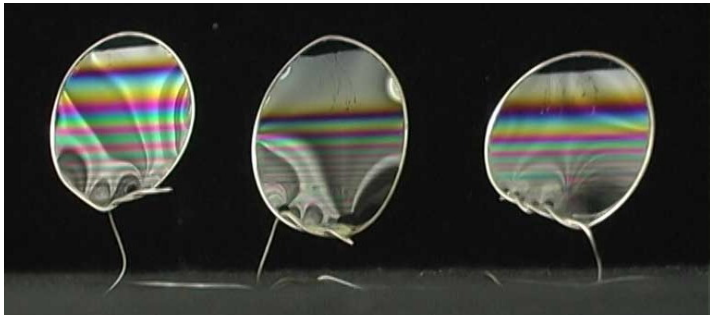

Figure 15: Soap-films spanning a wire-frame

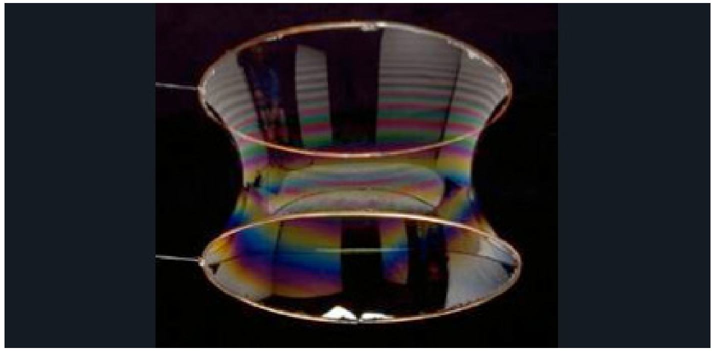

Figure 16: Catenoidal soap-film

- They have the property that the mean curvature =0 at every point on the surface. If the two principal curvatures are k_1 and k_2, then $k_1 + k_2 = 0$. If $k_1 = -k_2 \neq 0$, the two principal lines of curvature at every point curve in opposite directions – thus it has the shape of a saddle at every non-umbilical point. They have negative or zero Gaussian curvature. Umbilical points are where Gaussian curvature is zero.
- **Caution:** Soap bubbles (spherical) are not minimal surfaces since they have non-zero mean curvature at every point.

✦ Some Examples:

