5 April 2011

Recursion Relations for Tree Amplitudes

We will consider <u>tree</u> <u>amplitudes</u> in a non-Abelian gauge theory or gravity.

In YM, consider tree-level scattering of n-gluons.

Each gluon is labeled by:

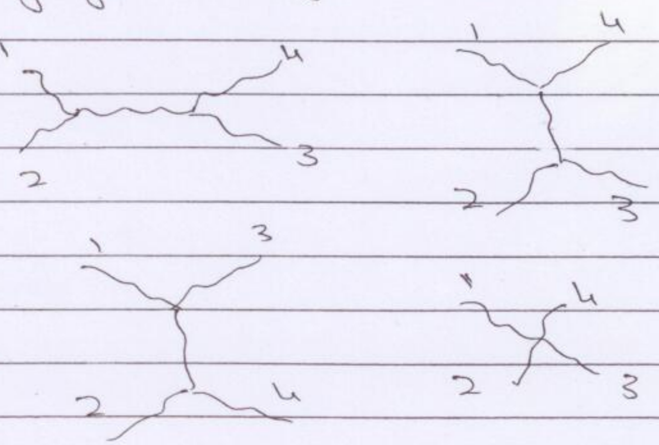
a) momentum: p_i ; $p_i^2 = 0$

b) helicity, which can be + or −
(for massless particles, helicity is a good quantum number).

→ $\boxed{\text{Scattering data: } n \text{ momenta } p_i;\ p_i^2=0;\ \Sigma p_i = 0 \text{ and } n \text{ helicities}}$

To reiterate. Say we want to compute a scattering amplitude of 4 particles in a non-Abelian gauge theory

$$\underbrace{\begin{matrix} p_1, + \\ p_2, + \\ p_3, - \\ p_4, - \end{matrix}}_{\text{Data}}$$



The Feynman diagrams give us:

$$M^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4)$$

The amplitude is given by

a) imposing $k_i^2 = 0$

b) $M(+,+,-,-) = \epsilon^{+}_{\mu_1}(k_1)\,\epsilon^{+}_{\mu_2}(k_2)\,\epsilon^{-}_{\mu_3}(k_3)\,\epsilon^{-}_{\mu_4}(k_4)\;M^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4)$

Now, we go back to our tree-amplitude.
To be concrete, take:

$p_1 = (1, 1, 0, 0)$
$p_n = (1, -1, 0, 0)$

particles coming straight at each other along the z-direction

Polarization vectors are:

$q = \epsilon_1^- = \epsilon_2^+ = (0, 0, 1, i)$
$q^* = \epsilon_1^+ = \epsilon_2^- = (0, 0, 1, -i)$.

We can see this easily. Little group rotations are rotations in the x-y plane

We now consider the following <u>deformation</u> of these momenta

$$P_1(w) \to P_1 + q\,w \qquad P_n(w) \to P_n - q\,w$$

$\Big\}$ w is a <u>complex</u> parameter to keep track of the deformation

This is an <u>unusual</u> deformation.

Our momenta now have <u>complex</u> components.

However, this is allowed because

a) momentum is conserved: $P_1(w) + P_n(w) = P_1(0) + P_n(0)$.

b) all momenta are still on shell: $P_1(w)^2 = P_n(w)^2 = P_1(0)^2 = P_n(0)^2 = 0$

We need to alter the polarization vectors to keep them orthogonal to the momenta:

$$\varepsilon_1^-(w) = \varepsilon_n^+(w) = q$$

$$\varepsilon_1^+(w) = q^* + P_n w; \quad \varepsilon_n^-(w) = q^* + P_1 w$$

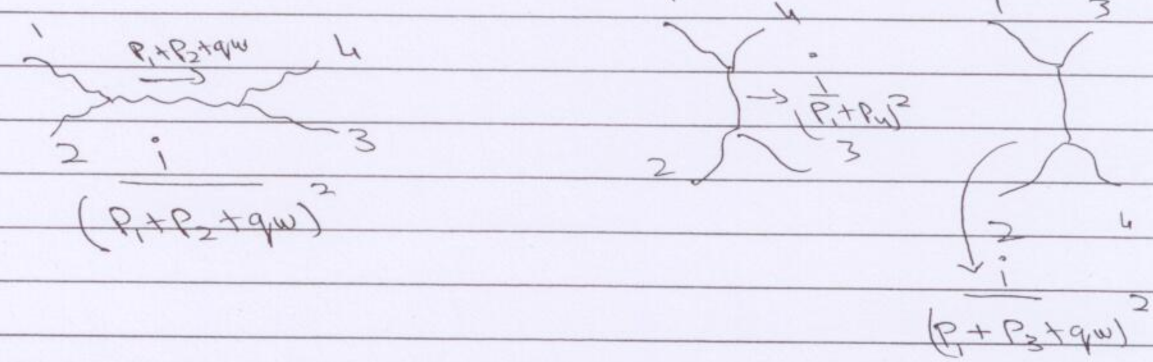
→ refers to undeformed P_1

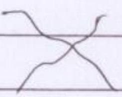
Note: $\varepsilon_1^+(w) \cdot P_1(w) = (q^* + P_n w)\cdot(P_1 + q w)$
$= q^* \cdot q\, w + P_1 \cdot P_n w = 0$

Since we have tree-level amplitudes, the amplitude is a rational function of w — $M(w)$.

