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6 April 2011

Recursion relations for Tree Amplitudes
Part II

1) Refer to an result on Page 10 in Part I

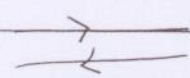
We will now present a simplification that dramatically reduces the number of terms in this expression by accounting for color-information.

Recall that our amplitude has n pieces of color-information. Now, at tree-level, we can write:

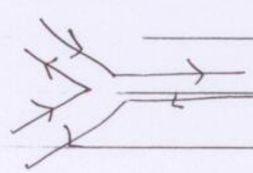
$$M = 2^{n/2} \sum_{\pi \in S_n/Z_n} A\left((h_{\pi_1}, p_{\pi_1}), (h_{\pi_2}, p_{\pi_2}) \ldots (h_{\pi_n}, p_n)\right) \mathrm{Tr}\left(T^{a_{\pi_1}} \ldots T^{a_{\pi_n}}\right)$$

A are called color-ordered amplitudes.

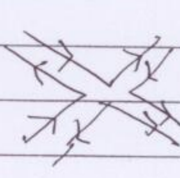
<u>Motivation</u>: For $U(N)$, the adjoint can be written as the product of the fundamental and the anti-fundamental. So, in a propagator we can keep track of these indices via



An interaction vertex involves



and



Every double-line graph has a cyclic ordering.

The color-ordered amplitudes correspond to a sum of all double-line graphs with the same cyclic ordering of external momenta.

<u>BCFW Recursion for color-ordered Amplitudes</u> :-

$$A(h_1, p_1 \ldots h_n, p_n) = \sum_{\substack{j=2 \\ h_{int}=\pm 1}}^{n-2} \frac{1}{\left(p_1 + \sum_{m=2}^{j} p_m\right)^2}$$

$$\times A(h_1, p_1 \ldots h_j, p_j, h_{int}, p_{int})$$

$$\times A(-h_{int}, -p_{int}, h_{j+1}, p_{j+1} \ldots h_n, p_n)$$

Significantly simpler since we don't have to worry about partitions.

e.g. $f^{abc} = \frac{-i}{\sqrt{2}} \left[Tr(T^a T^b T^c) - Tr(T^a T^c T^b) \right]$

We now turn to spinors.

Given a 4-dimensional momentum p, we can form:-

$$p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu_{\alpha\dot{\alpha}}$$

$$= \begin{pmatrix} p_0+p_3 & p_1 - ip_2 \\ p_1+ip_2 & p_0-p_3 \end{pmatrix}$$

which has determinant zero

Can write

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$$

eg. choose

$$\lambda_\alpha = \left(\sqrt{p_0+p_3}, \frac{p_1+ip_2}{\sqrt{p_0+p_3}} \right)$$

$$\tilde{\lambda}_{\dot{\alpha}} = \left(\sqrt{p_0+p_3}, \frac{p_1-ip_2}{\sqrt{p_0+p_3}} \right)$$

Check:

$$\lambda_0 \tilde{\lambda}_0 = p_0 + p_3$$

$$\lambda_0 \tilde{\lambda}_1 = p_1 - ip_2$$

$$\lambda_1 \tilde{\lambda}_0 = p_1 + ip_2$$

$$\lambda_1 \tilde{\lambda}_1 = \frac{p_1^2 + p_2^2}{p_0+p_3} = \frac{p_0^2 - p_3^2}{p_0+p_3} = p_0 - p_3$$

Under a Lorentz transformation — Λ, we have an $SL(2,C)$ element: U

corresponding to a rotation θ, we have the element

$$U = e^{-i\frac{\theta}{2}\vec{\sigma}\cdot\hat{n}}$$

corresponding to a boost we have:

$$U = e^{\frac{\phi}{2}\vec{\sigma}\cdot\hat{n}}$$

$$\lambda \to U\lambda ; \quad \tilde{\lambda} \to \tilde{\lambda}\, U^{+}$$

Under little group rotations; what happens?

eg. take

$$P = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right)$$

$$\lambda = (1, 0) ; \quad \tilde{\lambda} = (1, 0)$$

[Not being careful about writing λ as a row or column]