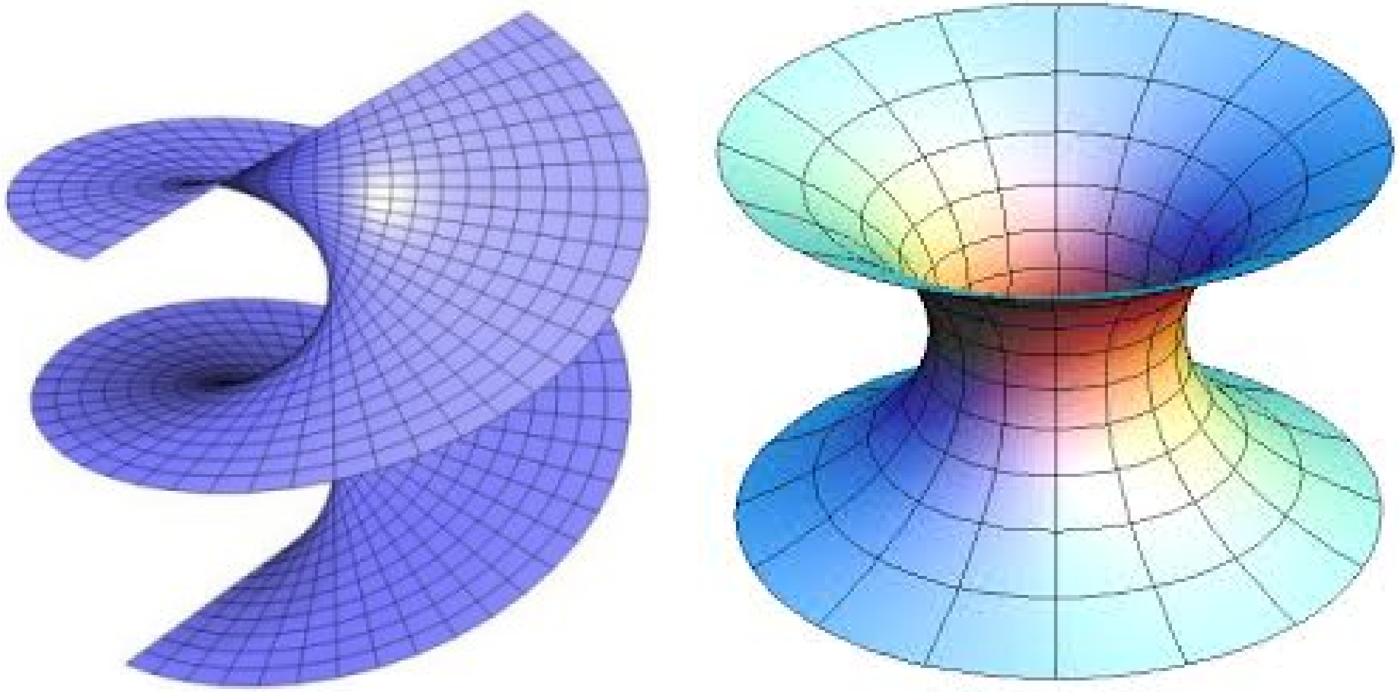

Figure 17: Helicoid and Catenoid: Conjugate minimal surfaces

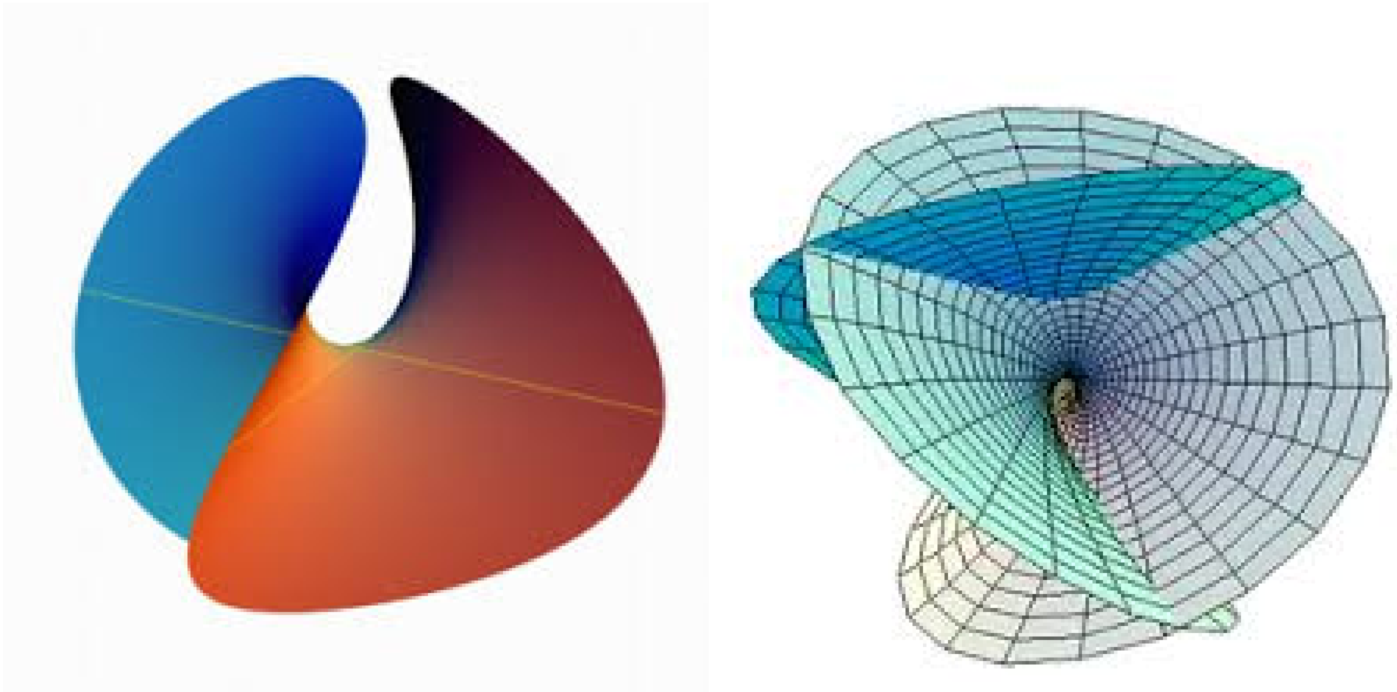

Figure 18: Enneper surface

