On proofs and definitions in mathematics

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Part-1 (warm-up)

Infinitude of primes

Definition, Statement, Proof - a first example

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Definition

A prime number is any natural number (except 1) that is not divisible by any number other than 1 and itself.

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Proof

If $p_1, \ldots, p_k$ are all the primes, any $n \geq 1$ can be written as

$$n = p_1^{e_1} \ldots p_k^{e_k}$$

for a unique k-tuple of non-negative integers $(e_1, \ldots, e_k)$.
As e_is range over 0 to $m - 1$, we must get all $n \leq 2^m - 1$. But the number of such k-tuples is m^k which is much smaller than $2^m - 1$. $\longrightarrow\!\longleftarrow$

The same proof, in more detail

If there were only $k = 15$ primes, we could make the table

	p_1=2	p_2=3	p_3=5	...	...	p_{15}
2	1	0	0	...	...	0
3	0	1	0	...	...	0
4	2	0	0	...	...	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	2	0	1	...	...	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^m	m	0	0	...	...	0

where each entry is at most m.

- On the left, 2^m distinct numbers.
- On the right, at most m^{15} numbers.
- For large m (eg., $m \geq 25$), the number m^{15} is smaller than 2^m.
- Hence the contradiction, showing that there must be at least 16 primes...

Food for thought

- The proof is so remarkable, that it can only be called a gem! So is the usual proof of Euclid...

- But one should also ask - Why did anyone think of the definition of prime number? Is it inevitable?

- The proof requires ingenuity. It is like problem solving, once the problem has been stated. The definition requires may be even deeper thought. In this case, it is not cleverness or ingenuity as much as clarity of thought and insight as to what concepts will turn out to be really meaningful.

Part-2: A great definition

Gamma function

Euler's extension of the factorial function

$1 + 2 + \ldots + n = \frac{1}{2}n(n+1)$. Analogous formula for $1 \times 2 \times \ldots \times n$?

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Euler's first observation

$$n! = \frac{(n+k)!}{(n+1)\ldots(n+k)} = \frac{k!(k+1)\ldots(k+n)}{(1+n)\ldots(k+n)} = \frac{k!\ k^n}{(1+n)\ldots(k+n)} \frac{(k+1)\ldots(k+n)}{k^n}.$$

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For $z\in\mathbb{C}\setminus\{-1,-2,\ldots\}$, define $z! = \lim_{k\to\infty}\frac{k!\ k^z}{(1+z)\ldots(k+z)}$

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For $z \in \mathbb{C}$ with $\Re z > 0$, define $z! = \int_0^\infty e^{-x} x^{z-1} dx$.

The integral does not make sense for $\Re z \leq 0$, hence the restriction on z.

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Fact

For $\Re z > 0$, the two definitions of $z!$ coincide.

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Can we find explicit value of $z!$ for at least one z that is not a positive integer?! Yes, eg., $(1/2)! = \sqrt{\pi}$.

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The Extended factorial function ("Gamma function") appears everywhere in mathematics. Almost as important as $\sin$ and $\cos$, $\exp$ and $\log$...

Part-2: Another great definition

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Recall the determinant of a 2×2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

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and for a 3×3 matrix

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} =$$

$$a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

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More generally, for an $n \times n$ array of numbers, we make a similar recursive definition. For example, if $n = 4$, we have

$$a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,2} & a_{4,3} & a_{4,4} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,3} & a_{4,4} \end{vmatrix} + \dots$$

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More succinctly,

If $A = (a_{i,j})1 \leq i,j \leq n$, then $\det(A) = \sum_{\pi \in \mathcal{S}_n} \mathrm{sgn}(\pi) \prod_{k=1}^{n} a_{k,\pi(k)}$.

Question

Why this definition? Why not some other combination like for example,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad + bc?$$

First use of determinant: Solving systems of linear equations

- Consider 2 simultaneous equations in 2 variables x, y (for fixed α, β)

$$
\begin{aligned}
ax + by &= \alpha \\
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$$\begin{aligned} ax + by &= \alpha \\ cx + dy &= \beta. \end{aligned}$$

Easy fact

Solution exists for any α, β if and only if $ad - bc \neq 0$, i.e., $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. In such a case, the solution is unique.

Same fact extends to n simultaneous linear equations in n variables.

Lesson

The solvability of a system of linear equations can be checked by computing the determinant of the array of coefficients.

Second use of determinant: Finding volumes

- The determinant of an $n \times n$ matrix is equal in absolute value to the parallelepiped formed by the columns of A (together with the origin).
- This fact can be used also to compute volume changes under non-linear transformations:

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Volume change formula

Let A be a region in the plane and let $T = (T_1(x,y), T_2(x,y))$ be a one-one, onto, differentiable function that maps A onto a region B. Then, $\text{Vol}(A) = \int_B |J_T(x,y)| dxdy$ where

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Same holds in higher dimensions.