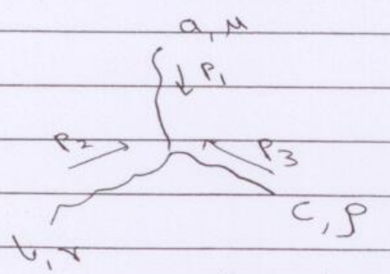
$M(0)$ is the original physical amplitude.

What is the analytic structure of $M(w)$. Let us see this by considering a 4-pt amplitude

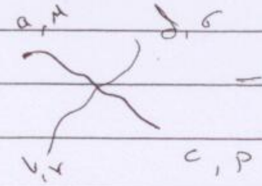


 ~ No denominator.

Recall, the vertex is:-



$$g f^{abc}\left[\eta^{\mu\nu}(p_1-p_2)^\rho + \eta^{\mu\rho}(p_3-p_1)^\nu + \eta^{\rho\nu}(p_2-p_3)^\mu\right]$$



$$-ig^2\left[f^{abe}f^{cde}\left(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}\right) + \dots\right]$$

Amplitude has simple poles at:-

$$w_s = -\frac{(p_1+p_2)^2}{2(p_2\cdot q)}$$

$$w_t = -\frac{(p_1+p_3)^2}{2p_3\cdot q}.$$

Derived from:-

$$(p_1+p_2+qw)^2 = 0 \Rightarrow (p_1+p_2)^2 + 2(p_2\cdot q)w = 0$$

$$\Rightarrow w = w_s.$$

What is the residue at such a pole

Recall the propagator looks like:-

$$\frac{i\eta^{\mu\nu}}{(p_1+p_2+qw)^2}$$

at $w=w_s$, we can replace:-

$$\eta^{\mu\nu} \longrightarrow \varepsilon^{+,\mu}(p_1+p_2+qw_s)\,\varepsilon^{-,\nu}(-p_1-p_2-qw_s) + \varepsilon^{-,\mu}(p_1+p_2+qw_s)\,\varepsilon^{+,\nu}(-p_1-p_2-qw_s)$$

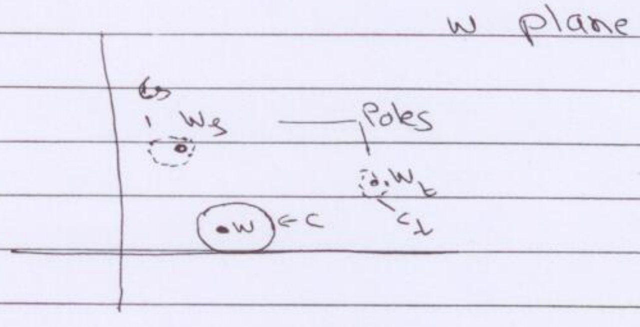
This ~~is~~ can be shown using the Ward identities. See Peskin and Schroeder

So, the residue is a product of smaller-point amplitudes.

$$R_s = \frac{i}{(P_3 - q)} \sum_{h_3} \Big[M(h_1, P_1 + q w_s, h_2, P_2, h_3, -P_1 - P_2 - q w_s) \; M(-h_3, P_1 + P_2 + q w_s, h_4, P_3, h_5, P_4) \Big]$$

One such pole for each partition of the set of momenta into two sets with P_1 in one and P_n in the other

Can we reconstruct $M(w)$ from these residues?



$$M(w) = \oint_c \frac{M(z)}{z - w} \frac{dz}{2\pi i} \quad \Big[\text{Cauchy's Integral Formula}\Big]$$

We can invert the contour. Think of the complex plane as S^2. Instead of circling

⑦

$$M(w) = -\oint_{C_s} \frac{M(z)}{z-w}\frac{dz}{2\pi i} - \oint_{C_t} \frac{M(z)}{z-w}\frac{dz}{2\pi i} \quad ?$$

+ missing contribution from $z = \infty$

the two contour integrals give

$$-\int_{C_s} \frac{M(z)}{z-w}\frac{dz}{2\pi i} = \frac{R_s}{w-w_s} \quad ; \quad R_s \text{ is residue at } w = w_s$$

$$-\int_{C_t} \frac{M(z)}{z-w}\frac{dz}{2\pi i} = \frac{R_t}{w-w_t}$$

We then have to consider

$$B = \oint_{z=\infty} \frac{M(z)}{z-w}\, dz$$

To analyze this, change variables to $t = \frac{1}{z}$

$$dz = -\frac{1}{t^2}\, dt$$

$$B = -\oint_{|t|=\varepsilon} \frac{M(t)}{1-tw}\frac{dt}{t}$$

this integral has a pole at $t = 0$

So, the question is:

Does $M(t)$ vanish as $t \to 0$?

or

Does $M(w)$ vanish as $w \to \infty$?