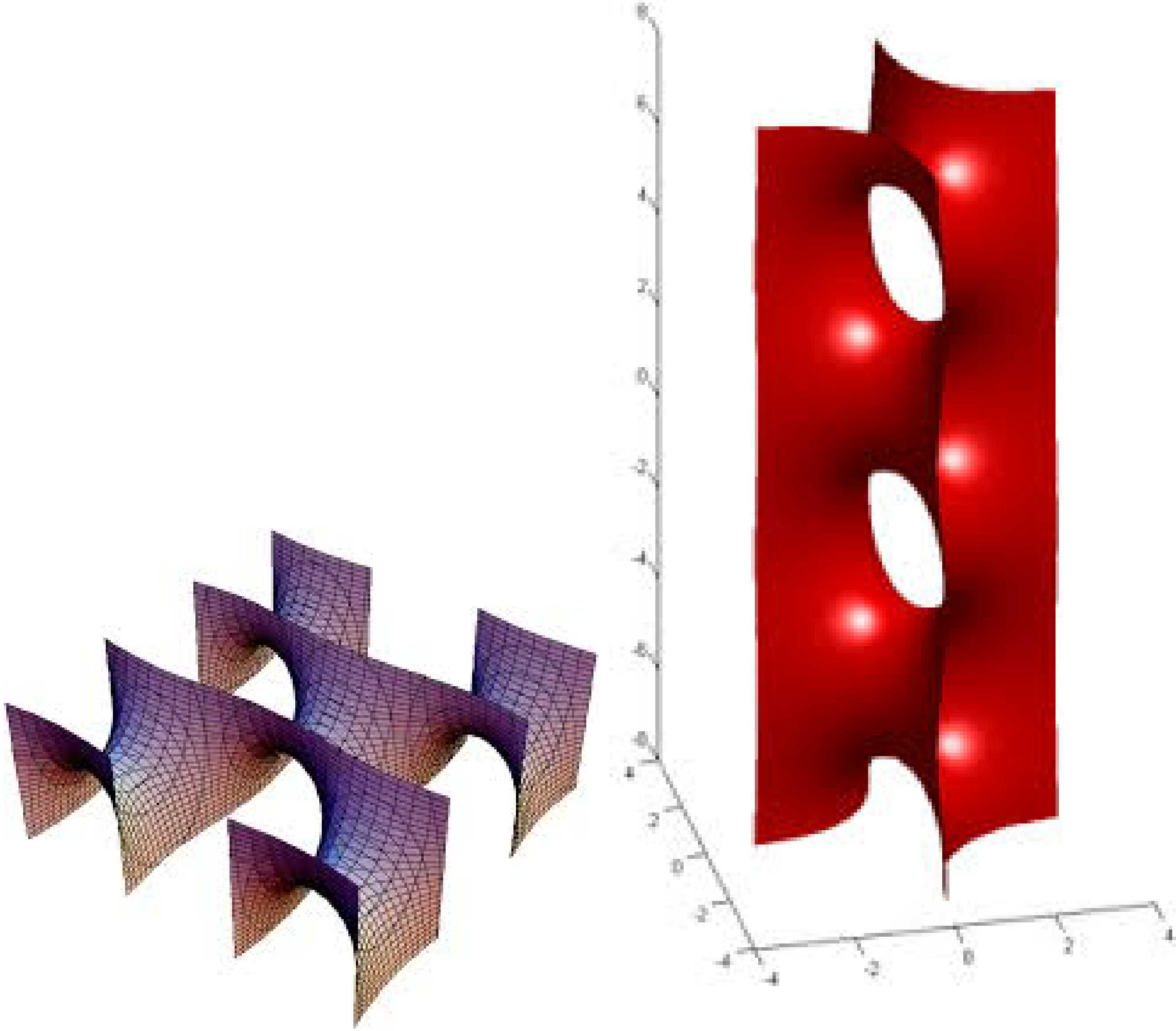

Figure 19: Scherk's surfaces

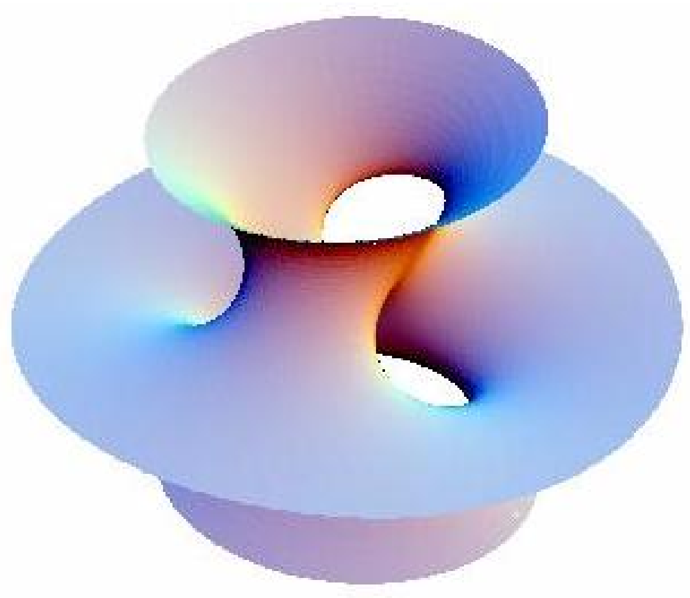

Figure 20: Genus 1 Costa surface: immersion of a torus with 3 punctures!

