Distributions and fluctuations in single file processes

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J. Cividini, WIS, Rehovot

D. Mukamel, WIS, Rehovot

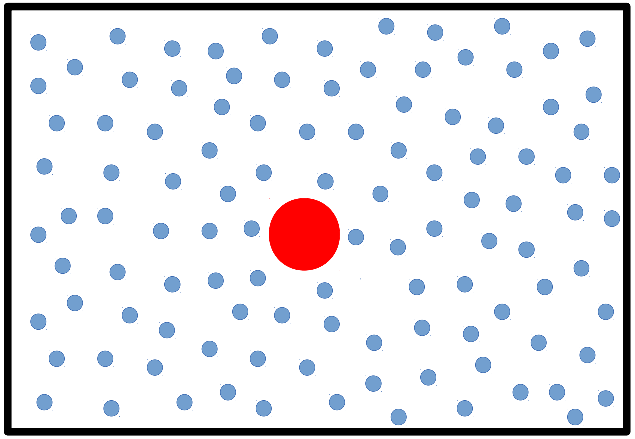
S. N. Majumdar, LPTMS, Orsay

J. Cividini and A. Kundu, arXiv:1704.04017, J. Stat. Mech. (2017), xxxxxx

A. Kundu, J. Cividini, Europhys. Lett. 115, 5, (2016).

J. Cividini, A. Kundu, S. N. Majumdar, D. Mukamel, J. Stat. Mech. (2016) 053212

Motion of particles in crowded medium is a frequent problem in physics, chemistry, biology...



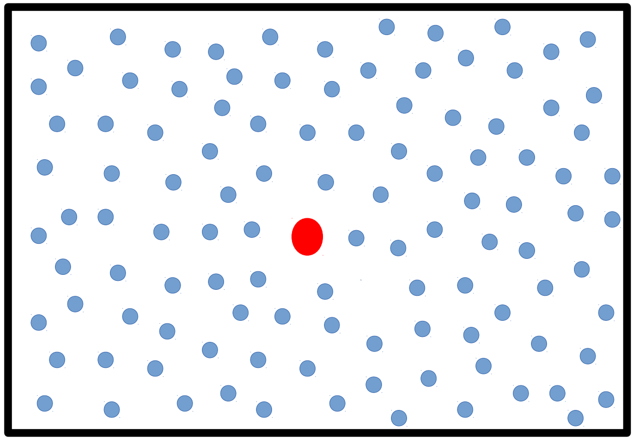
Langevin particle

Described phenomenologically

$$m\ddot{x} = F(x) - \gamma \dot{x} + \eta(t),$$

Time scale of the motion of ( ) is much smaller than the time scale of the motion of ( ).

Motion of particles in crowded medium is a frequent problem in physics, chemistry, biology...



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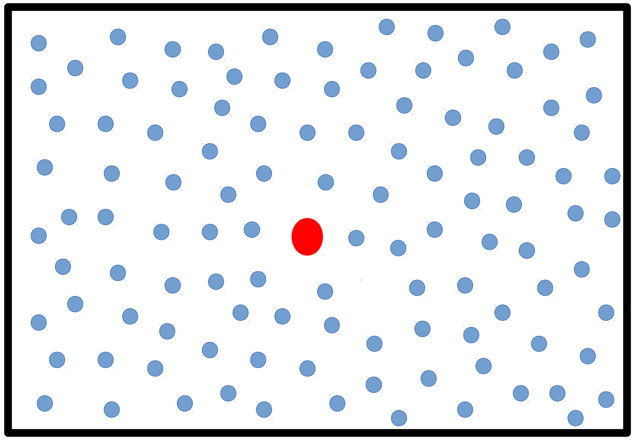
Described phenomenologically

$$m\ddot{x} = F(x) - \gamma\dot{x} + \eta(t),$$

Time scale of the motion of (•) is much smaller than the time scale of the motion of (●).

What happens if the sizes or the time scales are comparable ?

Motion of particles in crowded medium is a frequent problem in physics, chemistry, biology...



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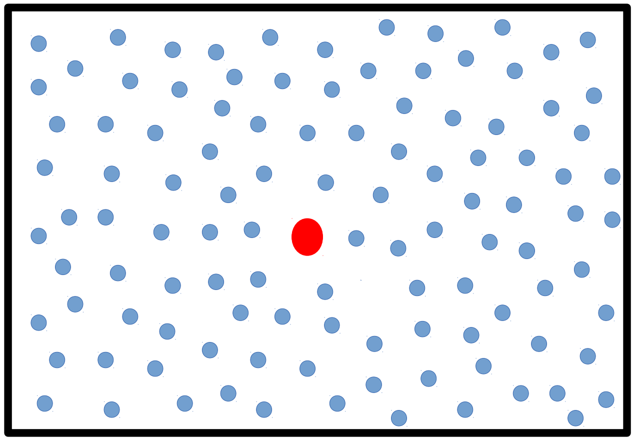
Above phenomenology does not provide accurate description and one needs to consider the full many interacting particle problem.

Motion of particles in crowded medium is a frequent problem in physics, chemistry, biology...



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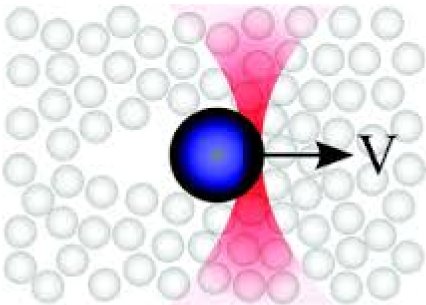
What happens if the sizes or the time scales are comparable ?

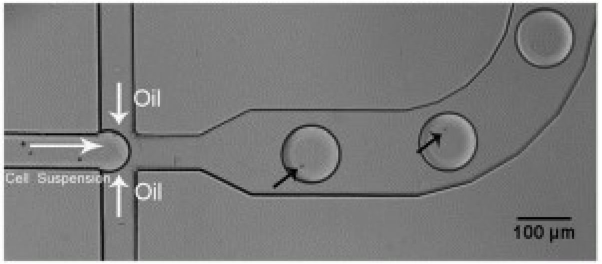
Above phenomenology does not provide accurate description and one needs to consider the full many interacting particle problem.

The interaction among the particles usually have strong short range repulsive part and weak attractive part at large distances.

Active microrheology

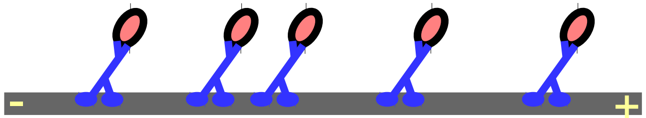


Microfluidic devices

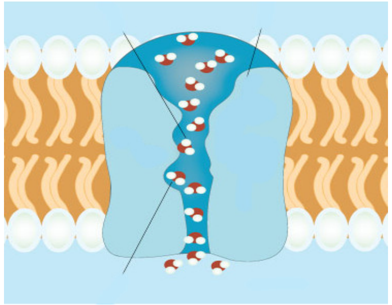


Traffic flow

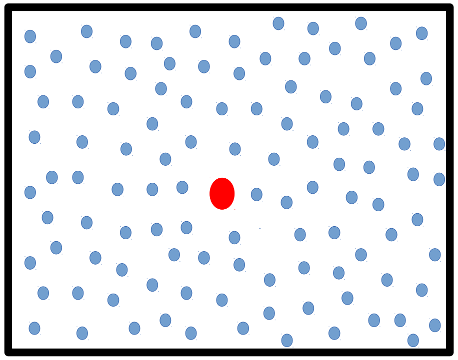
Motion of motor proteins along Actin filaments

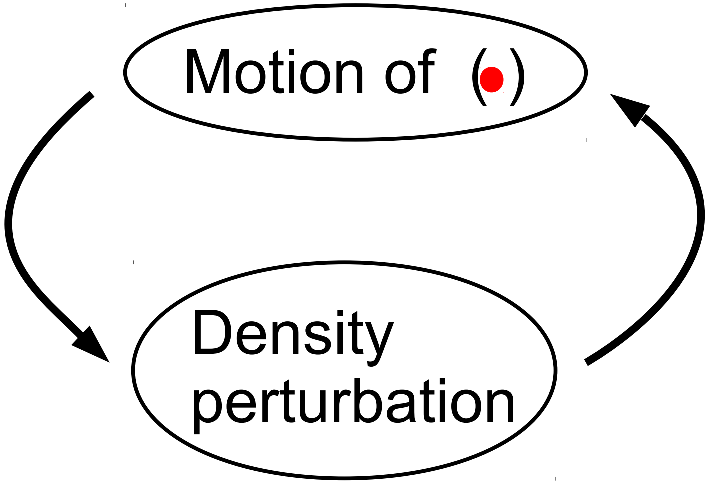


Ion transport through membrane pores



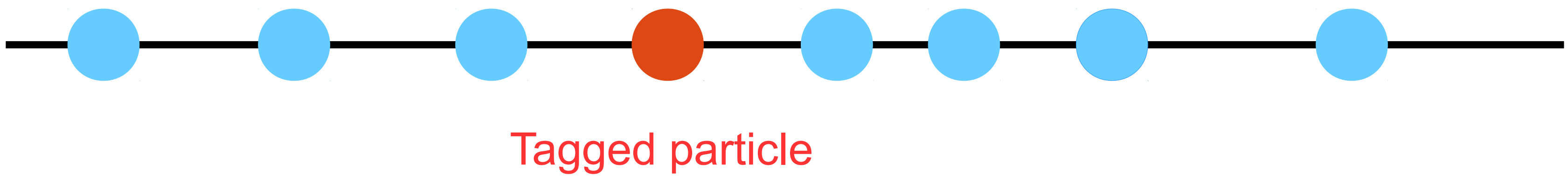
Active transport of vesicle in a crowded axon ….





Motion of different
Particles are strongly
Correlated.

One dimensional systems : Single file systems



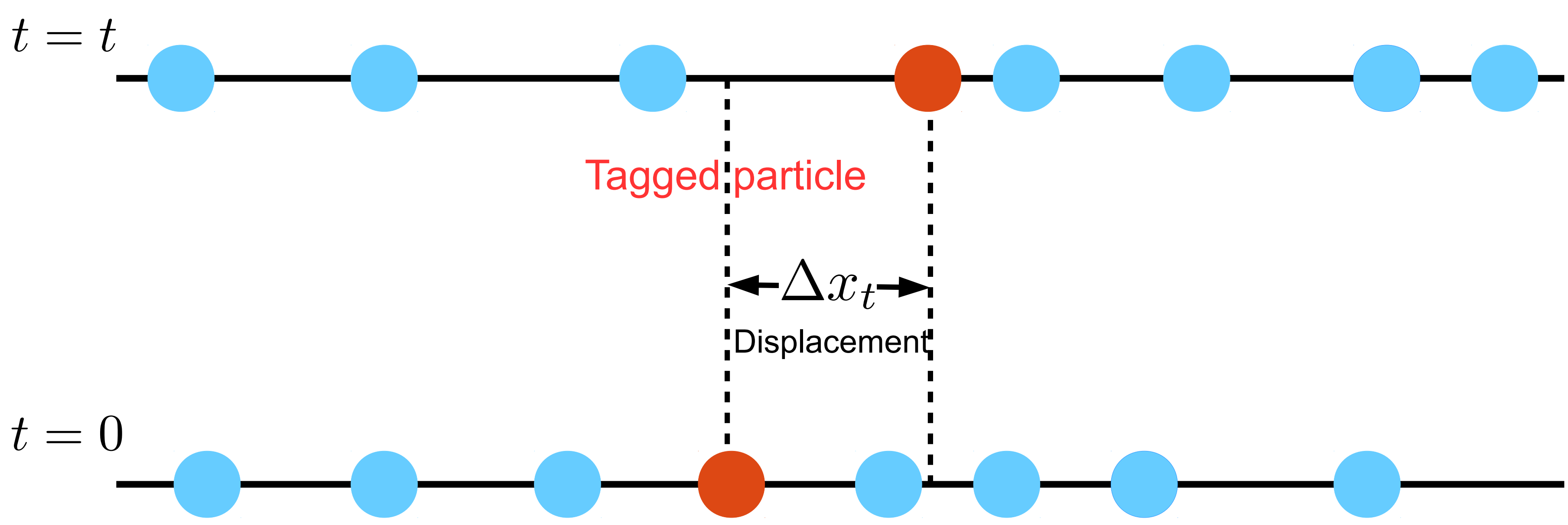
For Free Brownian particles :

$$\langle \Delta X_t^2 \rangle_{free} \sim t$$ Diffusive

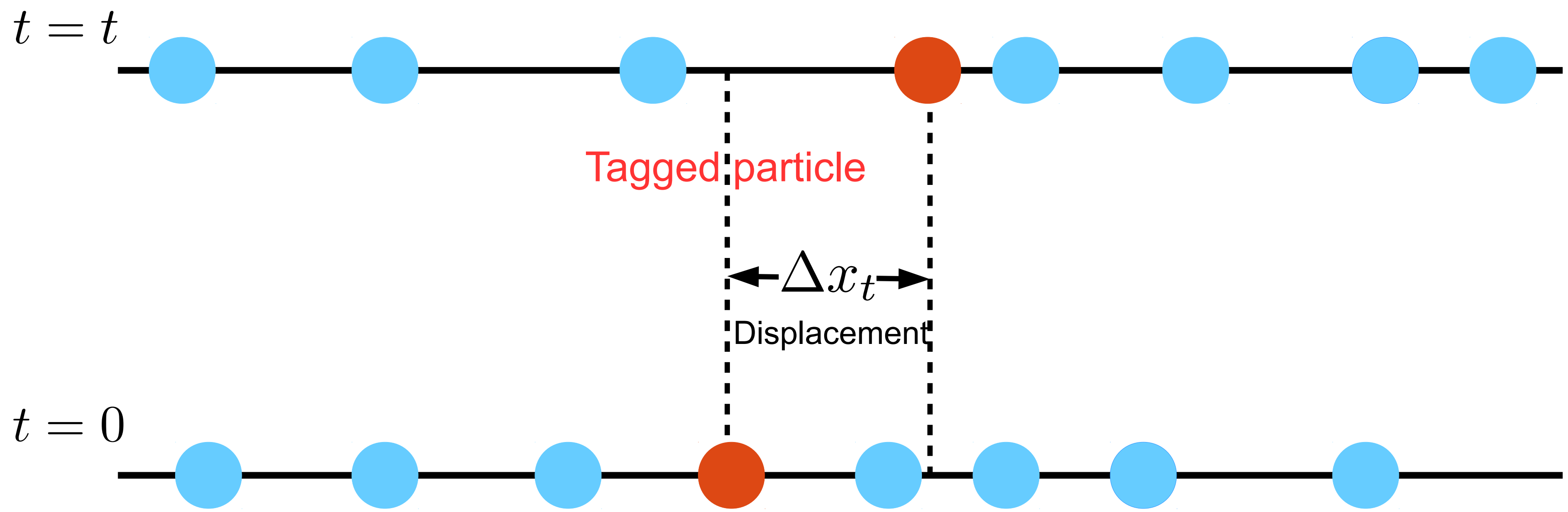
For hard point Brownian particles :

$$\langle \Delta X_t^2 \rangle_{tag} \sim \sqrt{t}$$ Sub-Diffusive

One dimensional systems : Single file systems



One dimensional systems : Single file systems



For hard point particles :

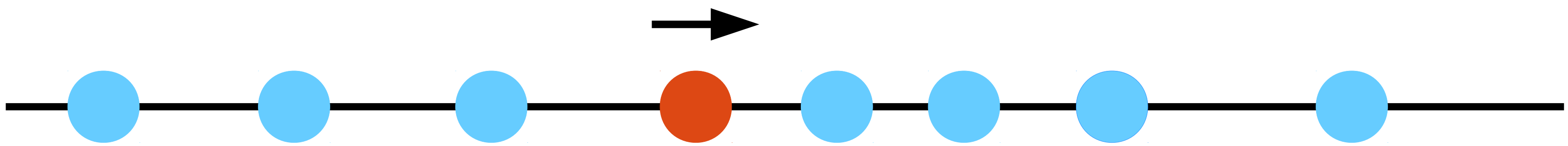
$$\langle \Delta X_t^2 \rangle_{tag} =$$

- Ballistic → $\sim \; t$ — Diffusive, Jepsen (1965)
- Brownian → $\sim \; \sqrt{t}$ — Sub-Diffusive, Harris (1965), Percus (1974)

Theoretical studies in homogeneous single file system :

- Jepsen (1965) [ballistic];
- Harris (1965) [diffusive]
- Lebowitz & Percus (1967) [ballistic]
- Lebowitz & Sykes (1972) [ballistic]
- Levitt (1973) [ballistic with random kicks from environment]
- Percus (1974) [ballistic with possibly random environment]
- Arratia (1983) [SSEP]
- van Beijeren, Kehr & Kutner (1983) [hardcore lattice gas]
- Alexander & Pincus (1978) [diffusive]
- Rödenbeck, Kärger & Hahn (1998) [using reflection principle]
- Majumdar & Barma (1991) [hardcore lattice gas with bias]
- Rajesh & Majumdar (2001) [Random average process]
- Lizana & Ambjörnsson (2008, 2009) [Bethe ansatz solution]
- Barkai & Silbey (2009, 2010) [in a force field]
- Kollmann (2003) [both diffusive and ballistic]
- Gupta, Majumdar, Godrèche & Barma (2007) [ASEP: finite size effect]
- Roy, Narayan, Dhar & Sabhapandit (2013, 2014) [various Hamiltonian systems]
- Sabhapandit (2007) [semi-infinite system]
- Hegde, Sabhapandit & Dhar (2014) [Large deviation, annealed IC]
- Krapivsky, Mallick & Sadhu (2014) [Large deviation, Hydrodynamic approach]

Theoretical studies in locally driven single file system :



Mean displacement, Force velocity relation, inhomogeneous density profile : (1992 - 2010)

- Burlatsky, Oshanin, Mogutov, Moreau (1992), (1996) [Mean displacement in 1D]
- De Conink, Oshanin, Moreau (1997) [Force velocity in 2D monolayer]
- Benichou, Cazabat, Lemarchand, Moreau, and Oshanin (1999, 2000a, 2000b, 2001)
  [FVR in adsorbed Monolayer and inhomogeneous density profile]

Fluctuations, correlations, distributions : (2010 -)

- Demery & Dean (2010, 2011) [motion coupled to classical fields]
- Illien, Bénichou, Mejía-Monasterio, Oshanin & Voituriez (2013) [bias on tagged particle]
- Bénichou, Illien, Mejía-Monasterio, Oshanin & Voituriez (2013) [Fluctuations]
- Demery, Benichou, Jacquin (2014) [Hydrodynamic approach in 2D]
- Illien, Benichou, Oshanin and Voituriez, (2015) [Distribution of tracer, dilute limit]

