Branching Random Walks: Two Predictions, Two Theorems and a Question

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Indian Statistical Institute

May 14, 2018

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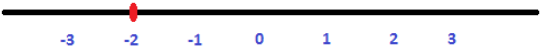

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5. If head appears, then take a step $+1$ and if tail appears, then take a step -1.
6. Repeat 4 and 5 again and again.

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In case of simple random walk, $p_1 = p_{-1} = \frac{1}{2}$ and all other p_i's are 0.

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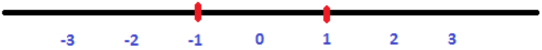

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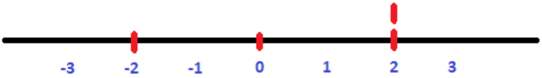

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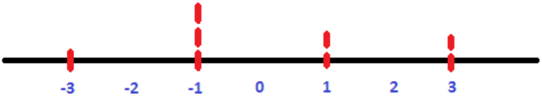

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This model was introduced by Hammerseley (1974), Kingman (1975) and Biggins (1976).

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- For the purpose of this talk, we shall restrict ourselves to the simple model and talk about the *long run configuration* of the positions of particles.

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- The *long run configuration* is of great importance in statistical physics, mathematical biology and probability theory.

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Key Question: What if the conditions are not satisfied?

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Key Question: What if we allow *bigger* step-sizes?

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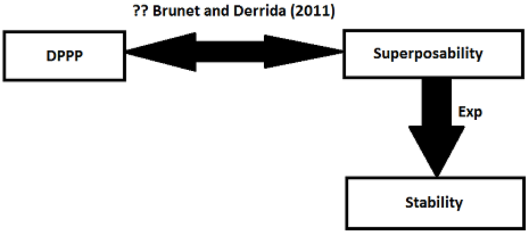

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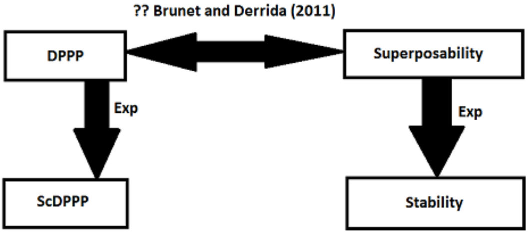

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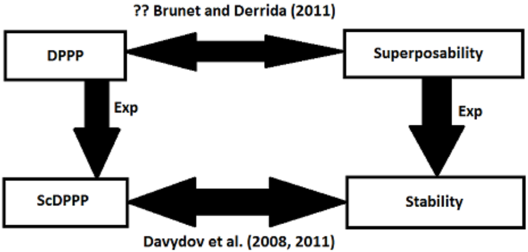

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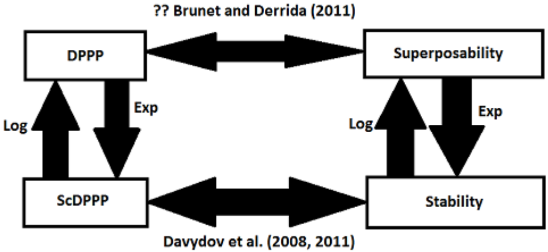

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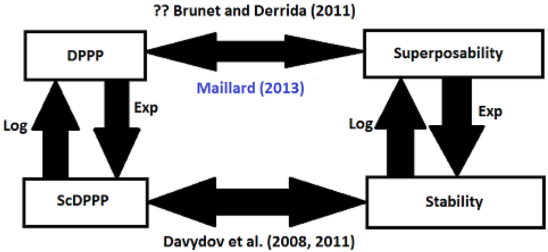

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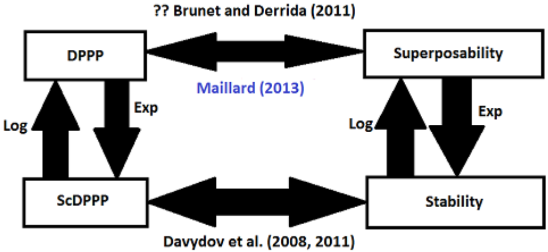

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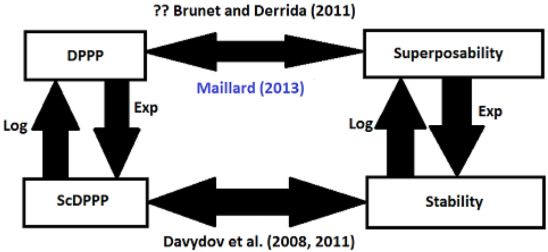

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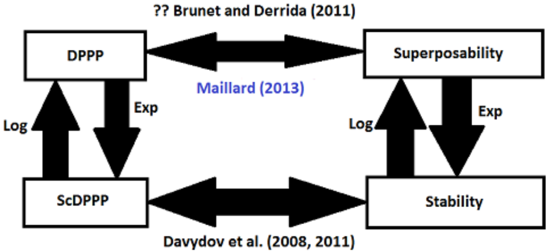

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Bhattacharya, Hazra and R. (2017, 2018): (1) Brunet-Derrida conjectures hold. (2) Long run configuration has been explicitly computed (missing when step-sizes are small).

My collaborators . . .

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