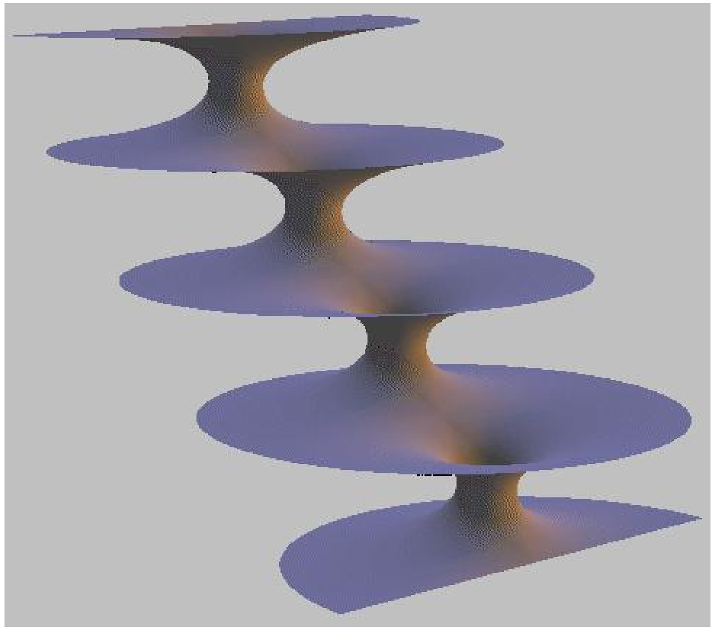

Figure 21: The Riemann Staircase

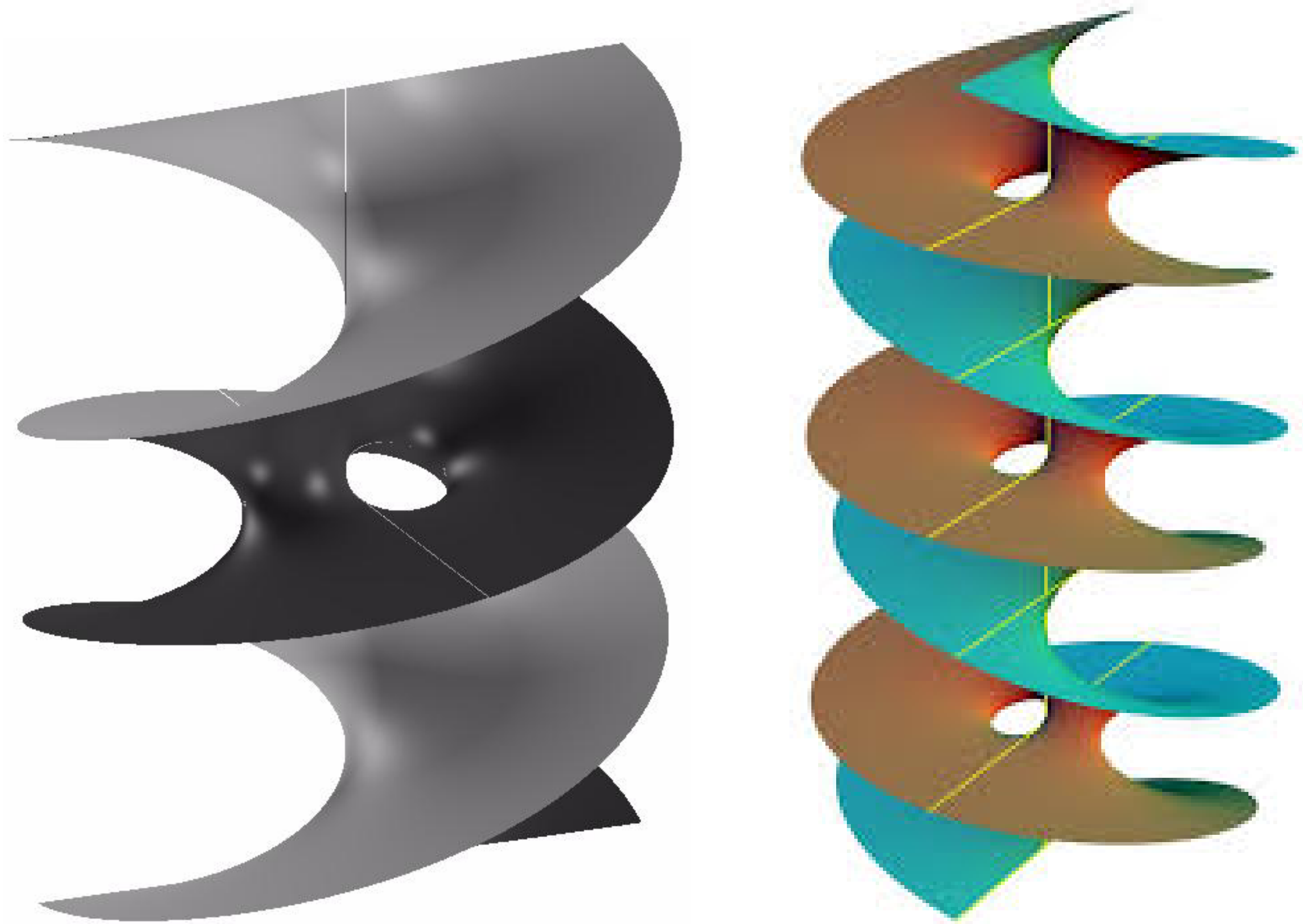

Figure 22: Genus 1 and Multiple genus helicoid

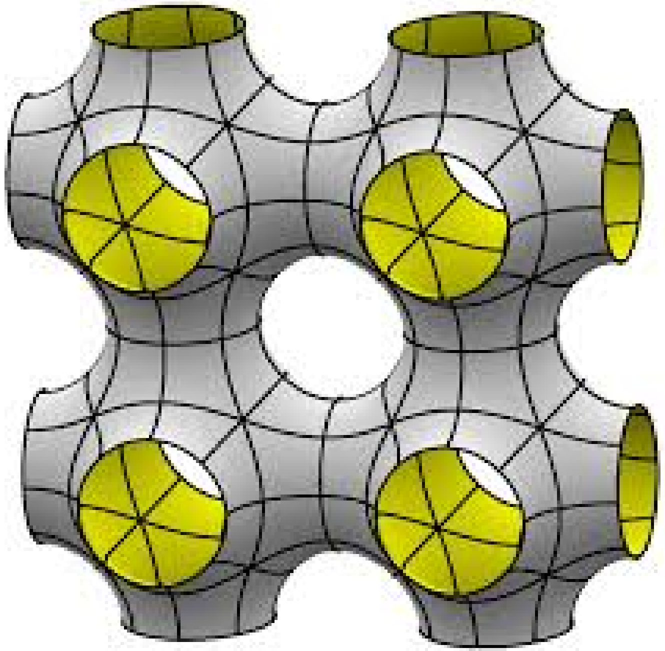

Figure 23: Triply periodic minimal surface

The Plateau's problem for the layman

✦ Given a small closed wire frame, if you dip it in soap solution, is there a soap film spanning it and if so how many? Douglas gave answer to the mathematical formulation of this question in the affirmative and subsequently this has been an active field of research. The question of how many minimal surfaces span a given curve has been studied by Tromba using bifurcation theory.