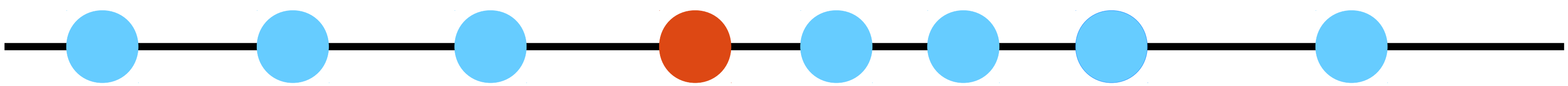
In this talk

1. Homogeneous single file motion :

Exact **full** probability distribution of the displacement of the tagged (tracer) particle for **arbitrary initial** configuration of the particle positions

J. CivIdini and A. Kundu,

arXiv:1704.04017



2. In-homogeneous single file motion : (Locally driven)

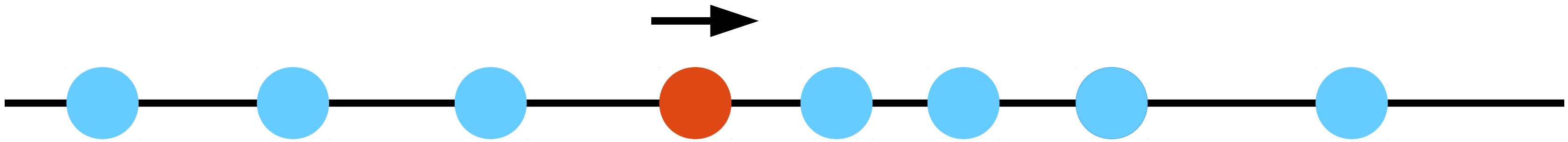
Fluctuations and correlations of the displacement of the tagged (tracer) particle for annealed and quenched initial conditions

A. Kundu, J. Cividini,

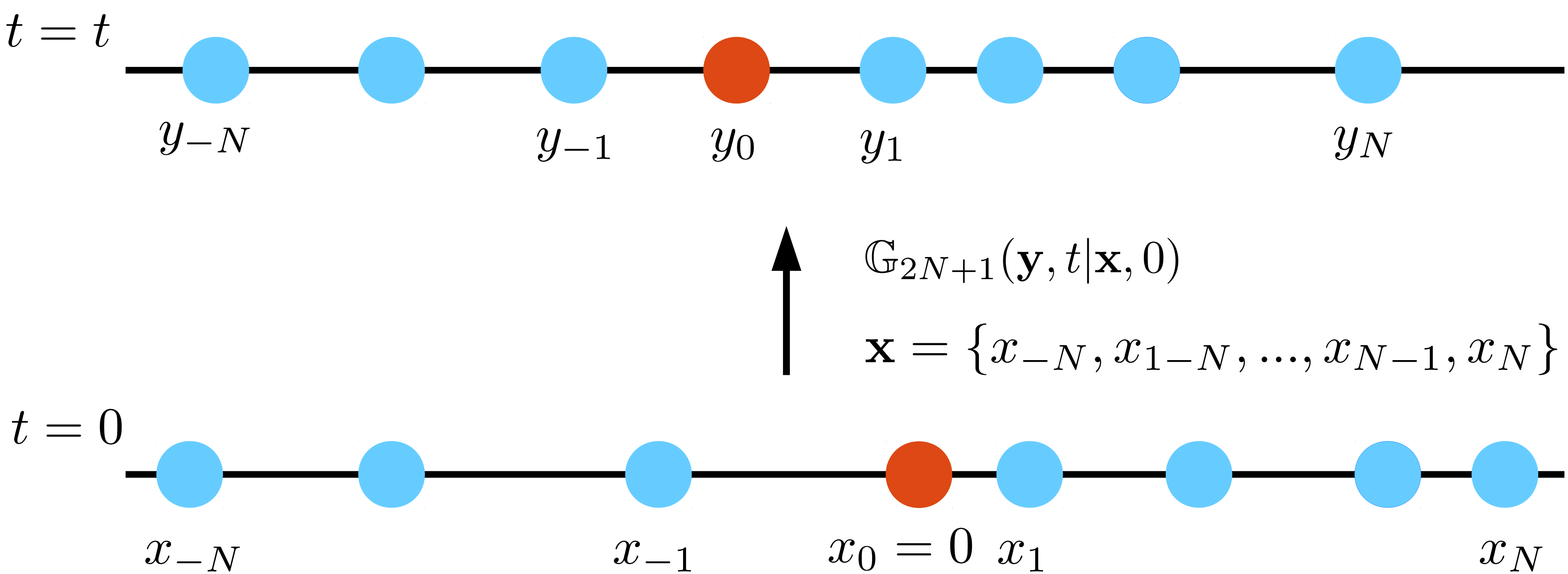
Euro. Phys. Lett. 115, 5, (2016).

J. Cividini, A. Kundu, S. N. Majumdar,

D. Mukamel, J. Stat. Mech. (2016) 053212



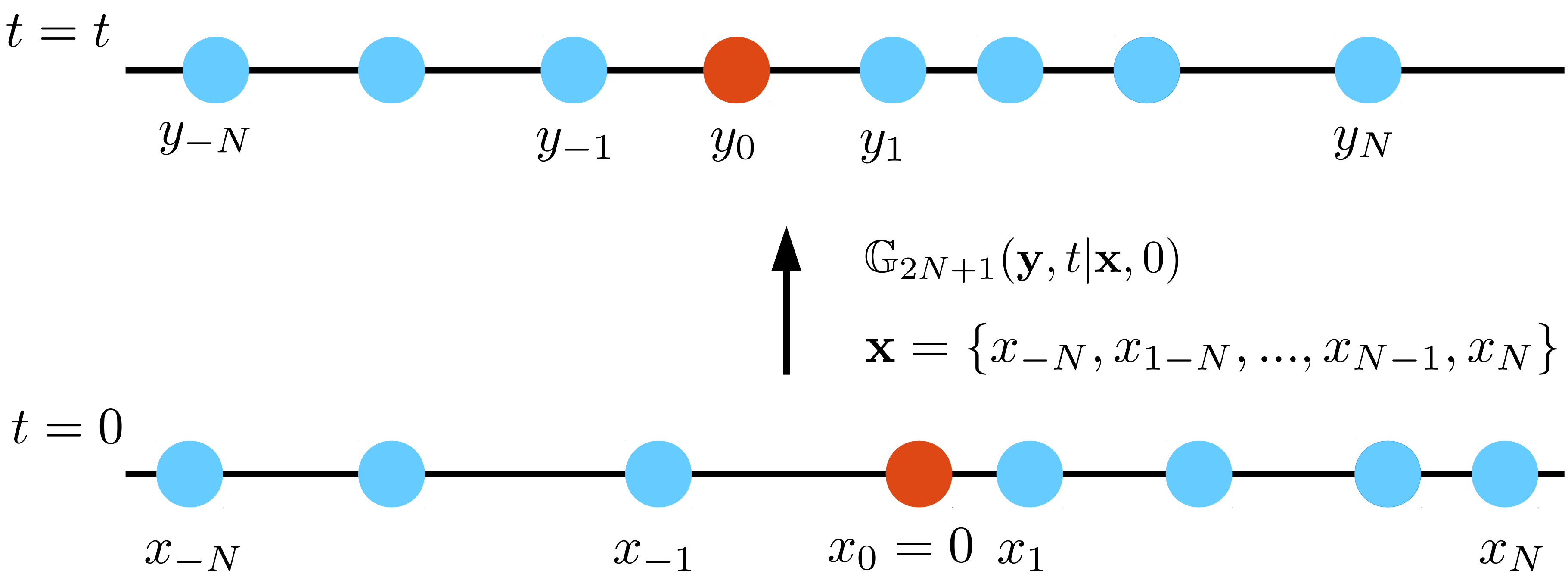
Multi-particle Propagator:



We consider (2N+1) particles with the middle particle at origin $x_0 = 0$
And N particles on the left and rest of the N particles on the right of it.

$$\text{Displacement} = \Delta X_t = y_0 - x_0 = y_0$$

Multi-particle Propagator:



$$\mathbb{G}_{2N+1}(\mathbf{y}, t|\mathbf{x}, 0) = \sum_{\tau \in \mathfrak{S}_{2N+1}} \prod_{k=-N}^{N} g(y_k, t|x_{\tau(k)}, 0), \qquad g(y, t|x, 0) = \frac{1}{\sigma_t} G\left(\frac{y-x}{\sigma_t}\right)$$

For brownian and ballistic particles : $G(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \qquad \sigma_t = \begin{cases} \sqrt{t}, & \text{Brownian} \\ t, & \text{Ballistic} \end{cases}$

Basic idea :

1. Integrate all the final positions except for the tagged particle

$$P(y_0,t|\mathbf{x}) = \underbrace{\int_{-\infty}^{y_0} dy_{-1} ... \int_{-\infty}^{y_{2-N}} dy_{1-N} \int_{-\infty}^{y_{1-N}} dy_{-N}}_{Integrations\ over\ particles\ on\ left} \underbrace{\int_{y_0}^{\infty} dy_N ... \int_{y_0}^{y_3} dy_2 \int_{y_0}^{y_2} dy_1}_{Integrations\ over\ particles\ on\ right} \mathbb{G}(\mathbf{y},t|\mathbf{x},0)$$

2. Average over the initial configurations

$$P(y_0,t) = \langle P(y_0,t|\mathbf{x}) \rangle_{ini}$$

3. Take the thermodynamic limit :

$$N \to \infty, \quad L \to \infty, \quad \text{keeping} \quad \frac{N}{L} = \rho, \quad \text{fixed}$$

Basic idea :

1. Integrate all the final positions except for the tagged particle

$$P(y_0, t|\mathbf{x}) = \underbrace{\int_{-\infty}^{y_0} dy_{-1} \ldots \int_{-\infty}^{y_{2-N}} dy_{1-N} \int_{-\infty}^{y_{1-N}} dy_{-N}}_{Integrations\ over\ particles\ on\ left}$$

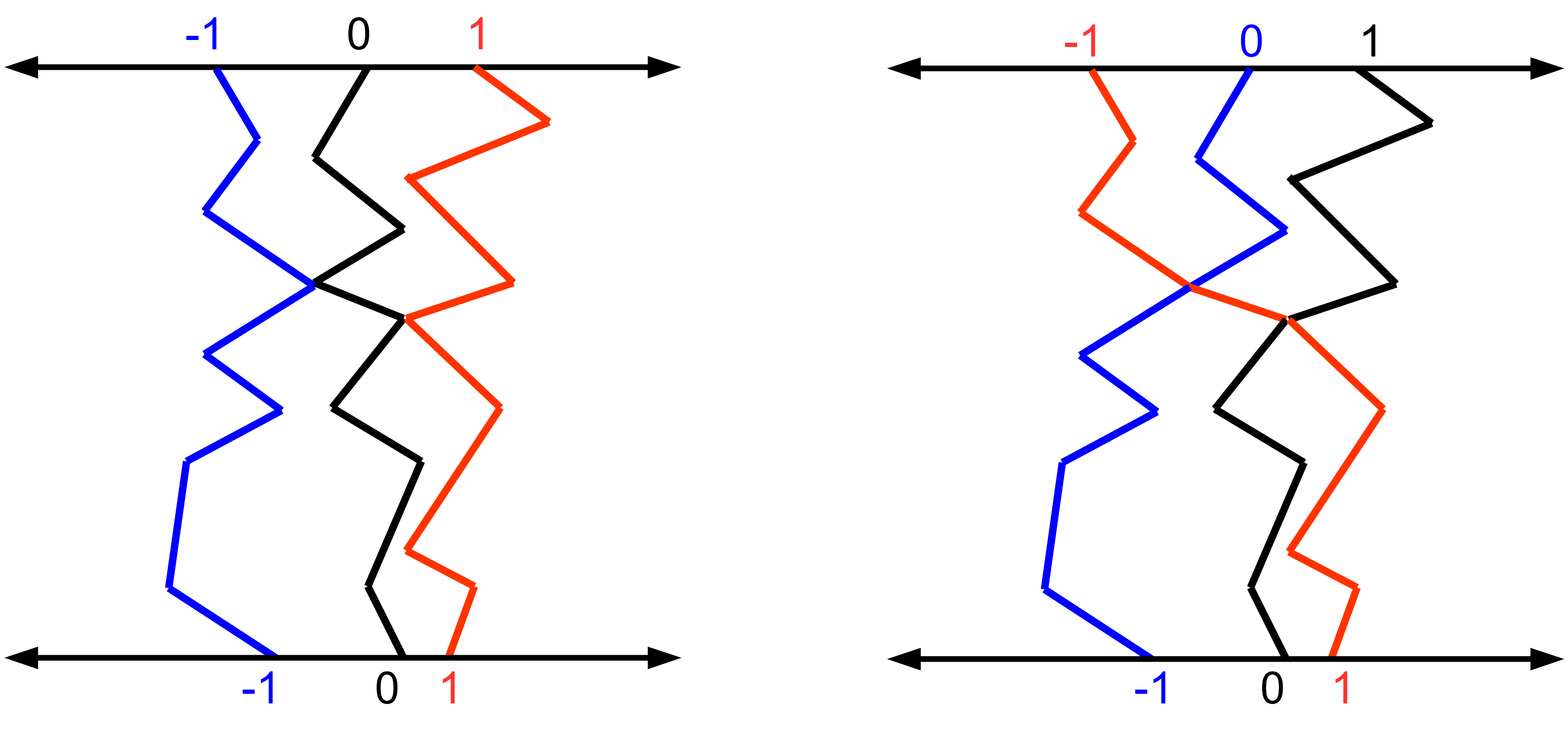
$$\times \underbrace{\int_{y_0}^{\infty} dy_N \ldots \int_{y_0}^{y_3} dy_2 \int_{y_0}^{y_2} dy_1}_{Integrations\ over\ particles\ on\ right} \quad \mathbb{G}(\mathbf{y}, t|\mathbf{x}, 0)$$

Mapping to non-interacting walkers :

For identical initial configurations

Prob.~[the displacement of the 0th particle in time t] = Prob.~[the difference between the positions of the 0th particle at time t and the 0th particle at t=0]

$$P(y_0, t|\mathbf{x}, 0) = P_{NI}(y_0, t|\mathbf{x}, 0), \quad \mathbf{x} = \{x_{-N}, ..., x_N\}$$



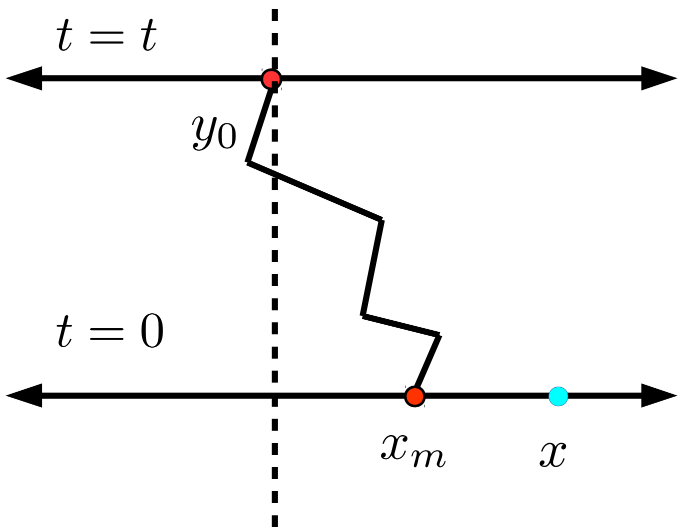
Interacting walkers

Non-interacting walkers

$$P(y_0, t|\mathbf{x}) = \sum_{m=-N}^{N} g(y_0, t|x_m, 0) \prod_{k=-N, k\neq m}^{N} \left(\sum_{\epsilon_k = \pm} \right) \delta_{\sum_{k=-N,k\neq m}^{N} \epsilon_k, 0}$$
$$\times \prod_{k=-N, k\neq m}^{N} g_{\epsilon_k}(x_k; y_0, t)$$

Prob.~[the particle starting from x
reaches on the left of y_0
in time t]

$$= g_-(x; y_0, t) = \int_{-\infty}^{y_0} dy' \; g(y', t|x, 0)$$



Prob.~[the particle starting from x
reaches on the right of y_0
in time t]

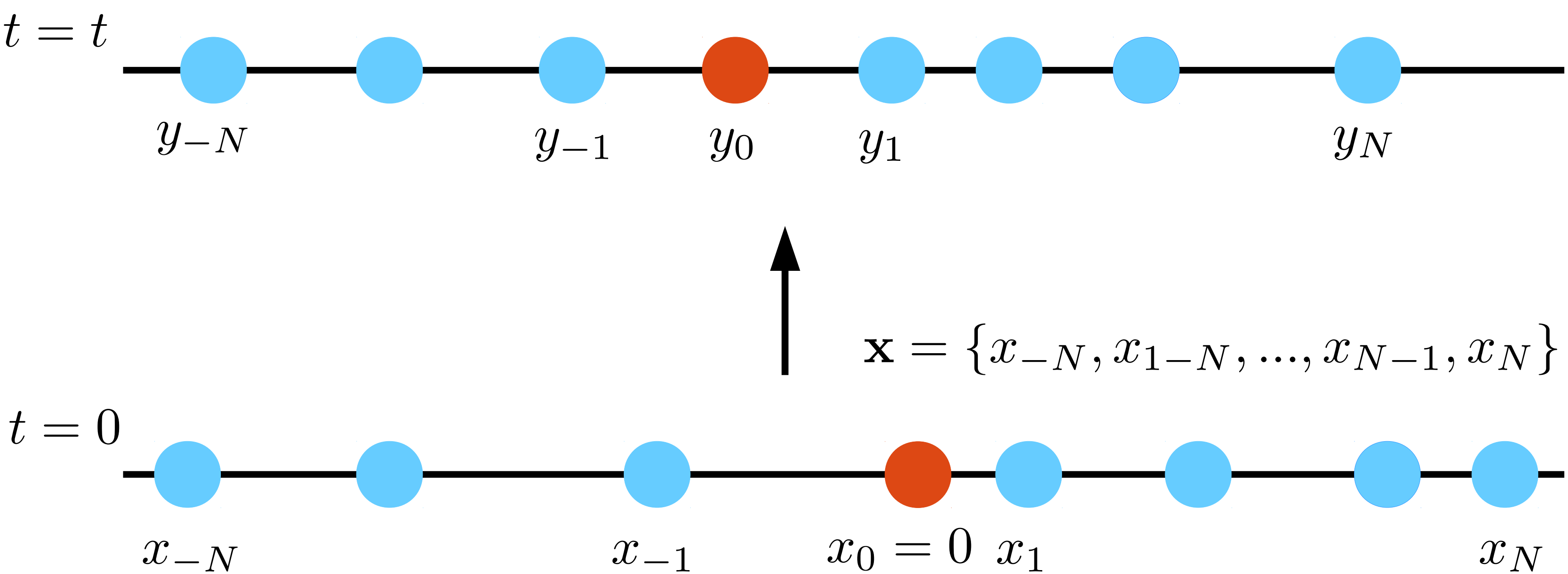
$$= g_+(x; y_0, t) = \int_{y_0}^{\infty} dy' \; g(y', t|x, 0)$$

$$P(y_0, t|\mathbf{x}) = \sum_{m=-N}^{N} g(y_0, t|x_m, 0) \prod_{k=-N, k\neq m}^{N} \left(\sum_{\epsilon_k=\pm} \right) \delta_{\sum_{k=-N,k\neq m}^{N} \epsilon_k, 0} \times \prod_{k=-N,k\neq m}^{N} g_{\epsilon_k}(x_k; y_0, t)$$

$$\delta_{n,0} = (1/2\pi) \int_{-\pi}^{\pi} e^{in\theta} d\theta$$

$$P(y_0, t|\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}y_0} \left[-\frac{1}{4\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{\sin(\theta/2)} \prod_{k=-N}^{N} \left(\mathrm{e}^{\mathrm{i}\frac{\theta}{2}} g_+(x_k; y_0, t) + \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}} g_-(x_k; y_0, t) \right) \right]$$