If it does, then the contour integral is 0 and

$$B = 0$$

If not, we need to worry about this boundary term.

What is the behaviour of $M(w)$ at $w \to \infty$

Go back to Feynman diagrams on page 1 and vertices on page 4, and BCFW extension on page 3

ε_1	ε_n	M_∞ (naive) (squared for gravity)	M_∞ (actual) YM	Grav
$-$	$+$	w (w^2)	$1/w$	$\frac{1}{w^2}$
$-$	$-$	w^2 (w^4)	$1/w$	$\frac{1}{w^2}$
$+$	$+$	w^2 (w^4)	$1/w$	$\frac{1}{w^2}$
$+$	$-$	w^3 (w^6)	w^3	w^6

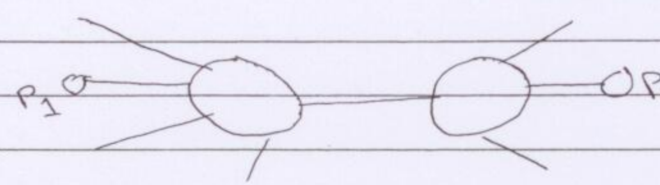
bracketed terms are for gravity

So for gauge and gravity theories but not for scalar theories, we can write down recursion relations.

This is because we can reconstruct the amplitude entirely from its residues at poles.
The residue at each pole is the product of lower order amplitudes.

List of poles

1) Every partition of the external momenta into two sets, with p_1 in one and p_n in the other corresponds to a pole because there are diagrams of the sort below.



2) The residue at each pole is the product of the left and right amplitudes.

Central result :-

$$M\begin{bmatrix} p_1 & p_2 & & p_n \\ h_1 & h_2 & \cdots & h_n \\ a_1 & a_2 & & a_n \end{bmatrix} = \sum_{\substack{h_{int}, a_{int} \\ \pi_j}} \frac{\tilde{M}^2_{\pi, h_{int}, a_{int}}}{(p_1 + p_{\pi_2} + \ldots p_{\pi_j})^2} ;$$

$$\tilde{M}^2_{\pi, h_{int}, a_{int}} \equiv M\begin{pmatrix} p_1 + q w_{int} & \ldots & p_{\pi_j} & -p_{int} \\ h_1 & & h_{\pi_j} & h_{int} \\ a_1 & & a_{\pi_j} & a_{int} \end{pmatrix}$$

$$\times M\begin{pmatrix} p_{int} & p_{\pi_{j+1}} & & p_n - q w_{int} \\ -h_{int} & h_{\pi_{j+1}} & \cdots & h_n \\ a_{int} & a_{\pi_{j+1}} & & a_n \end{pmatrix}.$$

$$p_{int} = p_1 + p_{\pi_2} + \ldots p_{\pi_j}$$

π_j corresponds to a partition of the momenta $1 \ldots n$ into sets of length $j+1$ [including p_1] and $n-j$ [including p_n]

Note the helicity in M_{left} and M_{right} is opposite.

We also need to sum over intermediate colors.

Helicities

Remember $M(w) \xrightarrow[w \to \infty]{} 0$ only for 3/4 helicities.

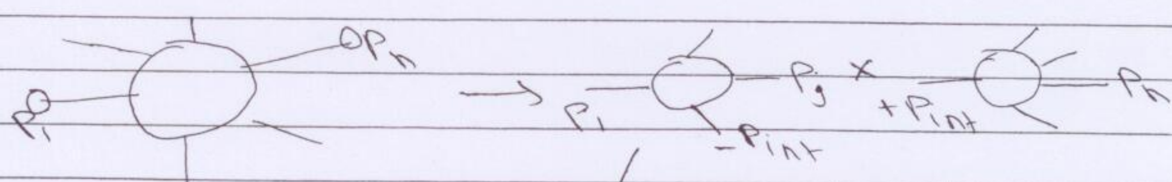
What if we are interested in the $+-$ answer [Refer to Page 8]

In that case, we just BCFW extend by q^*!

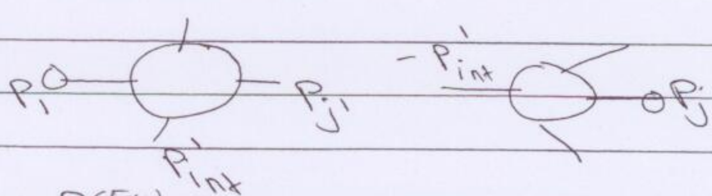
So, all tree amplitudes can be broken down using BCFW recursion

Three-pt Amplitudes

We can continue this process



breaks down to



by $\overset{\text{BCFW}}{\text{extending}}$ P_1 and P_j.

This process ends when we reach down to the 3-pt amplitude

eg.

$$M_5 = M_4 \times M_3$$

$$M_4 = M_3 \times M_3 .$$

This is the fundamental dynamical object in the theory.

Given the 3-pt on-shell amplitude, we can reconstruct all tree-amplitudes