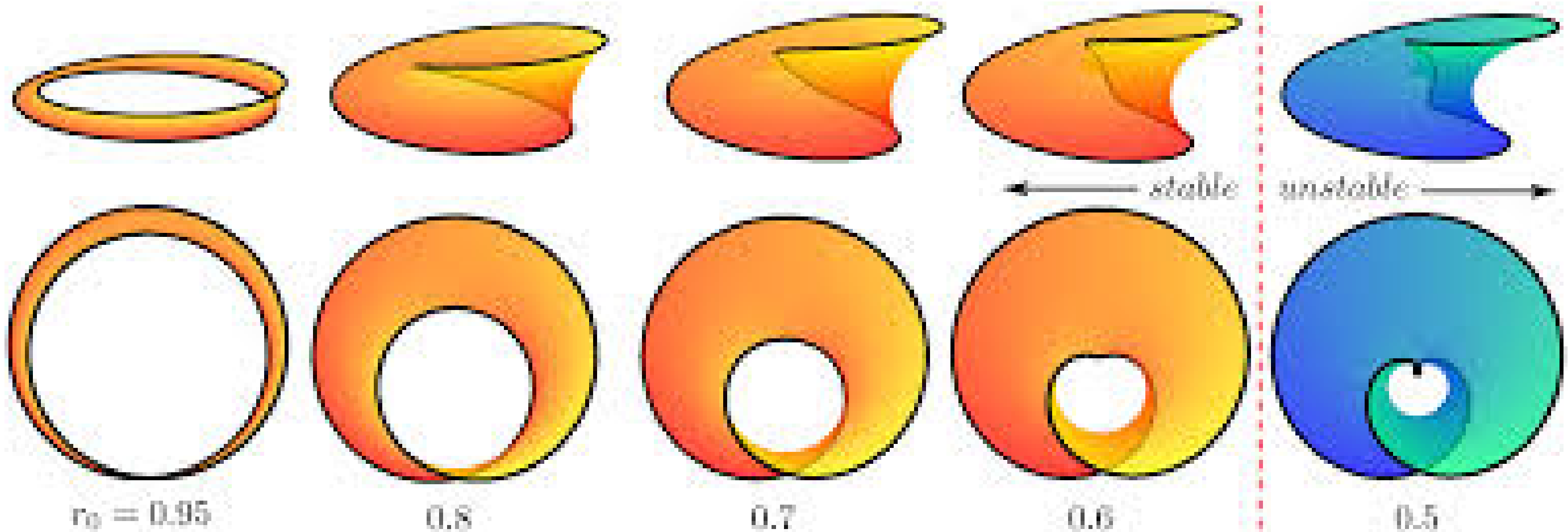

Figure 24: Two types of minimal surfaces spanning a mobius curve

Harmonicity of coordinates in Isothermal parameters

- Locally, if the minimal surface can be written as a graph of a function of two variables, namely $(x, y, \phi = \phi(x, y))$, then mean curvature $= 0$ implies minimal surfaces are solutions of the following non-linear equation:

$$(1 + \phi_x^2)\phi_{yy} - 2\phi_x\phi_y\phi_{xy} + (1 + \phi_y^2)\phi_{xx} = 0.$$

  The surface is locally given by $X(x, y) = (x, y, \phi(x, y))$.