Can be easily checked for the normalization



$$P(y_0,t|\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}y_0}\left[-\frac{1}{4\pi\mathrm{i}}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{\sin(\theta/2)}\prod_{k=-N}^{N}\left(\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}g_+(x_k;y_0,t)+\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}g_-(x_k;y_0,t)\right)\right]$$

Can be easily checked for the normalization

J. Cividini and A. Kundu,

arXiv:1704.04017

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$$\dots \int_{-\infty}^{y_0} dy_{-1} \int_{y_0}^{y_2} dy_1 \int_{y_0}^{y_3} dy_2 \dots \int_{y_0}^{\infty} dy_N \; \mathbb{G}(\mathbf{y}, t|\mathbf{x}, 0)$$

2. Average over the initial configurations

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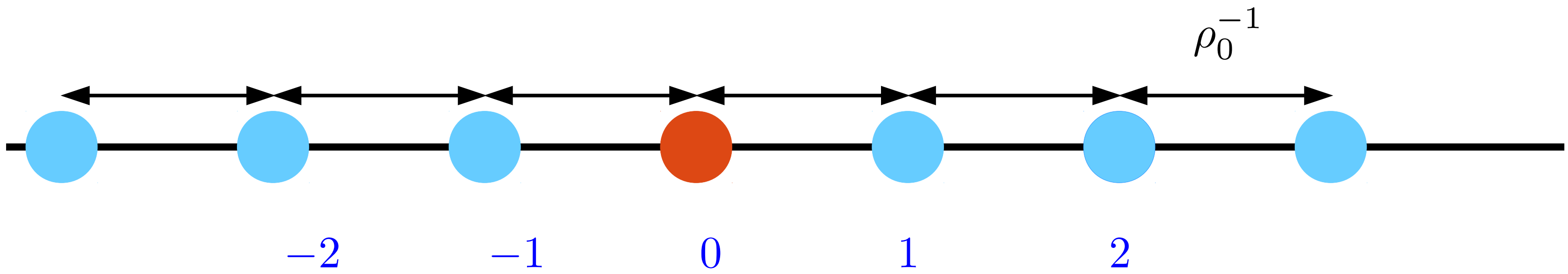
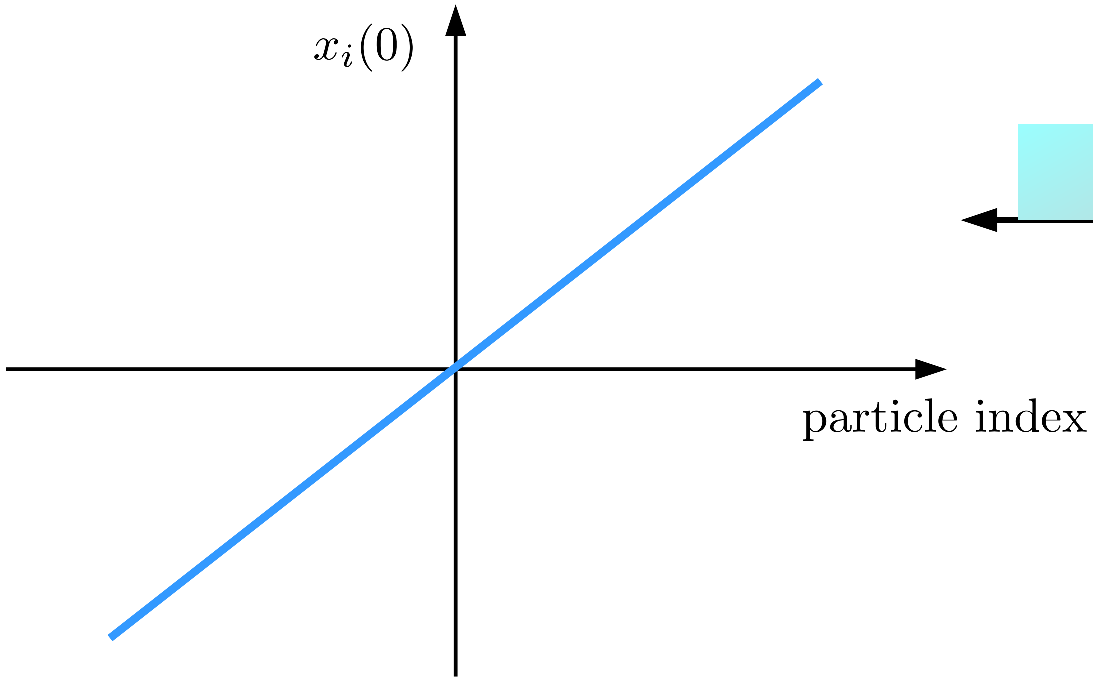
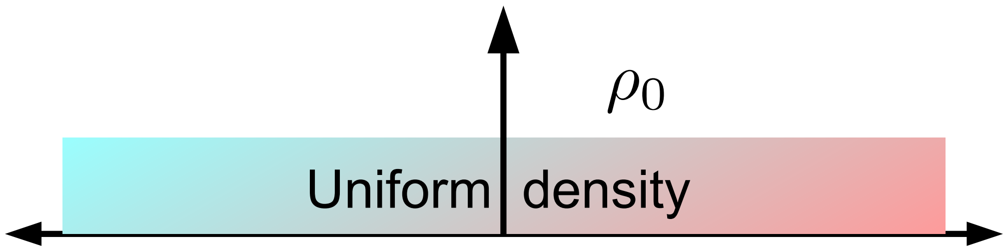
Initial Configuration : earlier studies



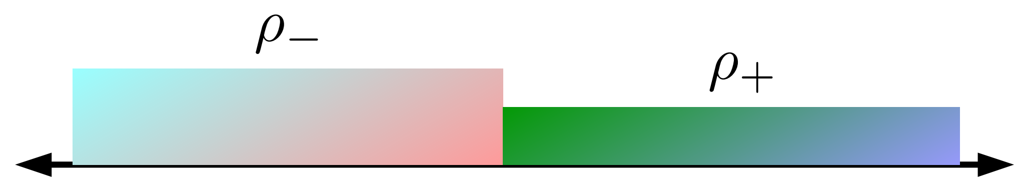
$$x_i(0) = \rho_0^{-1}\, i \;; \qquad \ldots, -2, -1, 0, 1, 2, \ldots \;, \qquad \rho_0 = \frac{N}{L}$$





Distribution:

Annealed : Step initial condition



$$P(y_0,t) = \frac{\mathrm{e}^{-\sigma_t\left((\rho_+ + \rho_-)Q + \frac{\rho_+ - \rho_-}{2}Y\right)}}{\sigma_t}\Bigg[\left(G + \sigma_t(\rho_+ + \rho_-)G_+G_-\right) I_0\left(\sigma_t\sqrt{\rho_+\rho_-}\sqrt{4Q^2 - Y^2}\right) + \sigma_t\left(\rho_+ G_-^2\sqrt{\frac{2Q-Y}{2Q+Y}} + \rho_- G_+^2\sqrt{\frac{2Q+Y}{2Q-Y}}\right) I_1\left(\sigma_t\sqrt{\rho_+\rho_-}\sqrt{4Q^2 - Y^2}\right)\Bigg].$$

$$Q(Y) = Y\int_{Z=0}^{Y} G(Z)\mathrm{d}Z + \int_{Z=Y}^{\infty} ZG(Z)\mathrm{d}Z,$$

J. Cividini and A. Kundu,

arXiv:1704.04017

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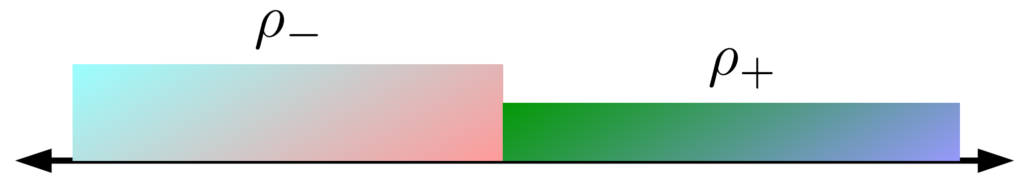
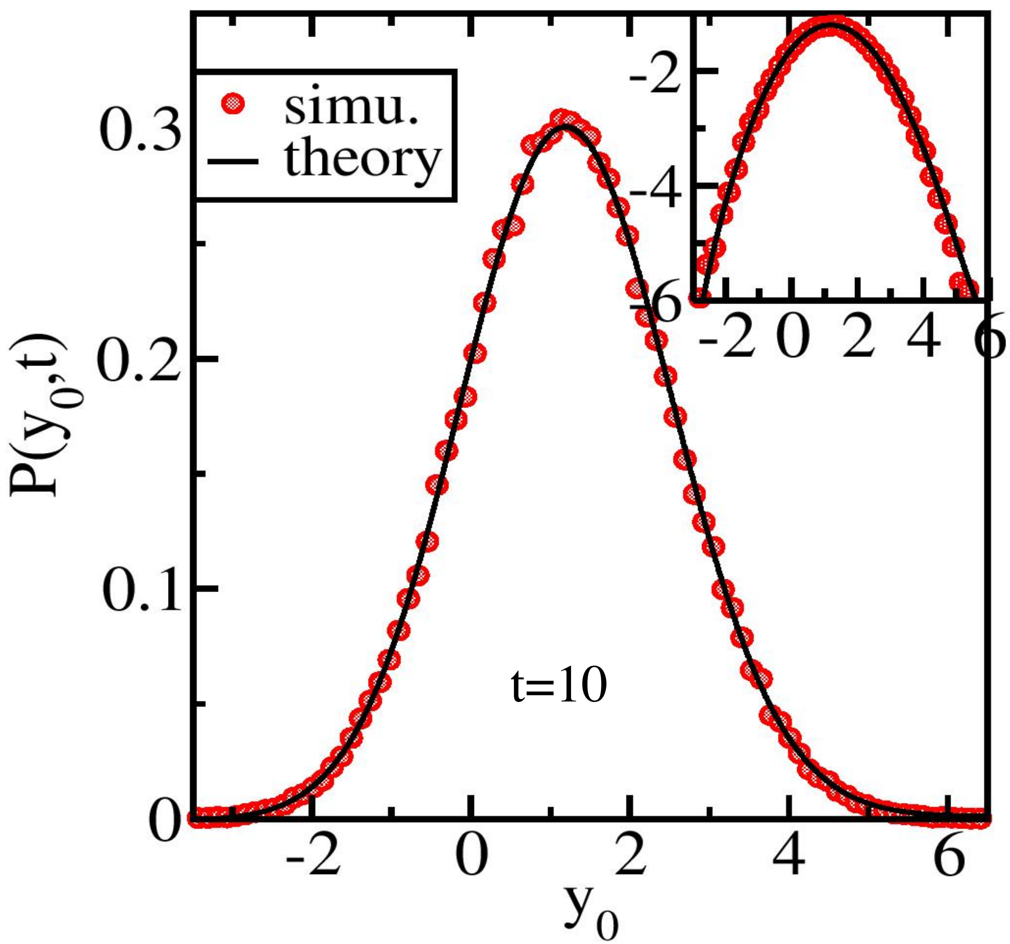
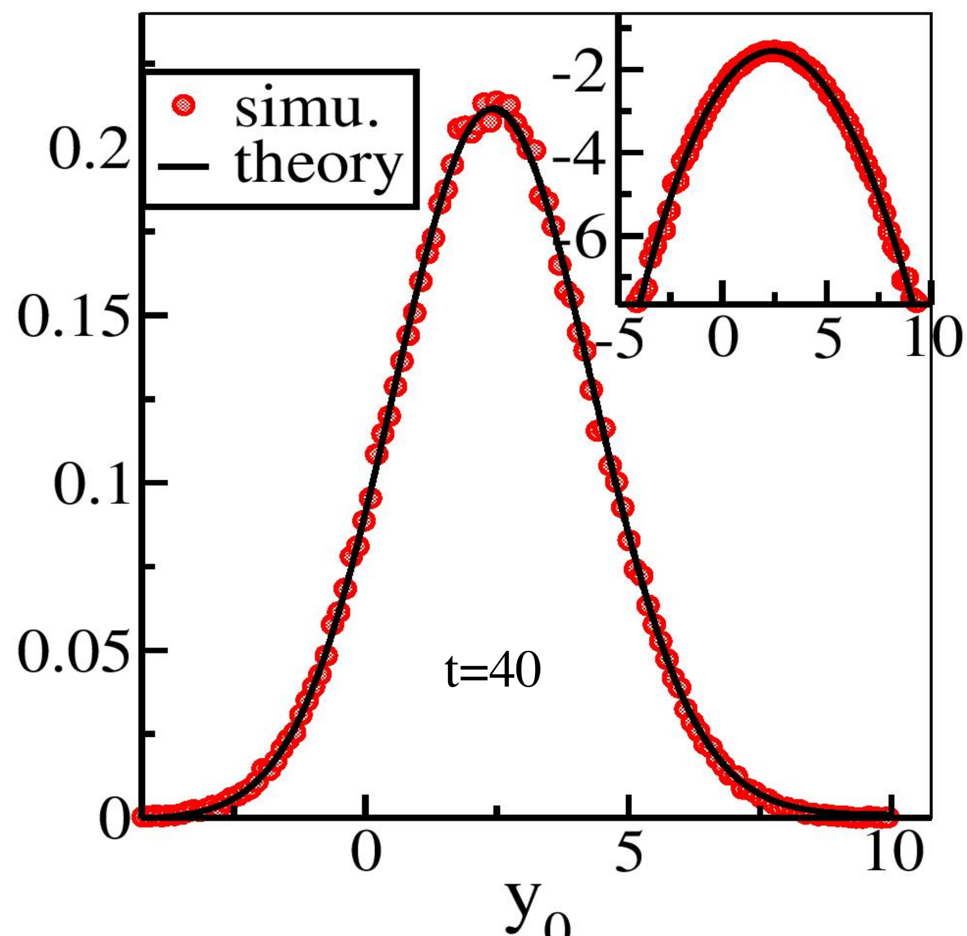
Annealed : Step initial condition



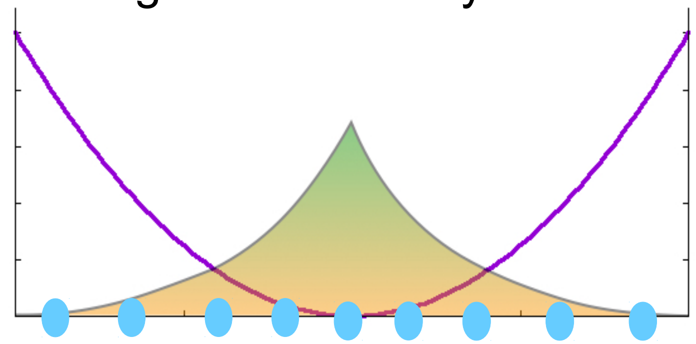
$P(y_0, t) =$ Exact explicit expression in terms of Bessel I_0 and I_1 functions





Initial configurations :

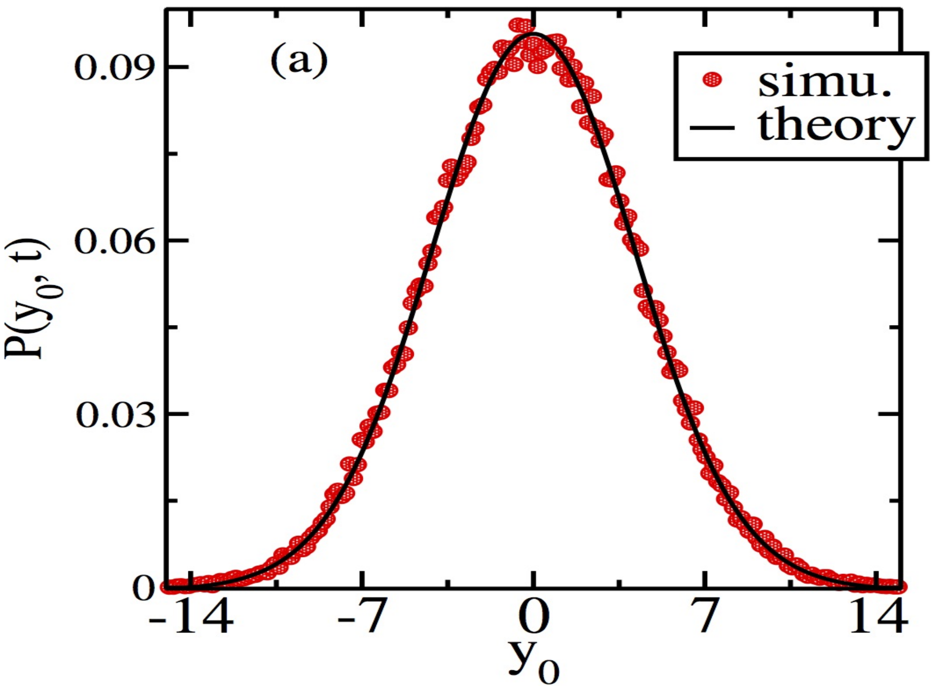
Inhomogeneous density:



$$x_k = A \ \mathrm{Sign}(k) \ |k|^\alpha, \ \alpha > 0$$

J. Cividini and A. Kundu,

arXiv:1704.04017



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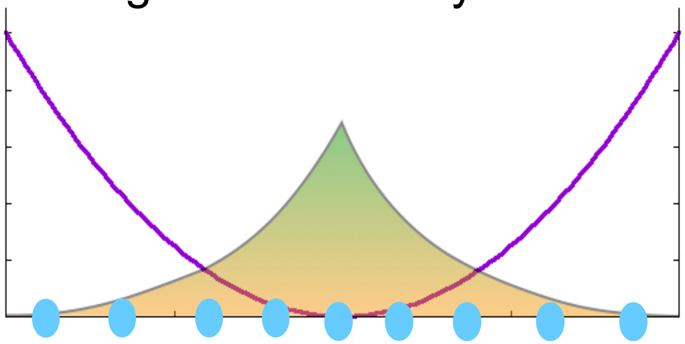