Under rotations with z-axis we have

$$U = \begin{pmatrix} e^{-i\theta/2} & \\ & e^{+i\theta/2} \end{pmatrix}$$

$$\lambda \to e^{-i\theta/2}\lambda ; \quad \tilde{\lambda} \to e^{i\theta/2}\tilde{\lambda}$$

Under a general Lorentz transformation

$$\lambda(\Lambda p) = e^{i\theta_{little}/2} U \lambda(p)$$

We can form inner products between two λ by

$$\langle \lambda_1, \lambda_2 \rangle = \epsilon^{\alpha\beta} \lambda_{1\alpha} \lambda_{2\beta}$$

eg

$$\lambda_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}; \quad \lambda_2 = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

$$\langle \lambda_1, \lambda_2 \rangle = \alpha\delta - \beta\gamma$$

$$\langle \lambda_1, \lambda_2 \rangle = - \langle \lambda_2, \lambda_1 \rangle$$ [antisymmetric product]

$$\langle \lambda, \lambda \rangle = 0$$

sometimes inner product of $\tilde{\lambda}$ written using square brackets.

$$[\tilde{\lambda}_1, \tilde{\lambda}_2]$$

don't get bothered by this. For us, there is no substantive difference.

$$2 P_1 \cdot P_2 = \langle \lambda_1, \lambda_2 \rangle [\tilde{\lambda}_1, \tilde{\lambda}_2]$$

$\boxed{\text{Verify}}$

Since the λ live in a 2d vector space, we have the identity

$$\lambda = \frac{\langle \lambda, \mu_1 \rangle \mu_2 - \langle \lambda, \mu_2 \rangle \mu_1}{\langle \mu_2, \mu_1 \rangle}.$$

Verify this

Exercise:

$$\lambda = \frac{\langle \lambda, \mu_1 \rangle \mu_2 \ \boxed{} \ \langle \lambda, \mu_2 \rangle \mu_1}{\boxed{}}$$

Fill in the top box with + or −
and bottom box with some expression

⑥

We can do many wonderful things with these spinors.

For example, gauge boson polarization vectors are given by

$$\varepsilon^-_{\alpha\dot\alpha} = \sqrt{2}\,\frac{\lambda_\alpha \tilde\mu_{\dot\alpha}}{[\tilde\lambda, \tilde\mu]} \quad , \tilde\mu \text{ is arbitrary}$$

$$\varepsilon^+ = \sqrt{2}\,\frac{\mu_\alpha \bar\lambda_{\dot\alpha}}{\langle\lambda, \mu\rangle} \quad , \mu \text{ is arbitrary}$$

Note little group transformation properties are obvious.

$\varepsilon \cdot p = 0$ [Verify].

Why does arbitrariness of μ not matter?

$$\lambda_\alpha \left(\frac{\tilde\mu_{\dot\alpha}}{[\tilde\lambda, \tilde\mu]} - \frac{\tilde\gamma_{\dot\alpha}}{[\tilde\lambda, \tilde\gamma]} \right)$$

$$= \lambda_\alpha \left(\frac{\tilde\mu_{\dot\alpha}[\tilde\lambda, \tilde\gamma] - \tilde\gamma_{\dot\alpha}[\tilde\lambda, \tilde\mu]}{[\tilde\lambda, \tilde\mu][\tilde\lambda, \tilde\gamma]} \right)$$

$$= \propto \lambda_\alpha \tilde\lambda_{\dot\alpha}$$

So, by giving the spinors and the helicity I fix the momentum and the polarization vectors!

BCFW extension in terms of spinors

$P_1 = \lambda_1 \tilde{\lambda}_1$

$P_n = \lambda_n \tilde{\lambda}_n$

Extension is:

Possibility 1: $\lambda_1 \to \lambda_1 + \lambda_n w \; ; \; \tilde{\lambda}_1 \to \tilde{\lambda}_1$
$\tilde{\lambda}_n \to \tilde{\lambda}_n - \tilde{\lambda}_1 w \; ; \; \lambda_n \to \lambda_n$

Exercise: What is q above?
What is possibility 2?