- However, for minimal surfaces (and in fact all smooth surfaces) there exists certain special parameters, called isothermal (or conformal) parameters.
  $X(u, v) = (x(u, v), y(u, v), z(u, v))$, u, v are the conformal parameters. (To be noted: each minimal surface has its own special confimal parameters)

- In these coordinates, a miracle happens! The minimal surface equation becomes,
  $\Delta_{uv} X = 0$ (harmonicity of the coordinates)
  and the isothermality condition, namely, $|X_u| = |X_v|$ and $< X_u, X_v >= 0$.

- Harmonicity of the coordinates makes it possible to relate the theory of minimal surfaces to complex analysis!! Real and imaginary parts of holomorphic functions are harmonic!

(Recall: holomorphic functions are such that their real and imaginary parts satisfy the Cauchy-Riemann equations).

- Complete minimal surfaces cannot be compact!!

The Weierstrass-Enneper Representation of Minimal surfaces

✦ Let $\zeta = u + iv$ and $\bar{\zeta} = u - iv$. For every non-planar minimal surface $X(u, v) = X(\zeta, \bar{\zeta}) = (x(\zeta, \bar{\zeta}), y(\zeta, \bar{\zeta}), \phi(\zeta, \bar{\zeta}))$, $\zeta \in \Omega$, a simply connected domain in $\mathbb{C}$, there is a holomorphic function f and a meromorphic function g in Ω with f, g not identically zero, such that fg^2 is holomorphic in Ω and that

$$
\begin{aligned}
x(\zeta, \bar{\zeta}) &= x_0 + Re \int_{\zeta_0}^{\zeta} \frac{1}{2} f(w)(1 - g^2(w))dw \\
y(\zeta, \bar{\zeta}) &= y_0 + Re \int_{\zeta_0}^{\zeta} \frac{i}{2} f(w)(1 + g^2(w))dw \\
\phi(\zeta, \bar{\zeta}) &= \phi_0 + Re \int_{\zeta_0}^{\zeta} f(w)g(w)dw
\end{aligned}
$$

Re-writing of Weierstrass-Enneper Representation of Minimal surfaces

✦ In a neighbourhood of a non-umbilical point (i.e. where the Gaussian curvature $\neq 0$), any minimal surface can be represented in terms of a non-vanishing meromorphic function $R(\omega)$:

$$
\begin{aligned}
x(\zeta,\bar{\zeta}) = &= x_0 + Re\int_{\zeta_0}^{\zeta}(1-w^2)R(w)dw \\
y(\zeta,\bar{\zeta}) &= y_0 + Re\int_{\zeta_0}^{\zeta} i(1+w^2)R(w)dw \\
\phi(\zeta,\bar{\zeta}) &= \phi_0 + Re\int_{\zeta_0}^{\zeta} 2wR(w)dw
\end{aligned}
$$

✦ Examples:

1. $R(w) = 1$ corresponds to Enneper surface
2. $R(w) = \frac{k}{2w^2}$, k real, corresponds to Catenoid
3. $R(w) = \frac{ik}{2w^2}$, k real, corresponds to Helicoid
4. $R(w) = \frac{ke^{i\alpha}}{2w^2}$, corresponds to the General Helicoid
5. $R(w) = \frac{2}{1-w^4}$, corresponds to the one of the Scherk's minimal surface which can be written in the graph form as

   $z = ln(\frac{cos(y)}{cos(x)})$, (valid in some domain for (x, y)).
6. $R(w) = \frac{-2aisin(2\alpha)}{1+2w^2cos2\alpha+w^4}$, $0 < \alpha < \pi/2, a > 0$ corresponds to the other type of Scherk's surface.

   It is given by $tanh(\frac{z}{a}) = tan(\frac{x}{acos\alpha})tan(\frac{y}{asin\alpha})$.

   Later the formula that will be important for us is when the above is re-written as follows:

   $\frac{x}{acos\alpha} = tan^{-1}(tanh(\frac{z}{a})cot(\frac{y}{asin\alpha}))$.
7. $R(w) = 1 - w^{-4}$, and (substituting $-y$ for y) lead to Henneberg's minimal surface.