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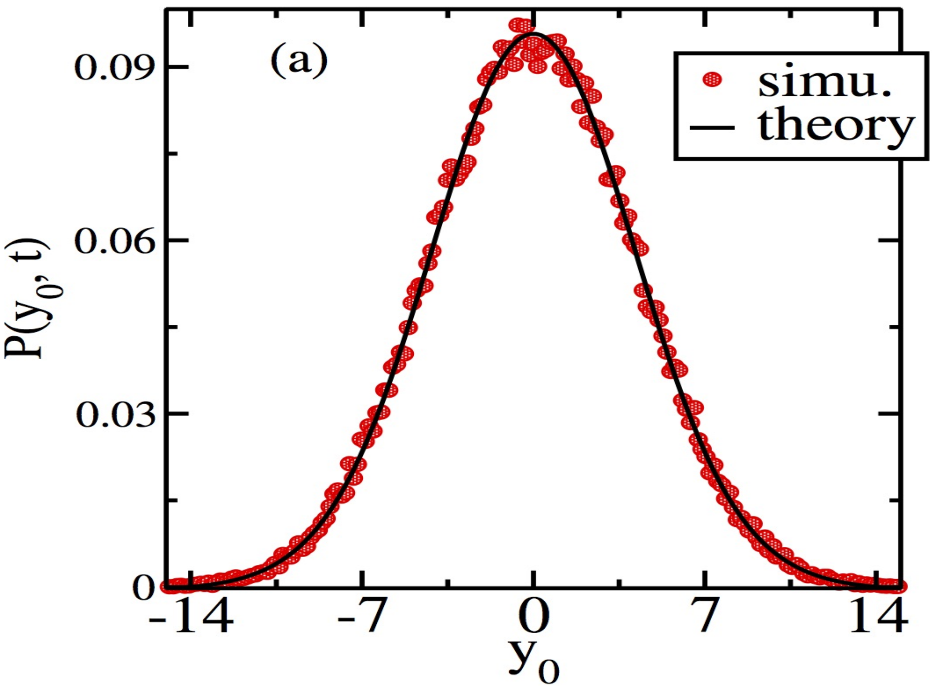
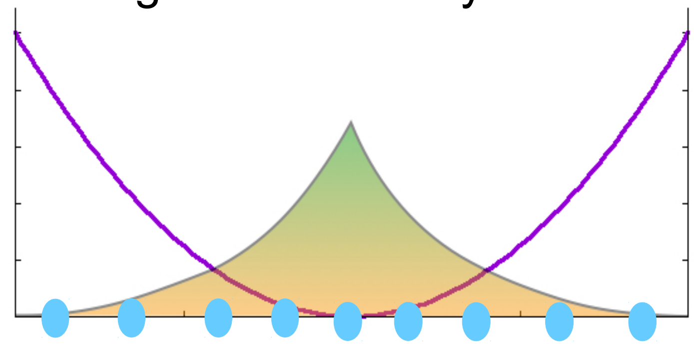


$$P_{\varrho}(y_0,t) \asymp \exp[-{\sigma_t}^{-2}\mathcal{F}_{\alpha}(y_0/\sigma_t)]$$

$$\mathcal{F}_{\alpha}(Y) \ = \ \frac{1}{A^{1/\alpha}\alpha}\left[\sqrt{\int_{Z=-\infty}^{0} \mathrm{d}Z|Z|^{1/\alpha-1}G_{+}(Y-Z)} - \sqrt{\int_{Z=0}^{\infty} \mathrm{d}Z|Z|^{1/\alpha-1}G_{-}(Y-Z)}\right]^2 .$$

Initial configurations :

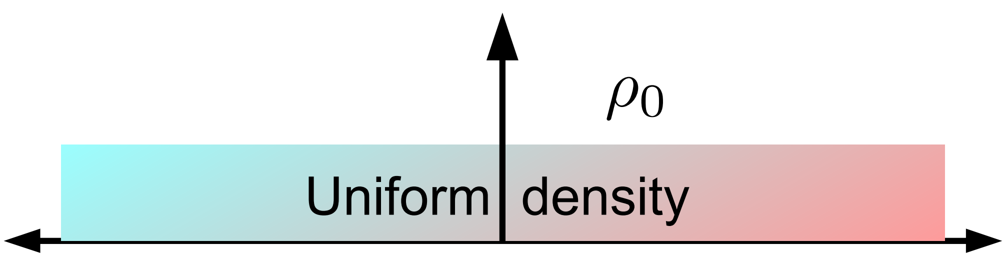
Inhomogeneous density:



$$x_k = A\ \mathrm{Sign}(k)\ |k|^{\alpha},\ \ \alpha > 0$$

$$\langle \Delta X_t^2 \rangle_{tag} \sim \sigma_t^{2-1/\alpha}.$$

For uniform initial configurations:



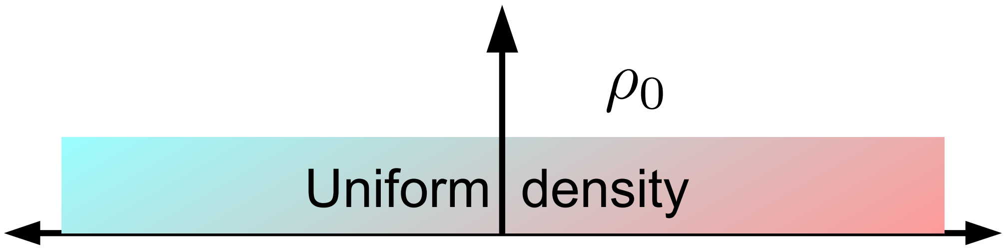
$$\langle \Delta X_t^2 \rangle_{tag} \sim \sigma_t$$

J. Cividini and A. Kundu,

arXiv:1704.04017

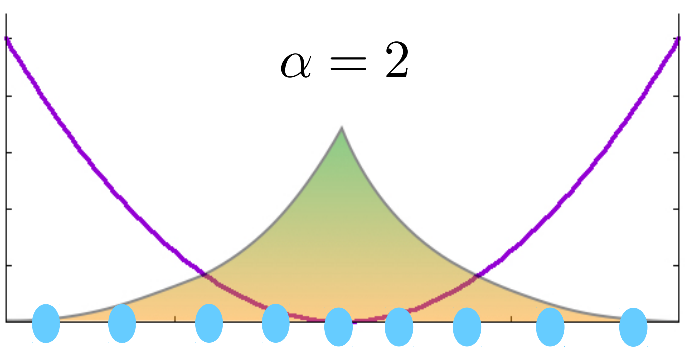
For uniform initial configurations:

for Brownian particles $\sigma_t \sim \sqrt{t}$

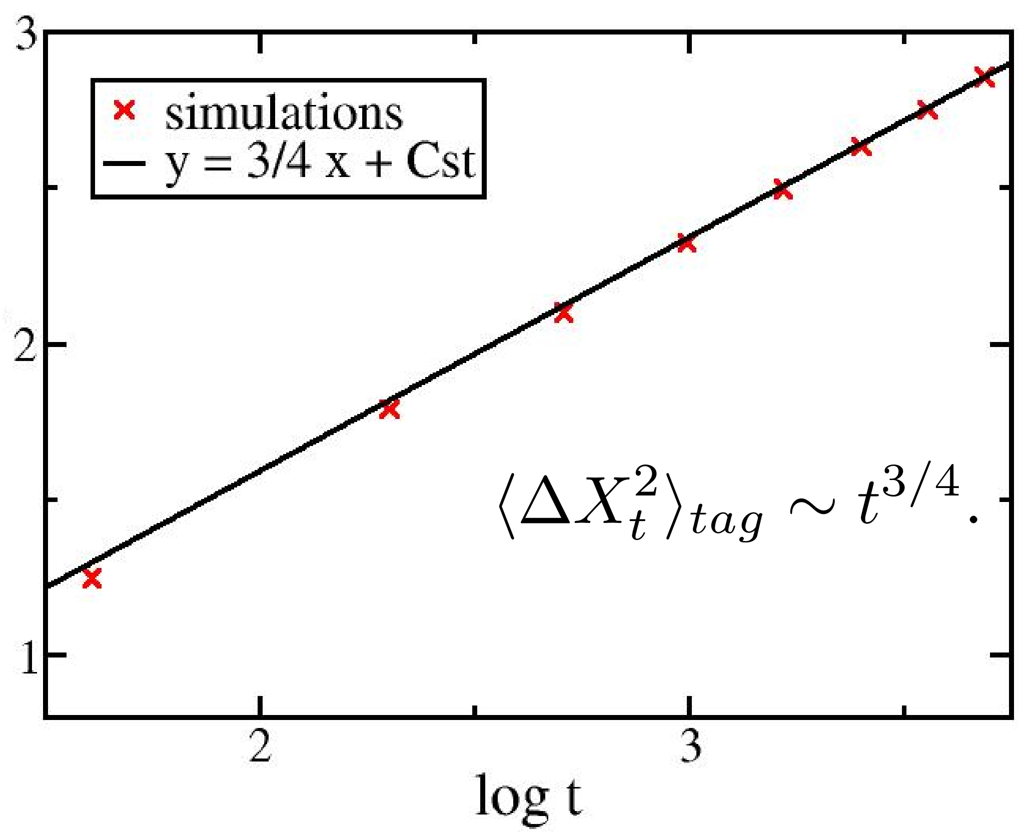


$$\langle \Delta X_t^2 \rangle_{tag} \sim t^{1/2}$$

For inhomogeneous initial configurations:



$$x_k = A \; \mathrm{sign}(k)|k|^2$$

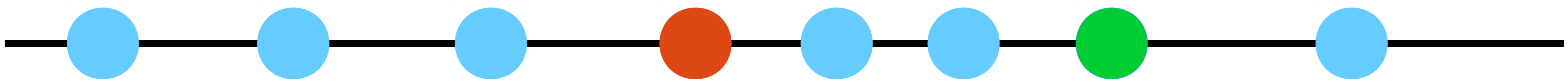


Summary:

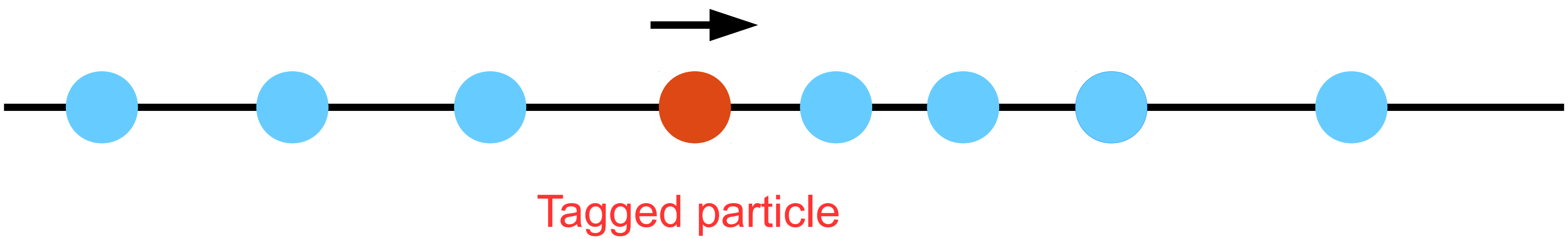
Exact distribution of the displacement of the tagged particle for arbitrary initial configuration.

Statistical properties of the tagged particle motion depends on how initially the particles are arranged.

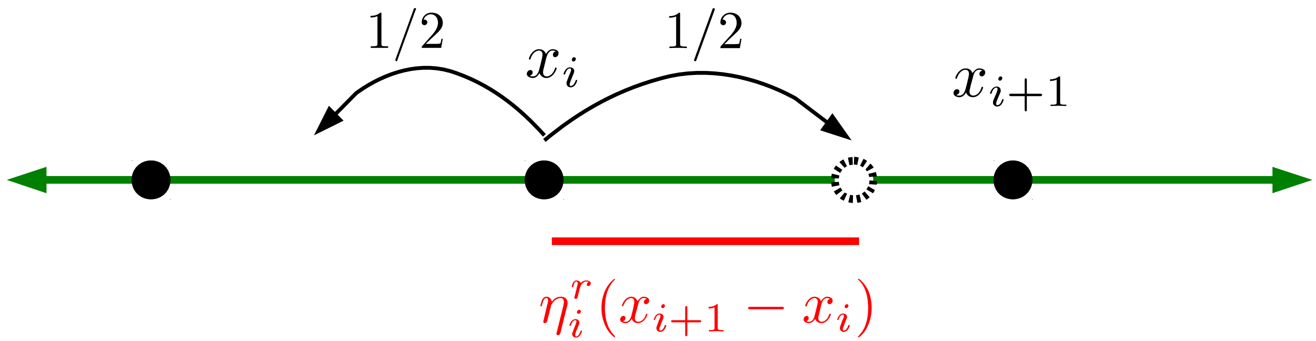
Can be easily extended to multi-tag case.



Single file with local drive



Random Average Process (RAP) :



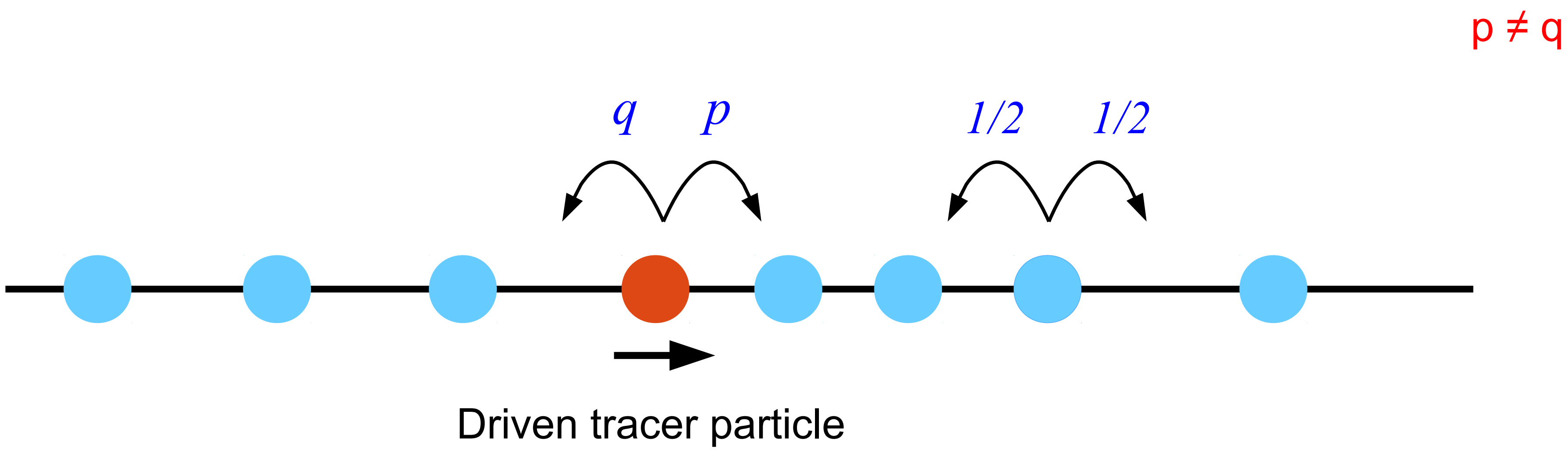
Ferrari & Fontes (1998)

$\eta \in [0,1)$ is a random variable chosen from jump distribution $R(\eta)$

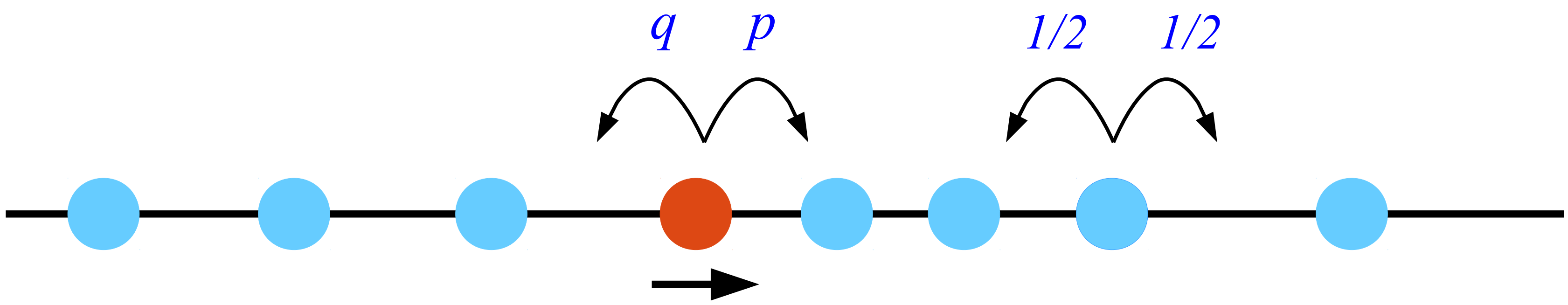
$$x_i(t+dt) \quad = \quad \begin{cases} x_i(t)+\eta_i^r(x_{i+1}(t)-x_i(t)), & \text{with Prob.} \quad R(\eta_i^r)d\eta_i^r \ \frac{dt}{2}, \\ x_i(t)+\eta_i^l(x_{i-1}(t)-x_i(t)), & \text{with Prob.} \quad R(\eta_i^l)d\eta_i^l \ \frac{dt}{2}, \\ x_i(t), & \text{with Prob.} \quad 1-dt, \end{cases}$$

No Single particle dynamics !!!