Ans: 1) $q = \lambda_n \tilde{\lambda}_1$

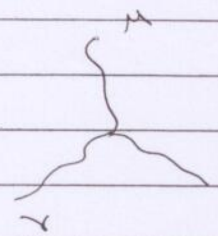
2) $\lambda_n \to \lambda_n + \lambda_1 w ; \; \tilde{\lambda}_n \to \tilde{\lambda}_n$
$\lambda_1 \to \lambda_1 ; \; \tilde{\lambda}_1 \to \tilde{\lambda}_1 - \tilde{\lambda}_n w.$

Talk about which polarization vectors grow in each case.

h_1	h_n	Large w behaviour Poss. 1	Poss. 2
$-$	$-$	$1/w$	$1/w$
$-$	$+$	w^3	$1/w$
$+$	$-$	$1/w$	w^3
$+$	$+$	$1/w$	$1/w$

Now, we turn to the fundamental dynamical object. The 3 point function!

Recall the 3-pt vertex:



$$= \left[\eta^{\mu\nu}(p_1-p_2)^\rho + \eta^{\mu\rho}(p_3-p_1)^\nu + \eta^{\rho\nu}(p_2-p_3)^\mu\right] g f^{abc}$$

The color-ordered amplitude is given by dotting the Lorentz structure with polarization vectors.

First, lets discuss some kinematics,
We have 3 - momenta,

$$p_1, p_2, p_3 \ ; \ p_i^2 = 0$$

$$p_i = \lambda_i \tilde{\lambda}_i \ ; \ \text{but } p_1 \cdot p_2 = 0 \text{ since } (p_1+p_2)^2 = p_3^2 = 0.$$

only solution is:

$$P_1 = \lambda \tilde{\lambda}_1 ; \quad P_2 = \lambda \tilde{\lambda}_2 ; \quad P_3 = \lambda \tilde{\lambda}_3$$

or

$$P_1 = \lambda_1 \tilde{\lambda} ; \quad P_2 = \lambda_2 \tilde{\lambda} ; \quad P_3 = \lambda_3 \tilde{\lambda}$$

In case 1: 2 helicities must be +
case 2: two hels must be −

Easy to see from term:

$$\eta^{\mu\nu} (P_1 - P_2)^{\rho} \varepsilon^1_{\mu} \varepsilon^2_{\nu} \varepsilon^3_{\rho}$$

Also, note $\lambda_3 = -\lambda_1 - \lambda_2$

So, there is only <u>one</u> independent inner product

$$\langle \lambda_1, \lambda_2 \rangle \, !$$

eg. take

$$\varepsilon^1 = \frac{\tilde{\lambda} \mu^1}{\langle \lambda^1, \mu^1 \rangle} ; \quad \varepsilon^2 = \frac{\lambda^2 \tilde{\mu}}{[\tilde{\lambda}, \tilde{\mu}]} ; \quad \varepsilon^3 = \frac{\lambda^3 \tilde{\mu}}{[\tilde{\lambda}, \tilde{\mu}]}$$

take $\mu^1 = \lambda^2$.

automatically 2 out of 3 terms vanish!

$$\eta^{\mu\rho} (P_2 - P_3)^{\nu} \cdot \varepsilon_{1\mu} \varepsilon_{3\rho} \varepsilon_{2\nu} = \# \langle \lambda_1, \lambda_2 \rangle \, !$$

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Can write this in the Parke-Taylor form:

$$M_{--++\cdots+} = \frac{\langle \lambda_1, \lambda_2 \rangle^4}{\langle \lambda_1, \lambda_2 \rangle \langle \lambda_2, \lambda_3 \rangle \cdots \langle \lambda_{n-1}, \lambda_n \rangle \langle \lambda_n, \lambda_1 \rangle}$$

$$M_{++--\cdots-} = \; ? \; \frac{[\tilde\lambda_1, \tilde\lambda_2]^4}{[\tilde\lambda_1, \tilde\lambda_2] \cdots [\tilde\lambda_n, \tilde\lambda_1]}$$

Fill in box!

Q) How does M for 4 particles have 2 expressions?

Note also, M is <u>holomorphic</u> in λ.