8. $R(w) = ia(w^2 - 1)/w^3 - ib/2w^2$,$a$ and b real, and setting $w = e^{-i\gamma/2}$ leads to another type of Enneper's minimal surface, and in particular for $a = 1, b = 0$ to Catalan's surface.

One can generate all examples this way away from the umbilical points.

Gaussian curvature of the surface is $K = -4|R(w)|^{-2}(1 + |w|^2)^{-4}$.

(See Nitsche, Lectures on Minimal Surfaces).

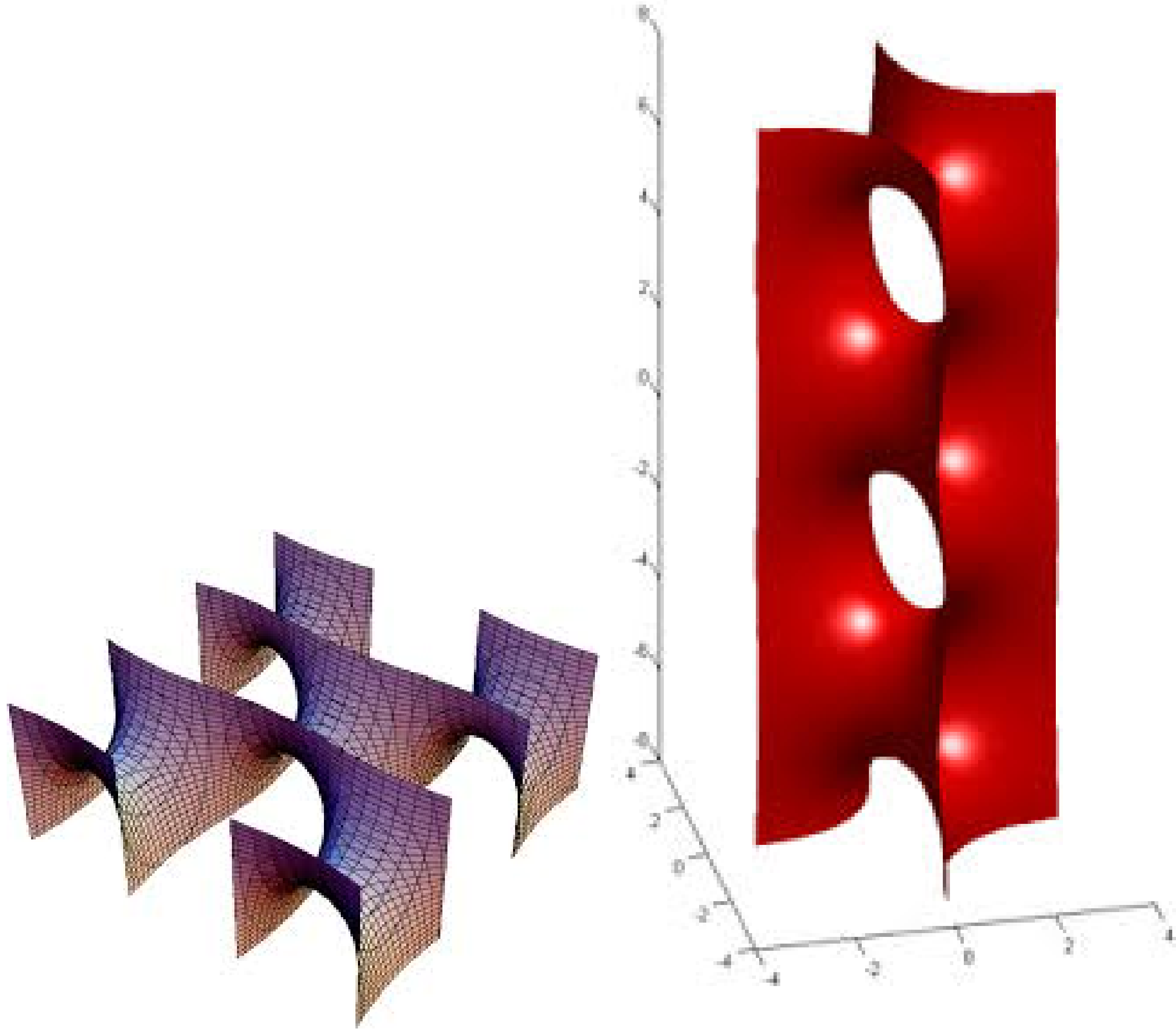

Figure 25: Scherk's surfaces

✦ W-E representation for Catenoid, $\phi = cosh^{-1}(\sqrt{x^2+y^2})$:

$R(w) = \frac{1}{2w^2}$, corresponds to Catenoid.

$$
\begin{aligned}
x(\zeta,\bar{\zeta}) &= \frac{1}{2} Re(\zeta + \frac{1}{\zeta}) \\
y(\zeta,\bar{\zeta}) &= \frac{1}{2} Im(\zeta - \frac{1}{\zeta}) \\
\phi(\zeta,\bar{\zeta}) &= -Reln(\zeta)
\end{aligned}
$$

This representation breaks down at $\zeta = 0$.