Traffic model, wealth distribution model, interface height models,
Force fluctuation in granular media etc.


p ≠ q
q
p
1/2
1/2
Driven tracer particle

p ≠ q



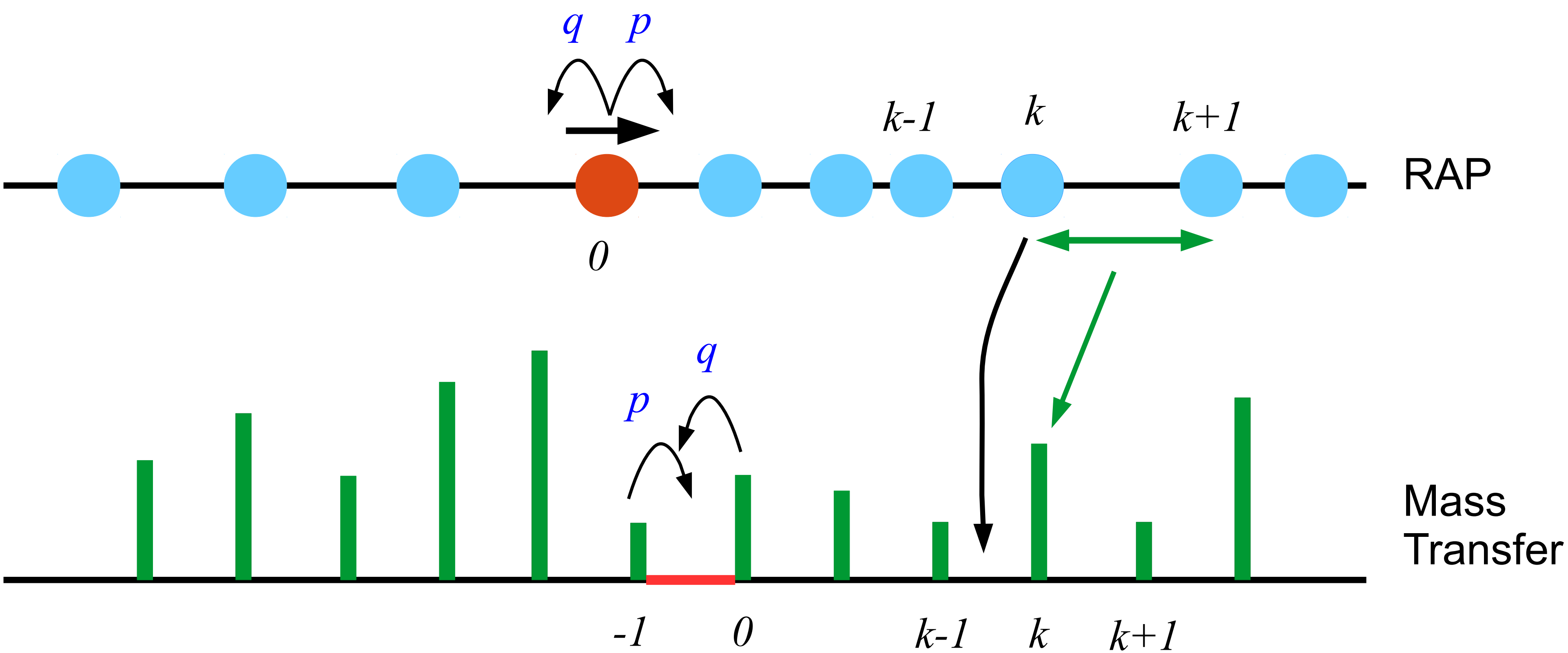
Dynamical Properties :

Mean displacement : $Z_i(t) = \langle x_i(t) \rangle - \langle x_i(0) \rangle = \ ?$

Variance : $V_i(t) = \langle \Delta x_i(t)^2 \rangle - \langle \Delta x_i(t) \rangle^2 = \ ?$

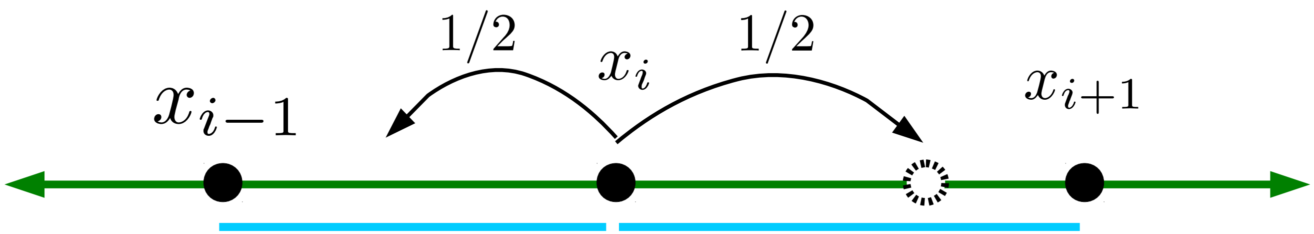
Correlations : $C_{i,j}(t) = \langle \Delta x_i(t) \Delta x_j(t) \rangle - \langle \Delta x_i(t) \rangle \langle \Delta x_j(t) \rangle = \ ?$

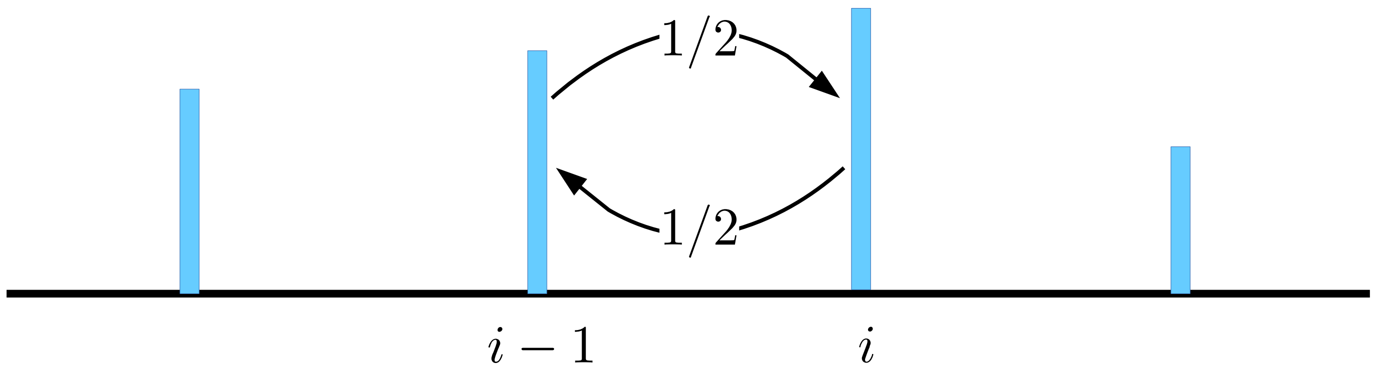
Mapping to mass transfer model :



- Corresponding to the motion of the particle in RAP, there exists a mass transfer process on a lattice
- k^{th} particle in RAP => link between sites (k -1, k) in MT picture. The tracer particle correspond to the special link (-1, 0).
- Mass at site k^{th} in MT = gki =(x_k+1 – x_k) = gap between k^{th} and $(k+1)^{th}$ particles in RAP

Random Average Process (RAP) :





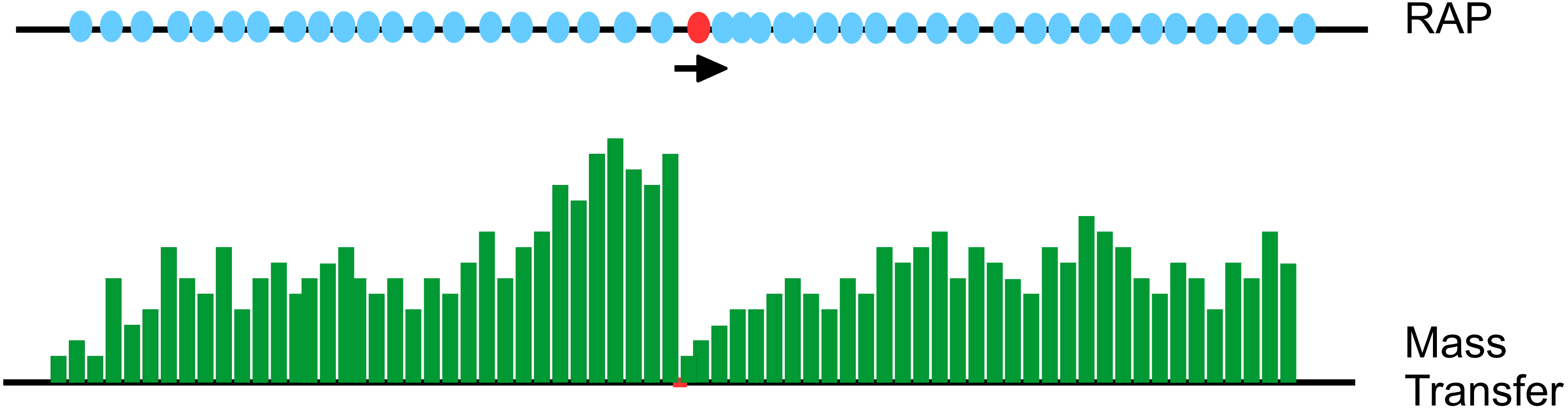
$$x_i(t+dt) = \begin{cases} x_i(t) + \eta_i^r (x_{i+1}(t) - x_i(t)), & \text{with Prob.} \quad R(\eta_i^r) d\eta_i^r \ \frac{dt}{2}, \\ x_i(t) + \eta_i^l (x_{i-1}(t) - x_i(t)), & \text{with Prob.} \quad R(\eta_i^l) d\eta_i^l \ \frac{dt}{2}, \\ x_i(t), & \text{with Prob.} \quad 1 - dt, \end{cases}$$

OR

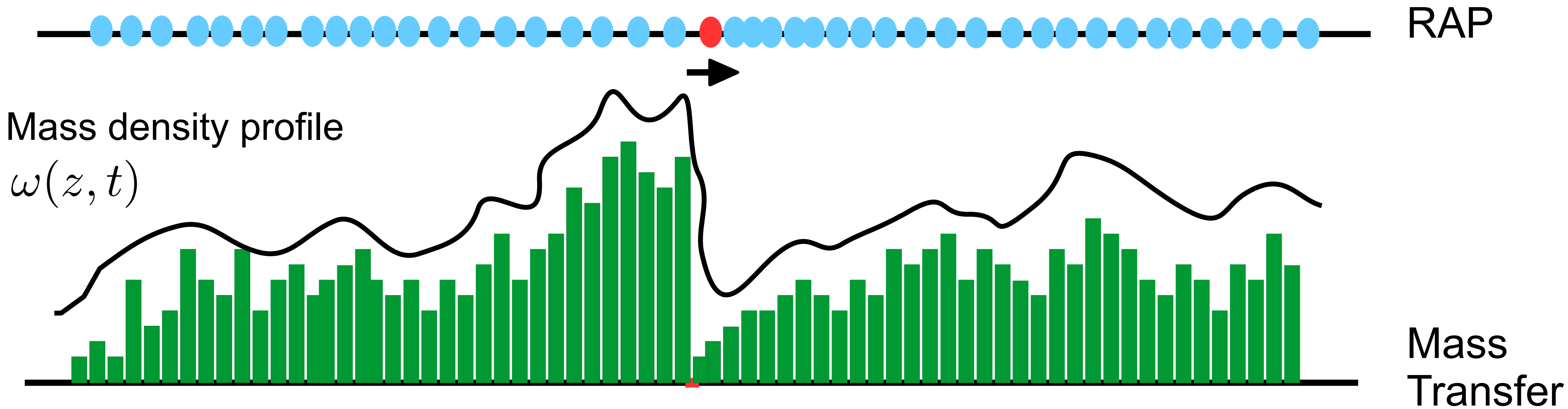
$$(g_i, g_{i+1})(t+dt) = \begin{cases} ((1-\eta)\ g_i, g_{i+1} + \eta g_i)(t) & \text{with prob.} \quad R(\eta) d\eta \ dt/2 \\ \\ (g_i + \eta g_{i+1}, (1-\eta)\ g_{i+1})(t) & \text{with prob.} \quad R(\eta) d\eta \ dt/2 \end{cases},$$

$\eta \in [0, 1)$ is a random variable chosen from jump distribution $R(\eta)$

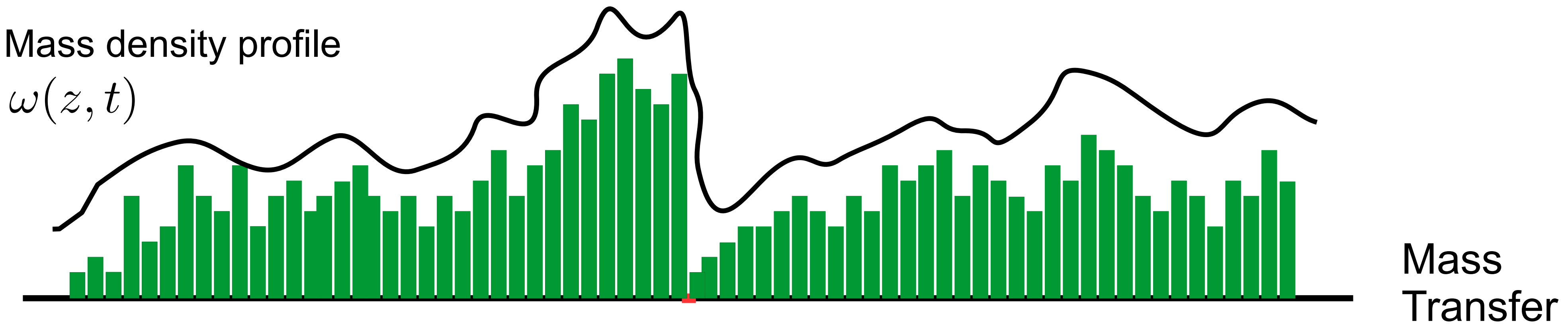
Continuum description :



Continuum description :

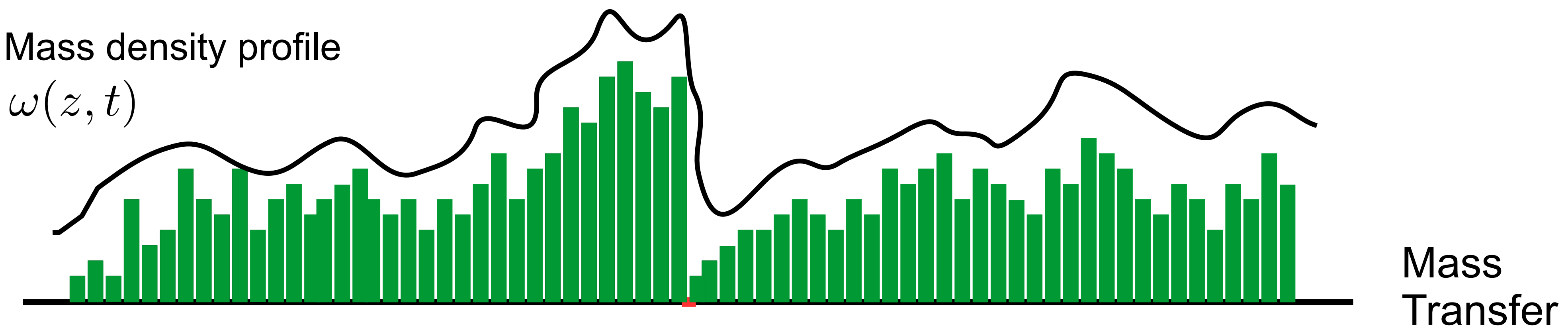


Continuum description :



$$\partial_t \omega(z,t) = -\partial_z j(z,t).$$

Continuum description :



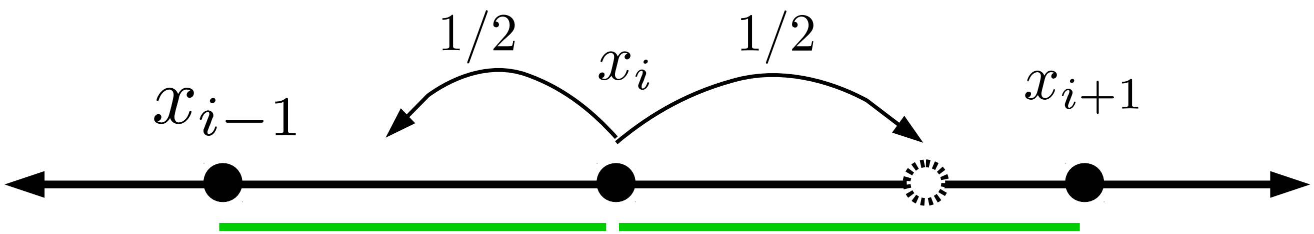
$$\partial_t \omega(z,t) = -\partial_z j(z,t).$$

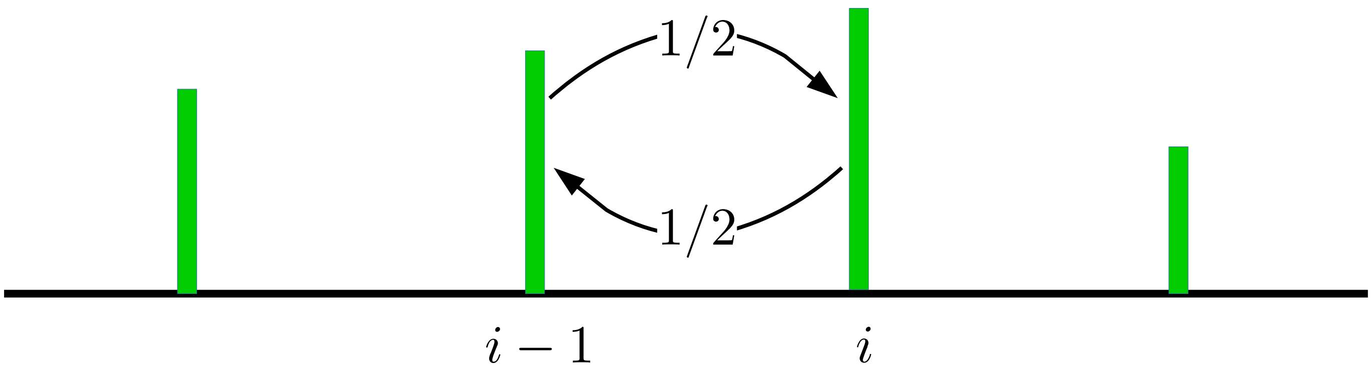
$$j(z,t) = -(\mu_1/2)\partial_z \omega(z,t) + \sqrt{\sigma(\omega(z,t))}\,\eta(z,t),$$

$$\langle \eta(z,t)\rangle = 0 \qquad \langle \eta(z,t)\eta(z',t')\rangle = \delta(z-z')\delta(t-t')$$

$$\sigma(\omega) = \frac{\mu_1\mu_2}{\mu_1-\mu_2}\omega^2 \qquad\qquad \mu_k = \int_0^1 d\eta\ \eta^k\ R(\eta)$$

Random Average Process (RAP) :





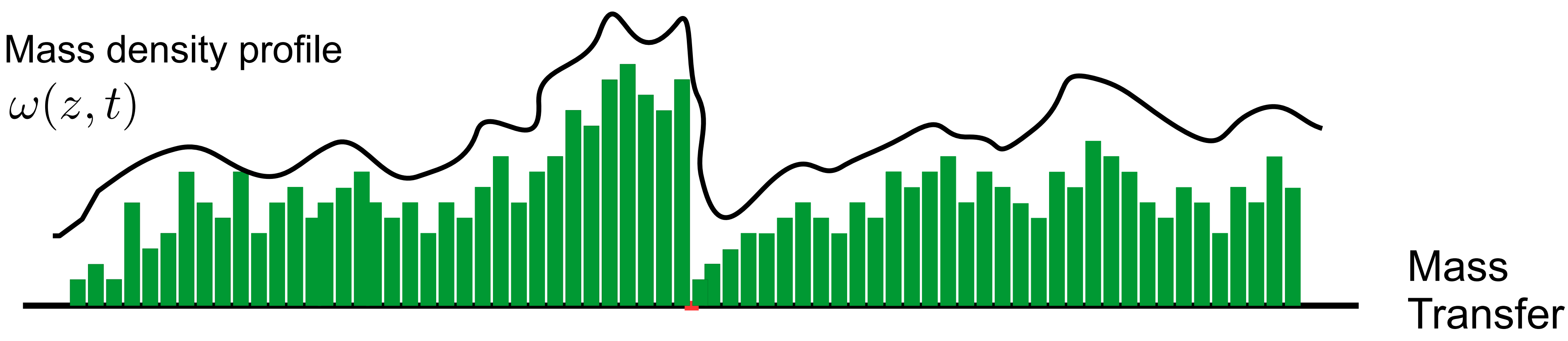
$$x_i(t+dt) = \begin{cases} x_i(t) + \eta_i^r(x_{i+1}(t) - x_i(t)), & \text{with Prob.} \quad R(\eta_i^r) d\eta_i^r \ \frac{dt}{2}, \\ x_i(t) + \eta_i^l(x_{i-1}(t) - x_i(t)), & \text{with Prob.} \quad R(\eta_i^l) d\eta_i^l \ \frac{dt}{2}, \\ x_i(t), & \text{with Prob.} \quad 1 - dt, \end{cases}$$