<u>Proof of the Parke-Taylor formula</u>:-

In the BCFW formula, say we take

$\lambda_1 \to \lambda_1 + \lambda_n w$; $\tilde\lambda_1 \to \tilde\lambda_1$

$\tilde\lambda_n \to \tilde\lambda_n - \tilde\lambda_1 w$; $\lambda_n \to \lambda_n$

only one division of

$- + + + + + + -$ is possible, into

$- + + \quad - + + + \cdots -$

hint

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What is the value of w at a pole?
Can only be when

$$\lambda_1 + \lambda_n w \propto \lambda_2$$

We know.

$$\lambda_2 = \frac{\langle \lambda_2, \lambda_1 \rangle \lambda_n - \langle \lambda_2, \lambda_n \rangle \lambda_1}{\langle \lambda_n, \lambda_1 \rangle}$$

$$= -\frac{\langle \lambda_2, \lambda_n \rangle}{\langle \lambda_n, \lambda_1 \rangle} \left(\lambda_1 - \frac{\langle \lambda_2, \lambda_1 \rangle}{\langle \lambda_2, \lambda_n \rangle} \lambda_n \right)$$

$$w = -\frac{\langle \lambda_2, \lambda_1 \rangle}{\langle \lambda_2, \lambda_n \rangle} \quad \text{at a pole.}$$

Amplitude on left:

Momenta on left?

$$-\frac{\langle \lambda_n, \lambda_1 \rangle}{\langle \lambda_2, \lambda_n \rangle} \lambda_2, \quad \tilde{\lambda}_1 \quad , \quad -$$

$$\lambda_2, \quad \tilde{\lambda}_2 \quad , \quad +$$

$$\lambda_2, \quad + \frac{\langle \lambda_n, \lambda_1 \rangle}{\langle \lambda_2, \lambda_n \rangle} \tilde{\lambda}_1 - \tilde{\lambda}_2, \quad +$$

(12)

Amplitude on left is:

$$\frac{\langle \lambda_n, \lambda_1 \rangle^3}{\langle \lambda_2, \lambda_n \rangle^3} [\tilde{\lambda}_1, \tilde{\lambda}_2]$$

Amplitude on right is:

$$\frac{\langle \lambda_n, \lambda_2 \rangle^4}{\langle \lambda_n \lambda_2 \rangle \langle \lambda_2, \lambda_3 \rangle \ldots \langle \lambda_{n-1}, \lambda_n \rangle}$$

Momentum factor is

$$\frac{1}{\langle \lambda_1, \lambda_2 \rangle [\tilde{\lambda}_1, \tilde{\lambda}_2]}$$

Putting everything together we get

$$\frac{\langle \lambda_1, \lambda_n \rangle^4}{\langle \lambda_1, \lambda_2 \rangle \langle \lambda_2, \lambda_3 \rangle \ldots \langle \lambda_n, \lambda_1 \rangle}$$

(13)

Gravity computation?

Ans is:

$$\frac{[\tilde{\lambda}_1, \tilde{\lambda}_4]^6 \langle \lambda_2, \lambda_3 \rangle}{[\tilde{\lambda}_1, \tilde{\lambda}_3][\tilde{\lambda}_2, \tilde{\lambda}_4][\tilde{\lambda}_1, \tilde{\lambda}_2][\tilde{\lambda}_2, \tilde{\lambda}_3][\tilde{\lambda}_3, \tilde{\lambda}_4]}$$

3-pt amplitude is just square of gauge-boson amplitude.

$$M_{--+} = \left(\frac{\langle \lambda_1, \lambda_2 \rangle^4}{\langle \lambda_1, \lambda_2 \rangle \langle \lambda_2, \lambda_3 \rangle \langle \lambda_3, \lambda_1 \rangle} \right)^2$$

$$M_{++-} = \left(\frac{[\tilde{\lambda}_1, \tilde{\lambda}_2]^4}{[\tilde{\lambda}_1, \tilde{\lambda}_2][\tilde{\lambda}_2, \tilde{\lambda}_3][\tilde{\lambda}_3, \tilde{\lambda}_1]} \right)^2$$

Motivate the answer:-
Starting with M_+