✦ W-E representation for the Helicoid, $\phi = tan^{-1}(y/x)$:

where $R(w) = \frac{i}{2w^2}$,

$$
\begin{aligned}
x(\zeta, \bar{\zeta}) &= -\frac{1}{2}\mathrm{Im}(\zeta + \frac{1}{\zeta}) \\
y(\zeta, \bar{\zeta}) &= \frac{1}{2}\mathrm{Re}(\zeta - \frac{1}{\zeta}) \\
z(\zeta, \bar{\zeta}) &= -\frac{\pi}{2} + \mathrm{Im}\ln(\zeta)
\end{aligned}
$$

Isometry between Conjugate Minimal surfaces

- Suppose $X = (x_1(\zeta_1, \zeta_2), y_1(\zeta_1, \zeta_2), z_1(\zeta_1, \zeta_2))$ and $Y = (x_2(\zeta_1, \zeta_2), y_2(\zeta_1, \zeta_2), z_2(\zeta_1, \zeta_2))$ are two minimal surfaces in isothermal coordinates ζ_1, ζ_2 such that X and Y are harmonic conjugates of each other (component-wise).

  In other words, $X + iY$ is an analytic function (each coordinate wise).

Then for all $\theta \in [0, \pi/2]$, $Cos(\theta)X + Sin(\theta)Y$ is a minimal surface again!! (This is a well known fact).

  - For example, the Helicoid and the Catenoid are harmonic conjugates of each other and $Cos(\theta)X_h + Sin(\theta)Y_c$ is again a minimal surface for each θ. It is in fact an isometric deformation!!

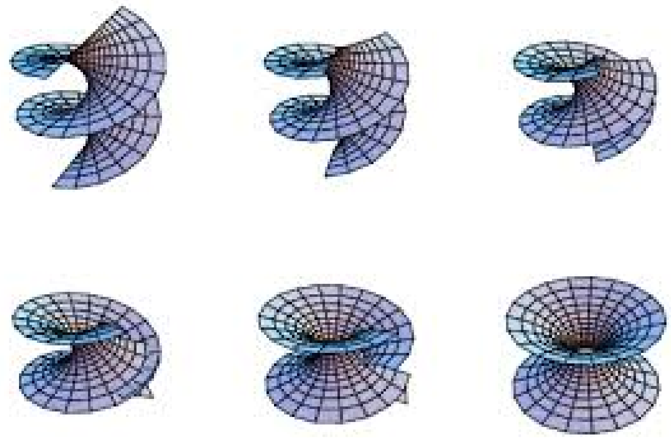

Figure 26: Isometric deformation from helicoid to catenoid

Geometrical interpretation of an identity of Ramanujan

As a consequence of a Ramanujan's identity:

$$\tan^{-1}(\tanh(y)\cot(x)) = \sum_{k=-\infty}^{k=\infty} \tan^{-1}(\frac{y}{x+k\pi}).$$

The left hand side is the height function of Scherk's first surface and the right hand side is sum of height functions of shifted helicoids – all minimal surfaces!

Said differently, $z = \tan^{-1}(\tanh(y)\cot(x))$ is the Scherk's first surface and $z = \tan^{-1}(\frac{y}{x+k\pi})$ is a shifted helicoid.

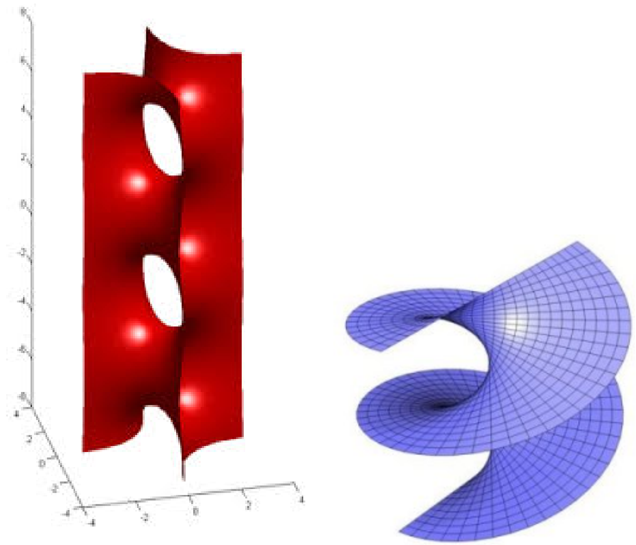

Figure 27: Scherk's first surface and helicoid

This last fact was discovered and used by a condensed matter physicist and his colla
orators –Randall Kamien et. al. , U . Penn.

Thank You