OR

$$(g_i, g_{i+1})(t+dt) = \begin{cases} ((1-\eta)\ g_i, g_{i+1} + \eta g_i)(t) & \text{with prob.} \quad R(\eta) d\eta \ dt/2 \\ \\ (g_i + \eta g_{i+1}, (1-\eta)\ g_{i+1})(t) & \text{with prob.} \quad R(\eta) d\eta \ dt/2 \end{cases},$$

$\eta \in [0, 1)$ is a random variable chosen from jump distribution $R(\eta)$

Continuum description :



$$\partial_t \omega(z,t) = -\partial_z j(z,t).$$

$$j(z,t) = -(\mu_1/2)\partial_z \omega(z,t) + \sqrt{\sigma(\omega(z,t))}\eta(z,t),$$

$$\Delta x_k(T) = x_k(T) - x_k(0) \quad = \quad \int_{-\infty}^{k} dz\ \omega(z,T) - \int_{-\infty}^{k} dz\ \omega(z,0),$$

Displacement of the k-th particle = Amount of mass transferred across the point-k in MT model

Procedure :

Stochastic evolution of the field

$$\partial_t \omega(z,t) = -\partial_z j(z,t).$$

$$j(z,t) = -(\mu_1/2)\partial_z \omega(z,t) + \sqrt{\sigma(\omega(z,t))}\eta(z,t),$$

Functional of the field

$$\Delta x_k(T) \quad = \quad \int_{-\infty}^{k} dz \; [\omega(z,T) - \omega(z,0)]$$

Given the stochastic evolution of the field one needs to find the statistical properties of the linear functional of the field

Outline of derivation :

Generating function: $\mu(\lambda_k, \lambda_l) = \langle \exp\{\lambda_k \Delta x_k(T) + \lambda_l \Delta x_l(T)\} \rangle_{\mathcal{P}[\omega,j]}$

Probability of mass density $\omega(z,t)$ and current $j(z,t)$ profiles:

$$\mathcal{P}[\omega, j] \asymp \exp\left[-\int_0^T dt \int_{-\infty}^{\infty} dz \frac{\left[j + (\mu_1/2)\partial_z \omega(z,t)\right]^2}{2\sigma(\omega(z,t))^2} \right] \mathbf{1}[\partial_t \omega(z,t) + \partial_z j(z,t)]$$

2-point Correlation function: $c_{k,l}(T) = \left[\frac{\partial^2 \ln[\mu(\lambda_k, \lambda_l)]}{\partial \lambda_k \partial \lambda_l} \right]_{\lambda_k=0, \lambda_l=0}$

Outline of derivation :

Generating function:

$$\begin{aligned}
\mu(\lambda_k, \lambda_l) &= \langle \exp\{\lambda_k X_k(T) + \lambda_l X_l(T)\}\rangle_{\mathcal{P}[\omega,j]} \\
&= \int \mathcal{D}[\omega_{in}] e^{-G[\omega_{in}]} \int\int \mathcal{D}[\omega(z,t)]\mathcal{D}[j(z,t)] \\
&\quad \times \ \exp\{\lambda_k X_k(T) + \lambda_l X_l(T)\} \ \mathcal{P}[\omega, j] \\
&= \int \mathcal{D}[\omega_{in}] e^{-G[\omega_{in}]} \int_{\omega|_{t=0}=\omega_{in}} \mathcal{D}[\omega, h] e^{-S[\omega,h]}, \\
&\sim \int \mathcal{D}[\omega_{in}] e^{-G[\omega_{in}]} \ e^{-S[\omega^*[\omega_{in},\lambda], h^*[\omega_{in},\lambda]]}
\end{aligned}$$

Optimal profiles:

$$\begin{aligned}
\partial_t \omega^*(z,t) &= (\mu_1/2)\partial_z^2 \omega^* - \partial_z(\sigma(\omega^*)\ \partial_z h^*), \\
\partial_t h^*(z,t) &= -(\mu_1/2)\partial_z^2 h^* - (\sigma'(\omega^*)/2)\ (\partial_z h^*)^2,
\end{aligned}$$

$$\sigma(\omega) = \frac{\mu_1 \mu_2}{\mu_1 - \mu_2}\omega^2$$

Optimal profiles:

$$\partial_t \omega^*(z,t) = (\mu_1/2)\partial_z^2 \omega^* - \partial_z(\sigma(\omega^*)\ \partial_z h^*),$$
$$\partial_t h^*(z,t) = -(\mu_1/2)\partial_z^2 h^* - (\sigma'(\omega^*)/2)\ (\partial_z h^*)^2,$$

$$\sigma(\omega) = \frac{\mu_1 \mu_2}{\mu_1 - \mu_2}\omega^2$$

Boundary conditions:

$$q\ \omega^*(z,t)|_{z\to 0^-} = p\ \omega^*(z,t)|_{z\to 0^+},$$

$$\frac{\mu_1}{2}\left[\partial_z \omega^*\right]_{0^+}^{0^-} = \left[\sigma(\omega^*)\partial_z h^*\right]_{0^+}^{0^-},$$

$$h^*(z,t)|_{0^+}^{0^-} = 0, \quad \left[\omega^* \partial_z h^*\right]_{0^+}^{0^-} = 0$$

Initial/final conditions:

$$\omega^*(z,0) = \omega_{in}(z),$$
$$\omega^*(z,t)|_{|z|\to\infty} = \bar{\omega}(z,t)_{|z|\to\infty},$$
$$h^*(z,T) = \lambda_k \Theta(k-z) + \lambda_l \Theta(l-z).$$

Dynamical properties :

Mean displacement : $Z_i(t) = \langle x_i(t) \rangle - \langle x_i(0) \rangle = \sqrt{2\mu_1 t}\ \mathcal{Y}\left(\frac{i}{\sqrt{2\mu_1 t}}\right)$

Variance : $\langle \Delta x_i(t)^2 \rangle - \langle \Delta x_i(t) \rangle^2 = \sqrt{2\mu_1 t}\ \mathcal{V}\left(\frac{i}{\sqrt{2\mu_1 t}}\right)$

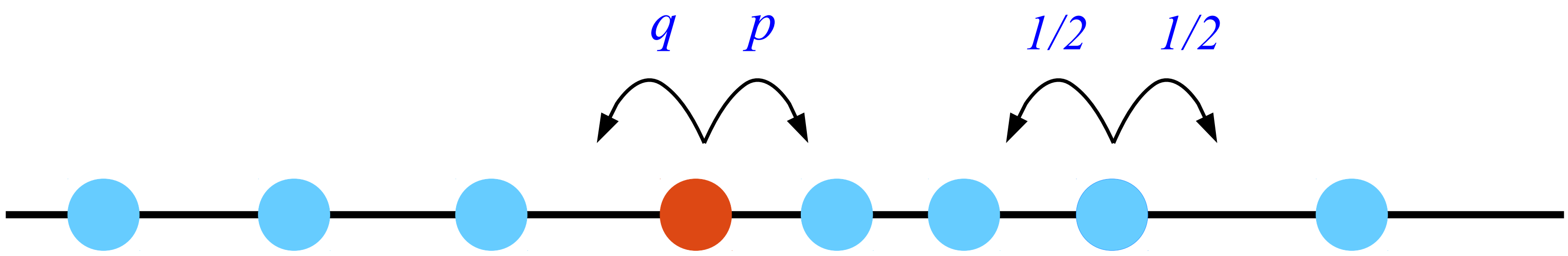
Correlations : $C_{i,j}(t) = \langle \Delta x_i(t) \Delta x_j(t) \rangle - \langle \Delta x_i(t) \rangle \langle \Delta x_j(t) \rangle$

$$C_{i,j}(t) = \rho_0^{-2} \sqrt{2\mu_1 t}\ \ \mathcal{C}\left(\frac{i}{\sqrt{2\mu_1 t}}, \frac{j}{\sqrt{2\mu_1 t}}\right)$$

$$\mu_1 = \int_0^1 d\eta\ \eta\ R(\eta)$$

Mean displacement of the driven particle *i.e.* particle *i=0* :

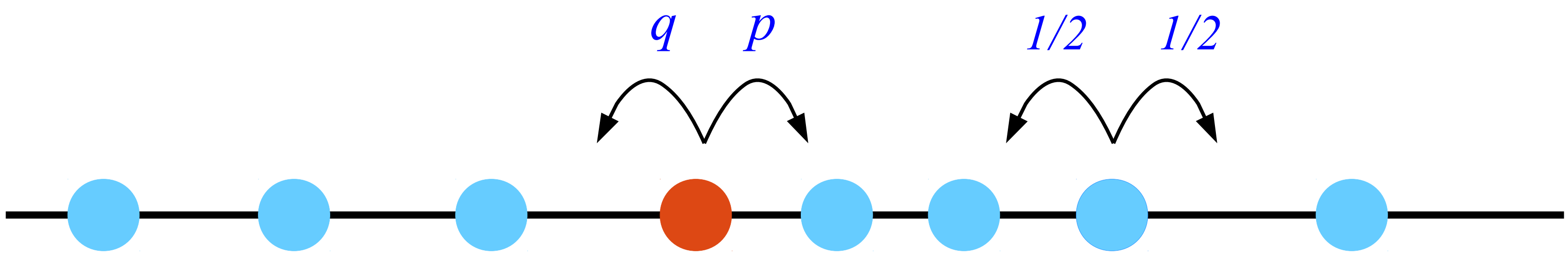
p ≠ q



$$Z_0(t) = \langle x_0(t) \rangle - \langle x_0(0) \rangle \sim t \quad ?$$

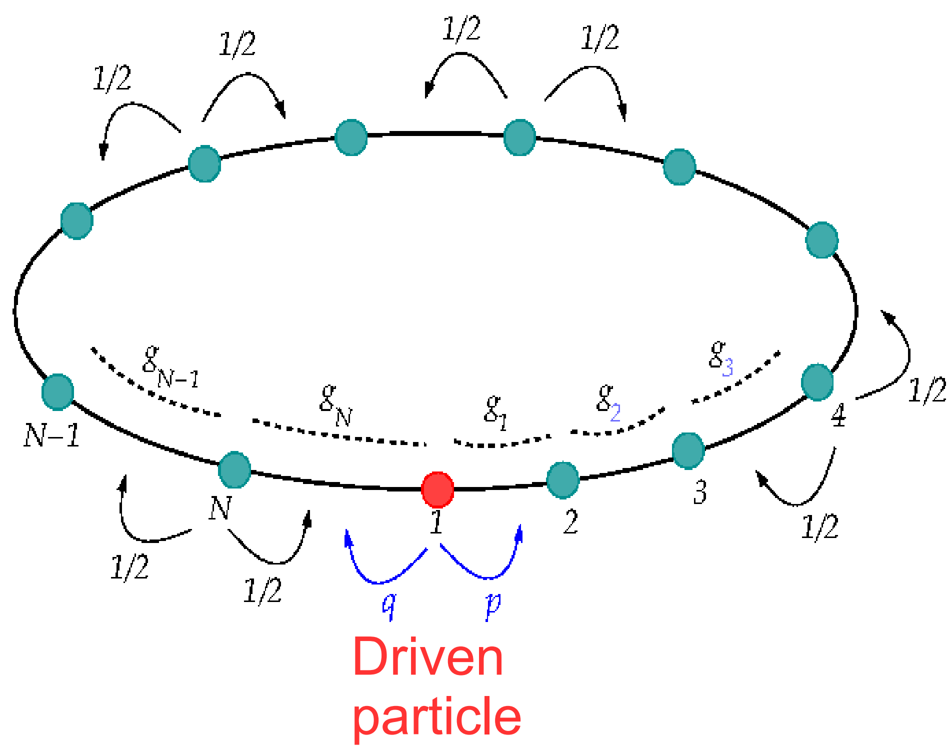
Mean displacement of the driven particle *i.e.* particle *i=0* :

p ≠ q



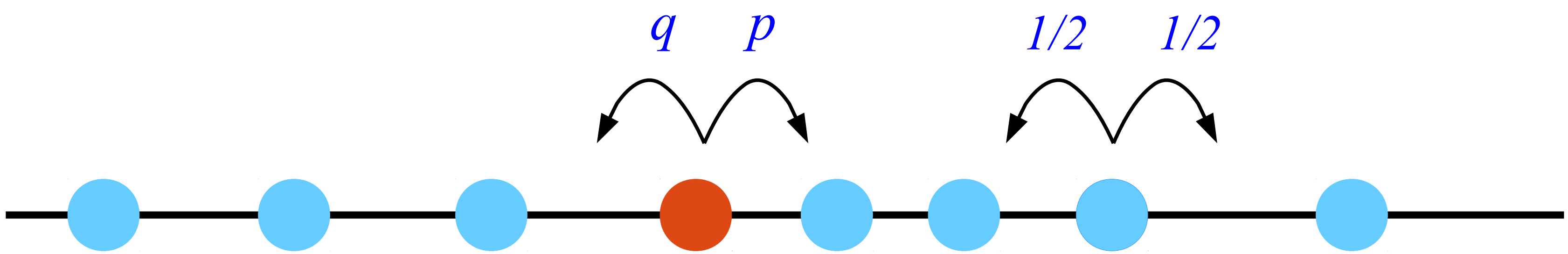
$$Z_0(t) = \langle x_0(t) \rangle - \langle x_0(0) \rangle \sim t \quad ?$$

In the simulation one can not study the problem on infinite line. Instead one starts with a finite system---say on a ring



Mean displacement of the driven particle *i.e.* particle *i=0* :

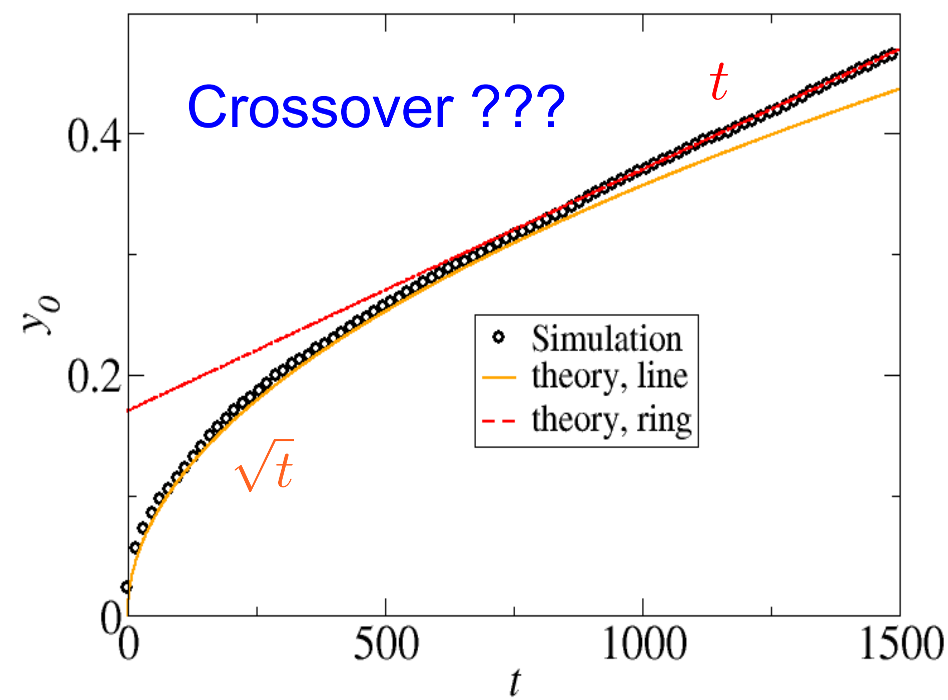
p ≠ q



$$Z_0(t) = \langle x_0(t) \rangle - \langle x_0(0) \rangle \sim t \quad ?$$

$$Z_0(t) \simeq \rho_0^{-1} \frac{(p-q)}{p+q} \left(\frac{\mu_1}{N}\, t + \sqrt{\frac{2\mu_1}{\pi}}\, \sqrt{t} + \mathcal{O}(1) \right)$$

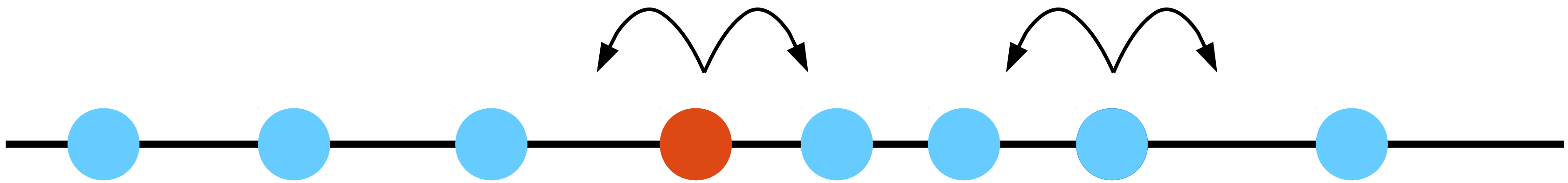
$$Z_0(t) \simeq \begin{cases} \sqrt{t}, & t \ll \mathcal{O}(N^2) \\ t, & t \gg \mathcal{O}(N^2) \end{cases}$$



Burlatsky et al. (1992)
Burlatsky et al. (1996)
Landim & Olla, (1998)
Oshanin et al. (2004)
Benichou et al (2013)

Mean displacement of the driven particle *i.e.* particle *i=0* :

p ≠ q



$$Z_0(t) = \langle x_0(t)\rangle - \langle x_0(0)\rangle \sim t \quad ?$$

$$Z_0(t) \simeq \rho_0^{-1}\frac{(p-q)}{p+q}\left(\frac{\mu_1}{N}\,t + \sqrt{\frac{2\mu_1}{\pi}}\,\sqrt{t} + \mathcal{O}(1)\right)$$

Crossover ???

$$Z_0(t) \simeq \begin{cases} \sqrt{t}, & t \ll \mathcal{O}(N^2) \\ t, & t \gg \mathcal{O}(N^2) \end{cases}$$

$$Z_0(t) = \sqrt{t}\;\Phi\left(\frac{t}{N^2}\right)$$

$$\Phi(\phi) \simeq \begin{cases} \sqrt{\phi}, & \phi \to \infty \\ Constant, & \phi \to 0 \end{cases}$$