Third use of determinant: A counting problem

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Answer

Label vertices so that $V = \{1, \ldots, n\}$, form the $n \times n$ matrix $\mathbb{L}$

$$\mathbb{L}(i,j) = \begin{cases} \deg(i) & \text{if } j = i, \\ -1 & \text{if } j \sim i, \\ 0 & \text{otherwise.} \end{cases}$$

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Determinant

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Example: Complete graph K_n has n^{n-2} spanning trees (Cayley's theorem).

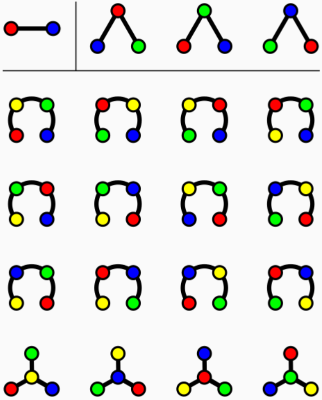

Figure: Illustration of Cayley's theorem (Picture taken from Wikipedia)

Part-3: A great proof

Cantor's diagonal argument

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$$
\begin{array}{l}
0\,1\,0\,0\,1\,0\,0\,\ldots\ldots \\
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1\,1\,0\,1\,0\,0\,0\,\ldots\ldots \\
0\,1\,0\,1\,0\,0\,1\,\ldots\ldots \\
0\,0\,0\,0\,1\,0\,0\,\ldots\ldots \\
1\,1\,1\,0\,1\,0\,1\,\ldots\ldots \\
0\,1\,0\,1\,0\,1\,0\,\ldots\ldots \\
\quad\vdots\ \vdots\ \vdots\ \vdots\ \vdots
\end{array}
$$

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0\,1\,0\,1\,0\,1\,0\ \ldots\ldots \\
\quad\ \vdots\ \ \vdots\ \ \vdots\ \ \vdots\ \ \vdots
\end{array}
$$

then some (in fact many) binary sequence must be missing from the list!

Cantor's diagonal argument

Proof?

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Proof? Consider the *diagonal* of the table

Cantor's diagonal argument

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$$
\begin{array}{ccccccccc}
\mathbf{0} & 1 & 0 & 0 & 1 & 0 & 0 & \ldots\ldots \\
0 & \mathbf{1} & 1 & 1 & 1 & 0 & 1 & \ldots\ldots \\
1 & 1 & \mathbf{0} & 1 & 0 & 0 & 0 & \ldots\ldots \\
0 & 1 & 0 & \mathbf{1} & 0 & 0 & 1 & \ldots\ldots \\
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and form a new sequence by switching zeros and ones to get

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This sequence is not present in the list!

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This is a remarkable proof technique found by Cantor. It occurs again and again in mathematics. Just to drop some names, you will see it in the context of *compactness of product topology*, in *Banach-Alaoglu theorem* in functional analysis, *Prohorov's theorem* in probability theory ...

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Exercise: Let $(x_{1,1}, x_{1,2}, \ldots)$, $(x_{2,1}, x_{2,2}, \ldots)$, $(x_{3,1}, x_{3,2}, \ldots)$, $\ldots$ be sequences of numbers between 0 and 1 (i.e., $0 \leq x_{i,j} \leq 1$ for all i, j). Then there is a common subsequence $n_1 < n_2 < n_3 < \ldots$ such that the sequences $(x_{1,n_1}, x_{1,n_2}, \ldots)$, $(x_{2,n_1}, x_{2,n_2}, \ldots)$, $(x_{3,n_1}, x_{3,n_2}, \ldots)$, $\ldots$ all converge.

Some remarks, from Gian Carlo Rota on the dichotomy between problem solving and theorizing in mathematics

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To the problem solver, the supreme achievement in mathematics is the solution to a problem that has been given up as hopeless....For him mathematics consists of a sequence of challenges to be met...The problem solver is the role model for budding young mathematicians. When we describe to the public the conquests of mathematics, our shining heroes are problem solvers.

The theorizer

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If I were a space engineer looking for a mathematician to help me send a rocket into space, I would choose a problem solver. But if I were looking for a mathematician to give a good education to my child, I would unhesitatingly prefer a theorizer.

Practical remarks relevant to studying mathematics at undergraduate level

Two broad aspects

- Problem solving
- Understanding concepts

We have been emphasizing the second aspect in this lecture
. . . It can be argued that it is the more important task of the two
. . . but -

It is vaguely defined and hence easy to fool oneself that one is thinking profound things while being fairly foolish about them. For a beginner, I offer the possibly less accurate but far more useful practical rule

If you cannot solve problems in a subject, you do not understand the concepts.

More practical suggestions

- At B.Sc. level, it suffices if you learn
  1. *Real analysis* (Apostol level),
  2. *Linear algebra* (Strang level),
  3. *Basic algebra* (Herstein level).
- + some useful skills: basic programming and basic statistics.
- Pick a good text. May also use resources such as lecture videos.

More practical suggestions

- Learning a subject means being able to
  1. *read the text,*
  2. *solve problems,*
  3. *write solutions.*
- If unable to solve problems, go back to reading the text again. Repeat as many times as needed.
- Test yourself by solving old papers of JAM, NBHM, GATE, ...
- An hour spent unsuccessfully trying to solve a problem is time well-spent.

Thank you!

And enjoy the rest of the summer school!