J. Cividini, A. Kundu, S. N. Majumdar, D. Mukamel,

J. Stat. Mech. (2016) 053212

Crossover :

p ≠ q

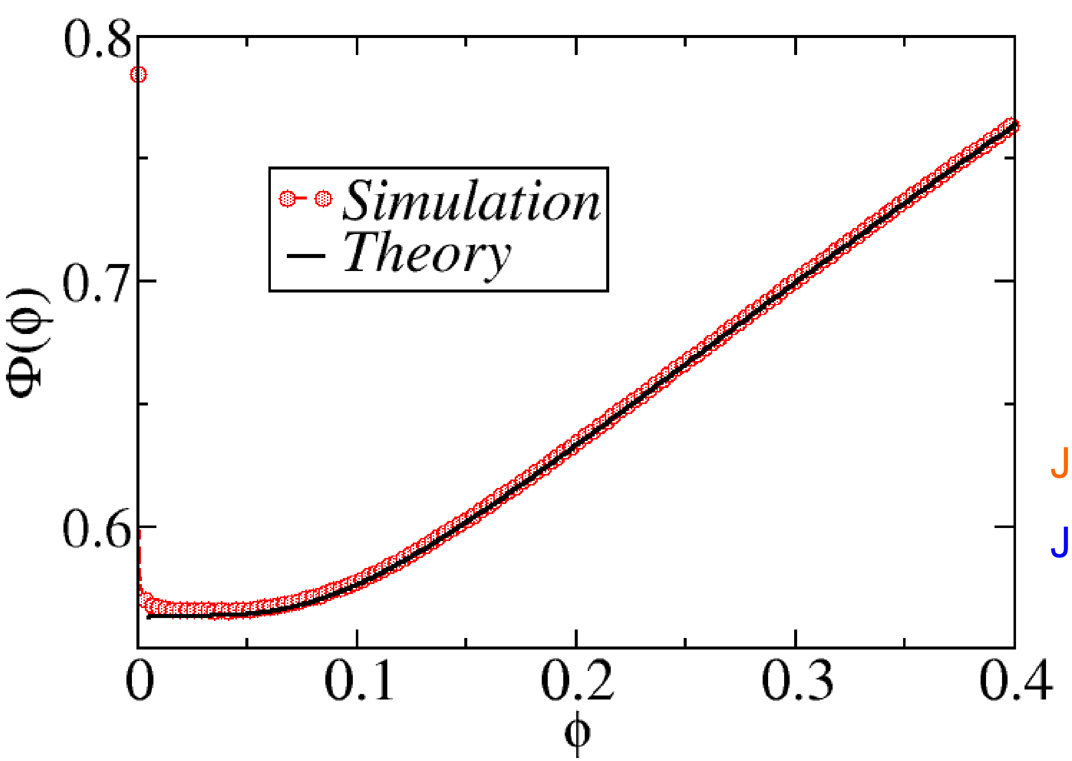
$$Z_0(t) = \rho_0^{-1} \frac{(p-q)}{p+q} \sqrt{2\mu_1 t}\ \Phi\left(\frac{\mu_1 t}{2N^2}\right)$$

$$\Phi(\phi) = \sqrt{\phi} + \frac{1}{2\pi^2\sqrt{\phi}} \sum_{k=1}^{\infty} \frac{1-\mathrm{e}^{-4\pi^2 k^2 \phi}}{k^2}.$$

$$\Phi(\phi) \simeq \begin{cases} \sqrt{\phi}, & \phi \to \infty \\ \sqrt{\pi}, & \phi \to 0 \end{cases}$$

$$\Downarrow$$

$$Z_0(t) \simeq \begin{cases} \sqrt{t}, & t \ll \mathcal{O}(N^2) \\ t, & t \gg \mathcal{O}(N^2) \end{cases}$$



J. Cividini, A. Kundu, S. N. Majumdar, D. Mukamel, J. Stat. Mech. (2016) 053212

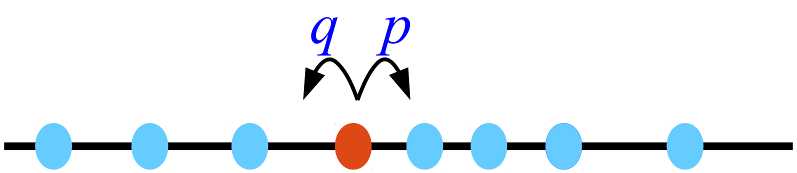
Mean displacement of the other particles *i.e.* particle $i \neq 0$:

p ≠ q

$$Z_i(t) = \langle x_i(t) \rangle - \langle x_i(0) \rangle = \quad ?$$

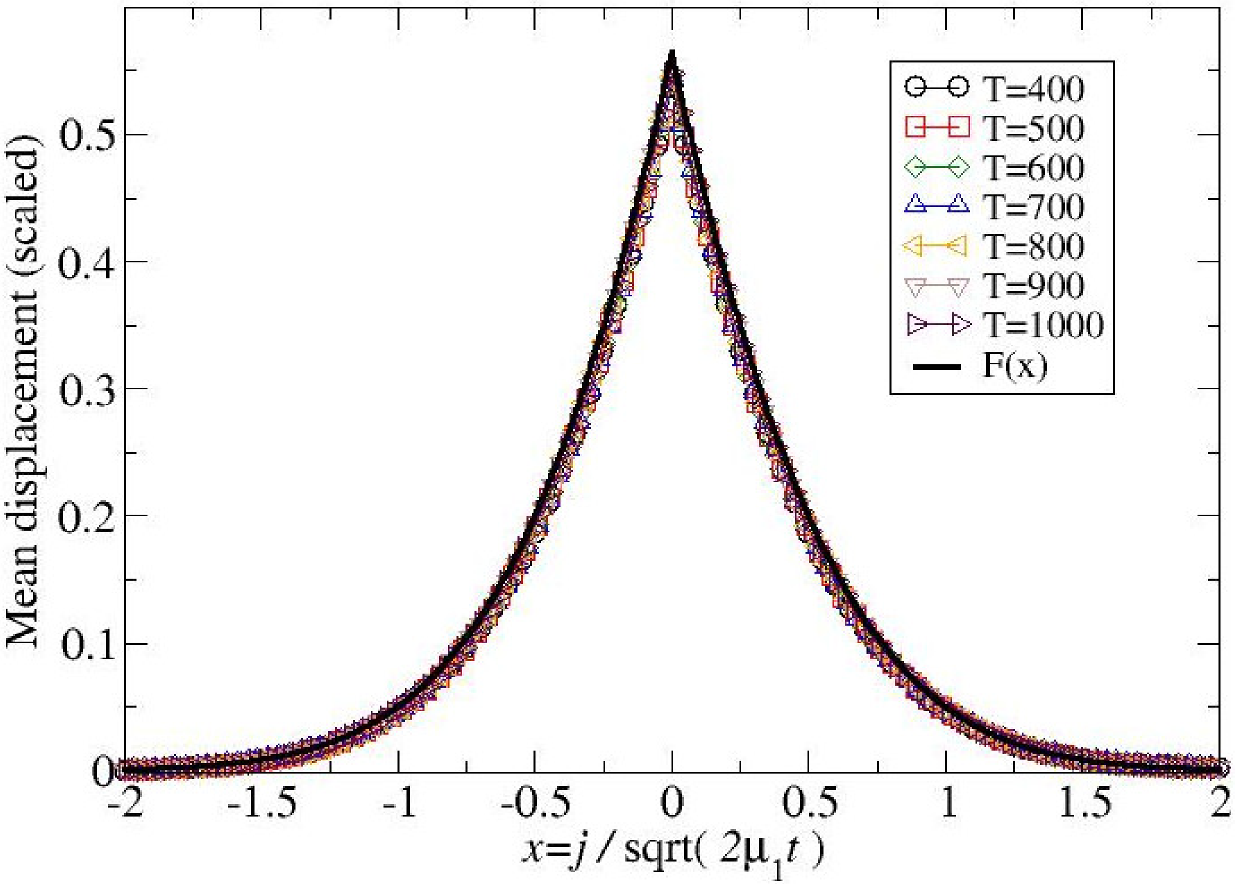
Results :

Mean displacement :



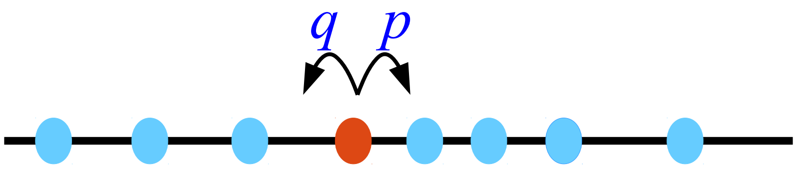
$$\langle x_i(t)\rangle - \langle x_i(0)\rangle = \sqrt{2\mu_1 t}\ \mathcal{Y}\left(\frac{i}{\sqrt{2\mu_1 t}}\right)$$

$$\mathcal{Y}(x) = \frac{p-q}{p+q}\left[\frac{e^{-x^2}}{\sqrt{\pi}} - |x|\ \mathrm{Erfc}(|x|)\right].$$



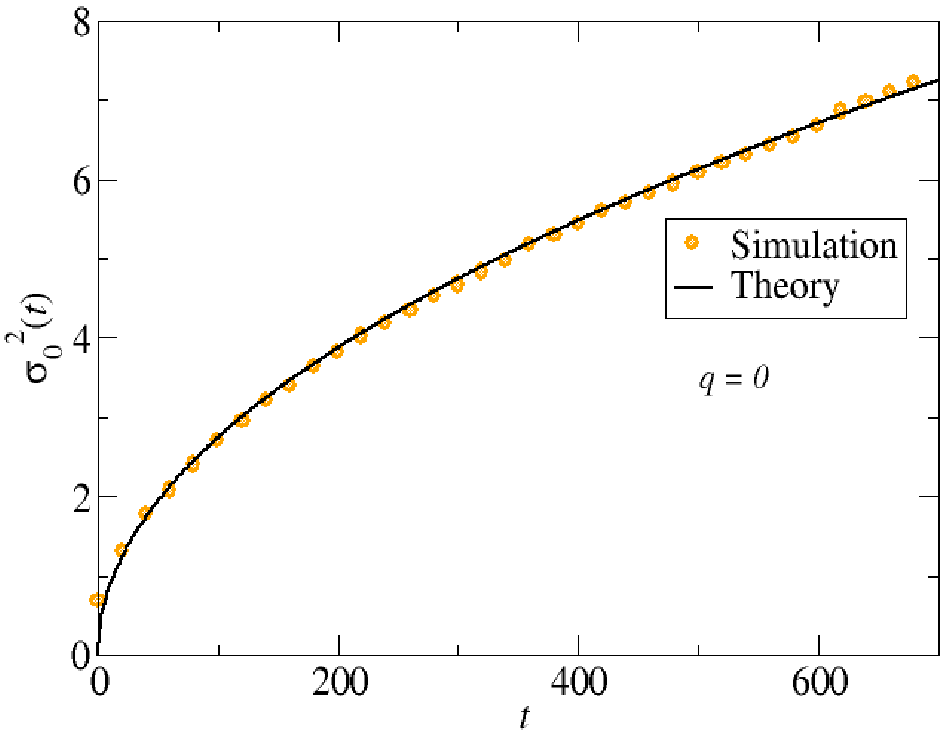
Cividini, AK, Majumdar, Mukamel (2016)
AK, Cividini (2016)

Results :

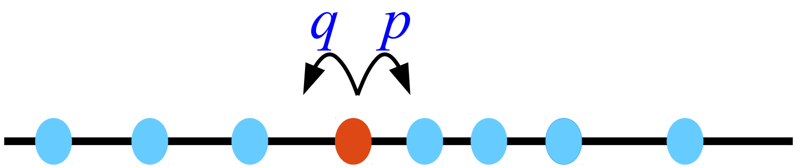


Variance :

$$\langle x_0(t)^2\rangle - \langle x_0(t)\rangle^2 \simeq \sqrt{t}$$



Results :



Variance :

$$\langle \Delta x_0(t)^2 \rangle - \langle \Delta x_0(t) \rangle^2 = \sqrt{t}\ \frac{2\rho_0^{-2}\mu_2\sqrt{\mu_1}(\sqrt{2}-1)^2}{\sqrt{\pi}(\mu_1-\mu_2)}\ \mathcal{A}(b)$$

Where $b = \frac{p-q}{p+q}$ quantifies the drive strength.

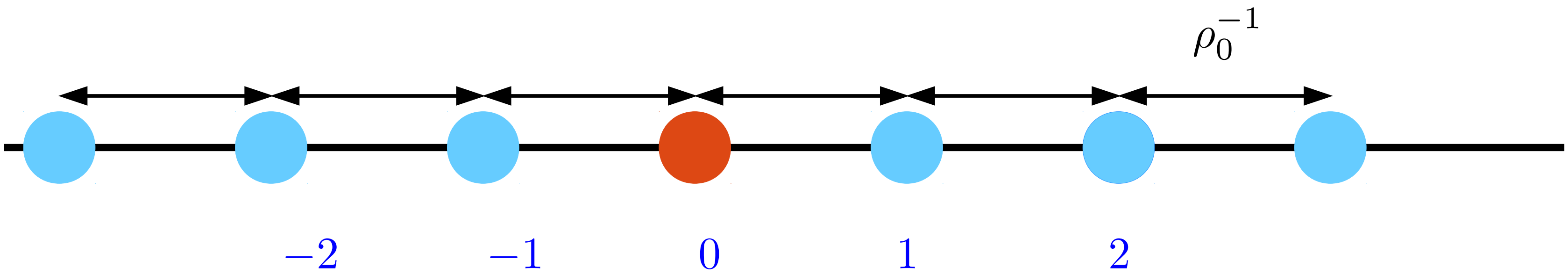
$$\mathcal{A}(b=0) = \begin{cases} (\sqrt{2}+1)^2/2, & \text{Quenched uniform} \\ (\sqrt{2}+1)^2/\sqrt{2}, & \text{Annealed uniform.} \end{cases}$$

Rajesh & Majumdar (2001)
Barkai & Silbey (2009, 2010)

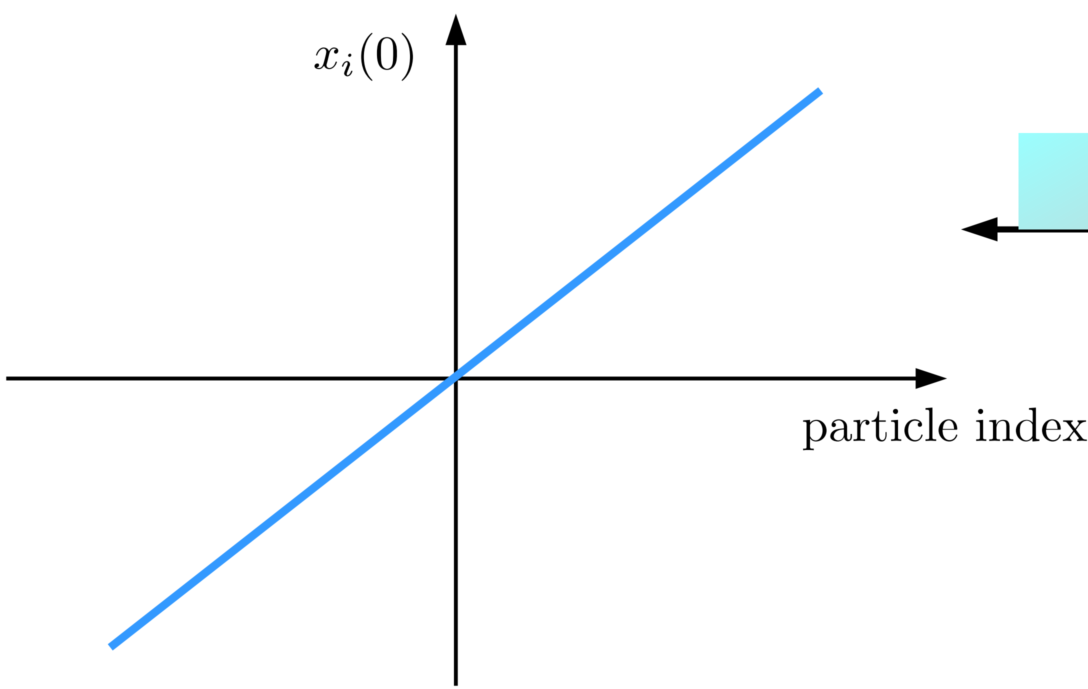
$$\mathcal{A}(b \neq 0) \ = \ ??$$

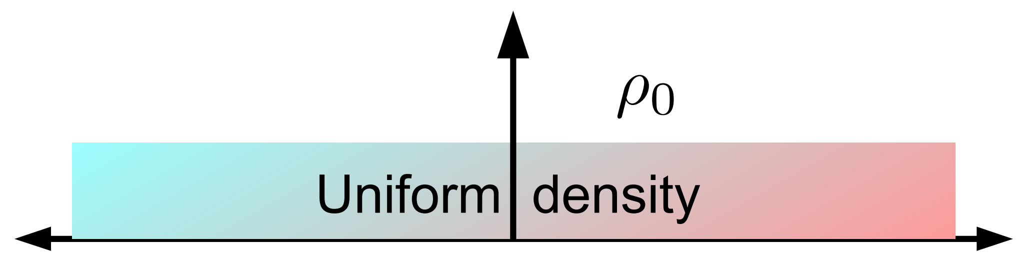
Initial Configuration : Quenched

p ≠ q



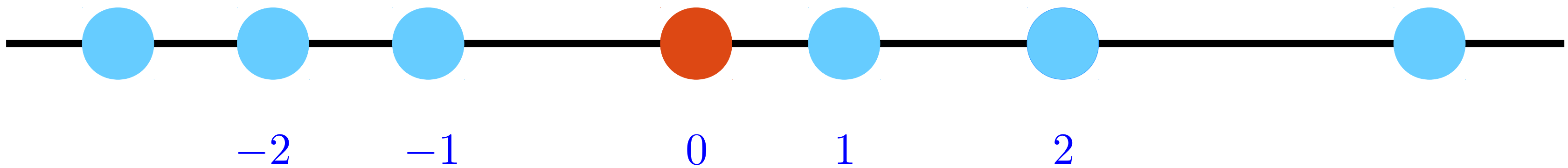
$$x_i(0) = \rho_0^{-1}\, i \; ; \qquad \ldots, -2, -1, 0, 1, 2, \ldots \;, \qquad \rho_0 = \frac{N}{L}$$





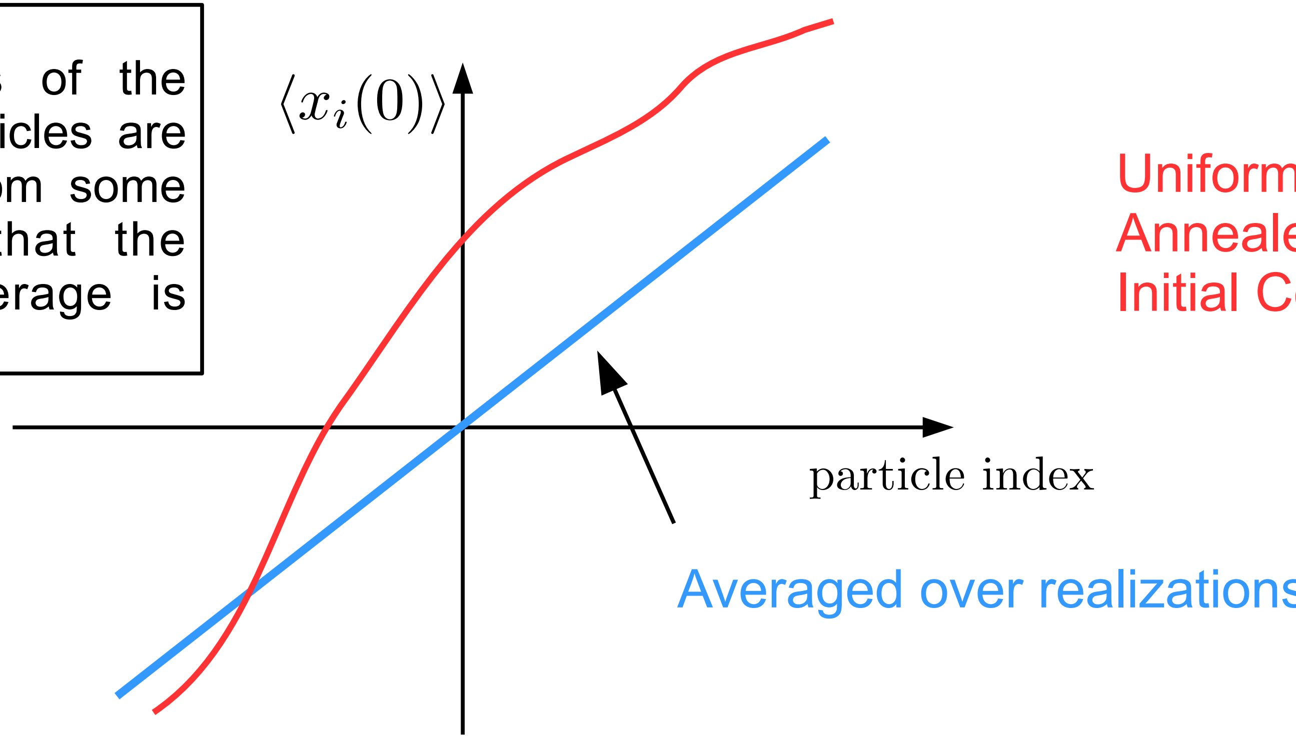
Initial Configuration : Annealed

p ≠ q



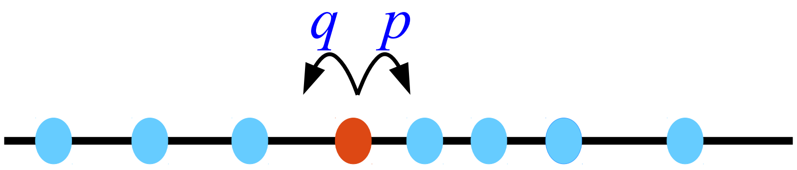
$$\langle x_i(0)\rangle = \rho_0^{-1}\, i \;; \qquad \ldots, -2, -1, 0, 1, 2, \ldots \;, \qquad \rho_{av} = \rho_0 = \frac{N}{L}$$

Annealed IC :
Initial configurations of the positions of the particles are randomly chosen from some distribution such that the density on an average is uniform



Uniform
Annealed
Initial Conditions

Results :



Variance :

$$\langle \Delta x_0(t)^2 \rangle - \langle \Delta x_0(t) \rangle^2 = \sqrt{t} \ \ \frac{2\rho_0^{-2}\mu_2\sqrt{\mu_1}(\sqrt{2}-1)^2}{\sqrt{\pi}(\mu_1-\mu_2)} \mathcal{A}(b)$$

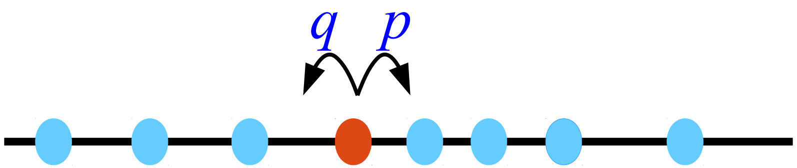
Where $b = \dfrac{p-q}{p+q}$ quantifies the drive strength.

$$\mathcal{A}(b=0) = \begin{cases} (\sqrt{2}+1)^2/2, & \text{Quenched uniform} \\ (\sqrt{2}+1)^2/\sqrt{2}, & \text{Annealed uniform.} \end{cases}$$

Rajesh & Majumdar (2001)
Barkai & Silbey (2009, 2010)

$$\mathcal{A}(b \neq 0) \ = \ ??$$

Results :



Variance :

$$\langle \Delta x_0(t)^2 \rangle - \langle \Delta x_0(t) \rangle^2 = \sqrt{t}\ \frac{2\rho_0^{-2}\mu_2\sqrt{\mu_1}(\sqrt{2}-1)^2}{\sqrt{\pi}(\mu_1-\mu_2)}\mathcal{A}(b)\ ;\ b = \frac{p-q}{p+q}$$

$$\mathcal{A}(b) = \begin{cases} (\sqrt{2}-1)(1-b^2) + \frac{1}{2}(1+b^2), & \text{Quenched} \\ (\sqrt{2}-1)(1-b^2) + \frac{2+\sqrt{2}}{2}(1+b^2), & \text{Annealed.} \end{cases}$$

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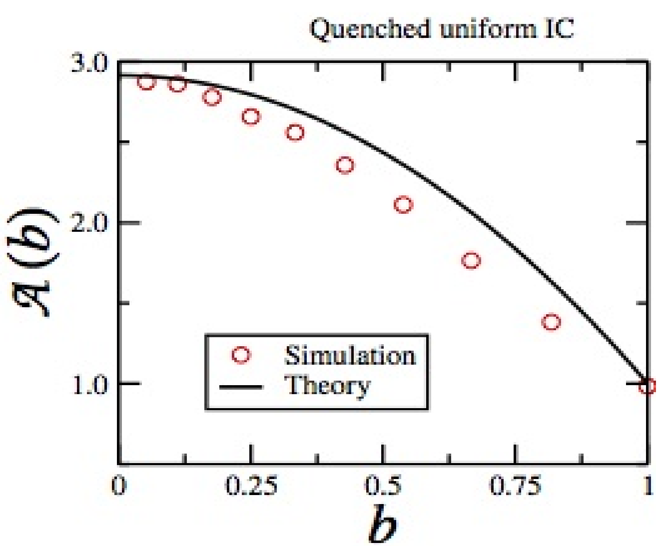
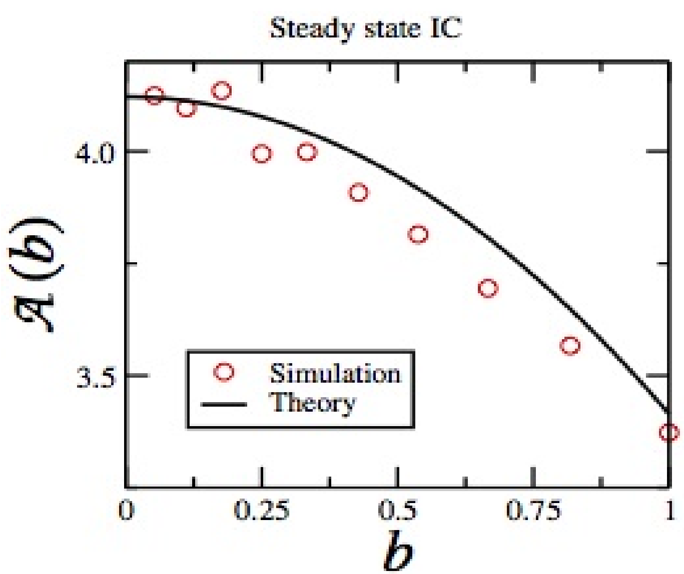
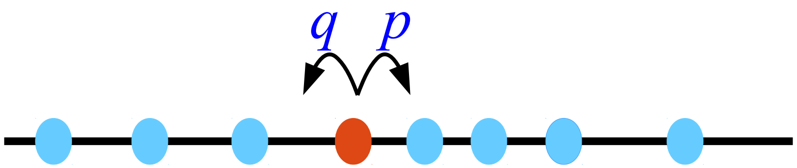




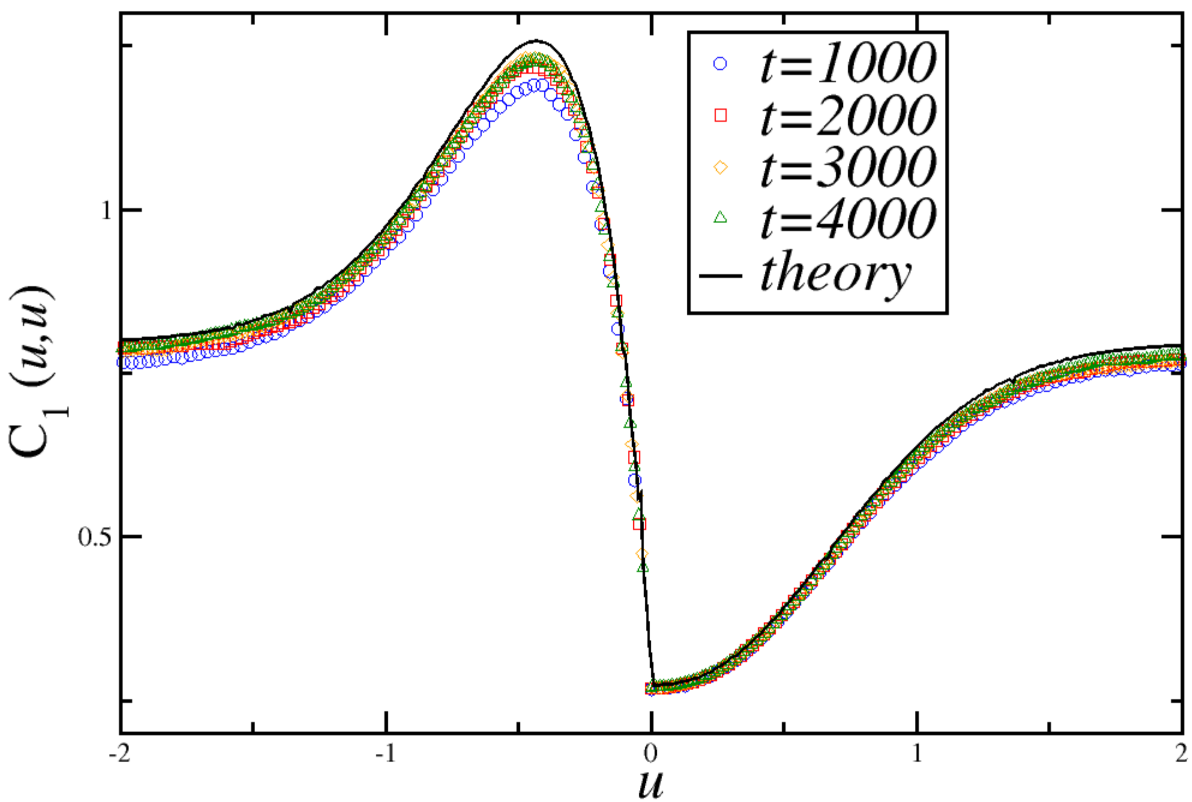
Results :



Variance :

$$\langle \Delta x_k(t)^2 \rangle - \langle \Delta x_k(t) \rangle^2 = \sqrt{2\mu_1 t} \;\; \mathcal{V}\left(\frac{k}{\sqrt{2\mu_1 t}} \right)$$

$$\frac{\langle \Delta x_k(t)^2 \rangle - \langle \Delta x_k(t) \rangle^2}{\sqrt{2\mu_1 t}} \;\; vs. \;\; u = \frac{k}{\sqrt{2\mu_1 t}}$$



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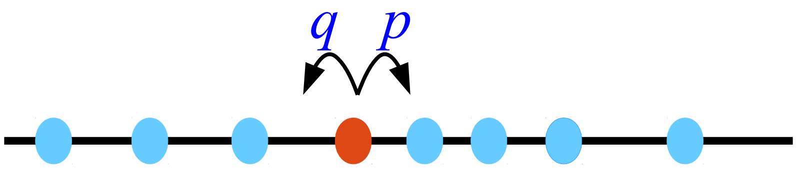
Results :

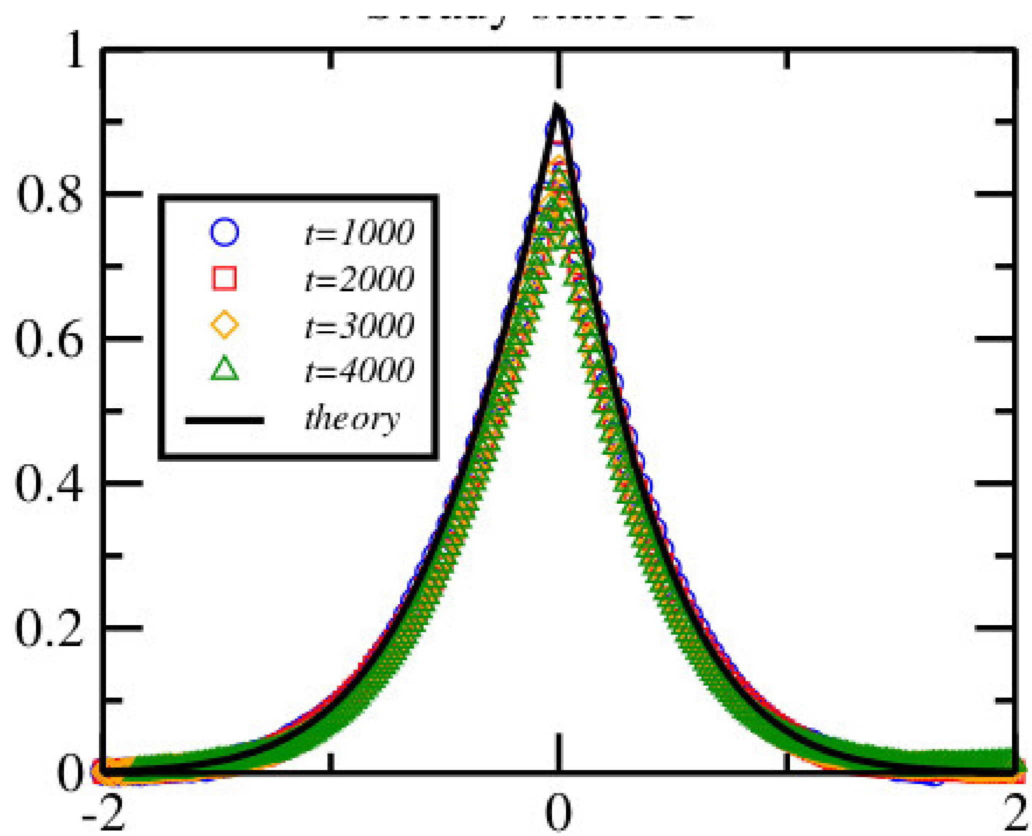


Correlations :

$$C_{i,j}(t) = \langle \Delta x_i(t) \Delta x_j(t) \rangle - \langle \Delta x_i(t) \rangle \langle \Delta x_j(t) \rangle$$

$$C_{i,j}(t) = \rho_0^{-2} \sqrt{2\mu_1 t} \;\; \mathcal{C}\left(\frac{i}{\sqrt{2\mu_1 t}}, \frac{j}{\sqrt{2\mu_1 t}} \right)$$

$$\frac{C_{0,j}(t)}{\sqrt{2\mu_1 t}} \;\; vs. \;\; v = \frac{j}{\sqrt{2\mu_1 t}}$$



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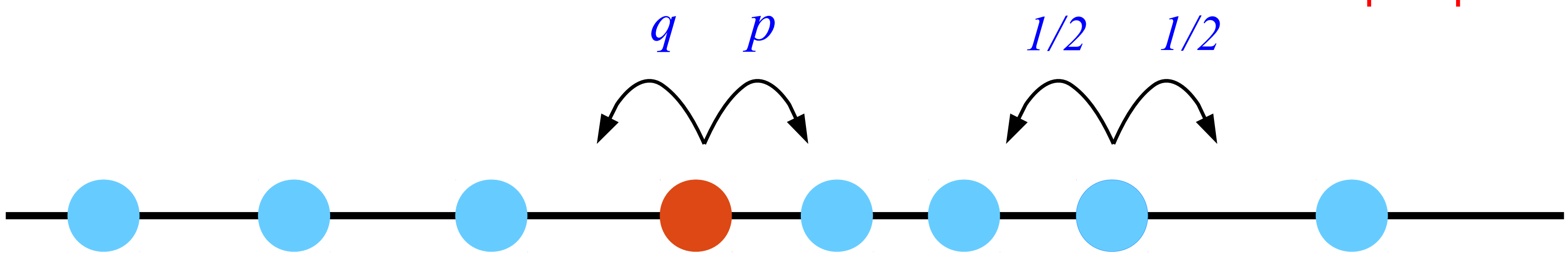
Summary - part 2

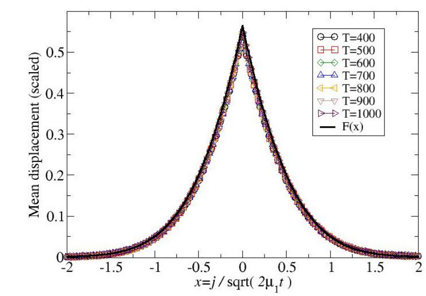
p ≠ q case



Dynamical properties :

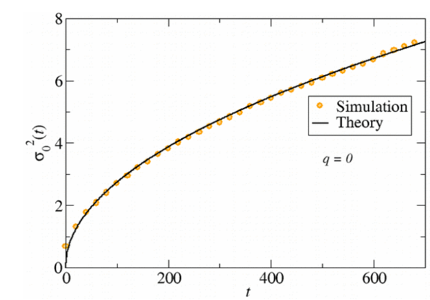
Mean displacement :

$$Z_i(t) = \langle x_i(t) \rangle - \langle x_i(0) \rangle = \sqrt{2\mu_1 t}\ \mathcal{Y}\left(\frac{i}{\sqrt{2\mu_1 t}}\right)$$



Variance :

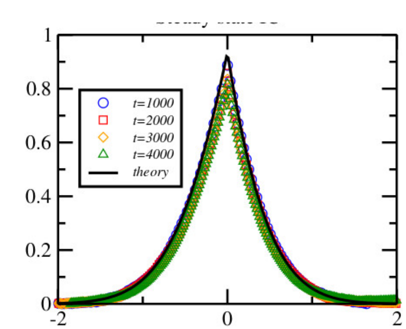
$$\langle \Delta x_0(t)^2 \rangle - \langle \Delta x_0(t) \rangle^2 = \sqrt{t}\ \mathcal{B}(b)$$



Correlations :

$$C_{i,j}(t) = \langle \Delta x_i(t) \Delta x_j(t) \rangle - \langle \Delta x_i(t) \rangle \langle \Delta x_j(t) \rangle$$

$$C_{i,j}(t) = \rho_0^{-2} \sqrt{2\mu_1 t}\ \ \mathcal{C}\left(\frac{i}{\sqrt{2\mu_1 t}}, \frac{j}{\sqrt{2\mu_1 t}}\right)$$



Thank you