Coulomb branches for quiver gauge theories with symmetrizers

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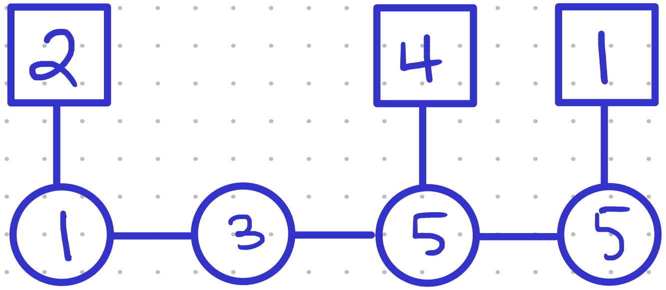
Quantum fields, geometry and representation theory 2021

ICTS

* 3d $\mathcal{N}=4$ gauge theories are extremely rich from both physical & mathematical perspectives

* Especially significant: quiver gauge theories



Type A_4

Coulomb branch $\mathcal{M}_C$

- affine Grassmannians
- geometric Satake

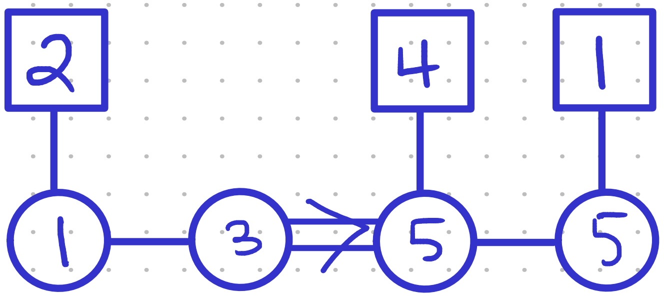
) at least for finite ADE

Higgs branch $\mathcal{M}_H$

- Nakajima quiver varieties

* Promising setting for geometric representation theory of symmetric Kac-Moody types

* What about non-symmetric types?



Type F_4

$\mathcal{M}_C = ??$ our topic today

$\mathcal{M}_H = ??$

* Expectations from physics (e.g. Hanany and collaborators)

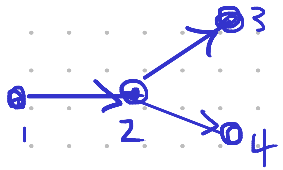
* With Nakajima:

  - Mathematical construction of $\mathcal{M}_C$ for symmetrizable types
  - Finite BCFG: $\mathcal{M}_C \simeq$ generalized affine Grassmannian slice

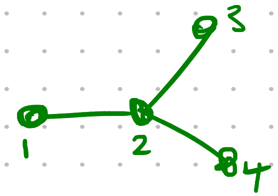
I) Review of usual quiver gauge theories

1. Start with quiver $Q=(I,E)$

vertices edges



type D_4

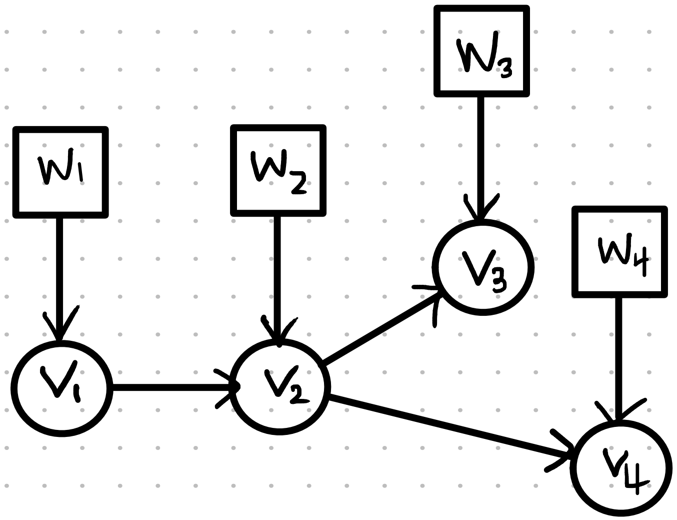


2. Choose graded $\mathbb{C}$-vector spaces

$$V=\bigoplus_{i\in I} V_i\,, \qquad W=\bigoplus_{i\in I} W_i$$

$v_i=\dim_{\mathbb{C}} V_i$

$w_i=\dim_{\mathbb{C}} W_i$



$$G=\prod_{i\in I} GL(V_i)$$

$$N=\bigoplus_{i\to j}\mathrm{Hom}(V_i,V_j)\oplus\bigoplus_{i\in I}\mathrm{Hom}(W_i,V_i)$$

$N\oplus N^*$

3. Coulomb branch $\mathcal{M}_C(G,N) \overset{\text{def}}{=} \operatorname{Spec} H_*^{G(O)}(R)$

Theorem: (BFN) For Q finite ADE type:

$$\mathcal{M}_C \simeq \overline{\mathcal{W}}^{\lambda}_{\mu} \longleftarrow \text{generalized affine Gr slice (of same ADE type)}$$

* $G = G_Q$ — semisimple algebraic group (adjoint type)

$$\lambda = \sum_i w_i \underbrace{\varpi_i^\vee}_{\text{fundamental coweights}}, \qquad \lambda - \mu = \sum_i v_i \underbrace{\alpha_i^\vee}_{\text{simple coroots}} \qquad \underbrace{\lambda, \mu : \mathbb{C}^\times \to G}_{\text{coweights}}$$

* $\mathcal{M}_C \simeq \overline{\mathcal{W}}^{\lambda}_{\mu}$ is "geometric avatar" of weight space

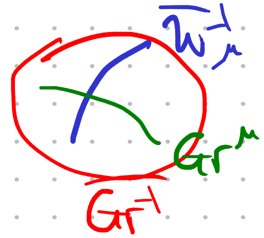
$$V(\lambda)_\mu \subset V(\lambda) \circlearrowleft \text{Langlands dual } G^\vee$$

This is a consequence of Geometric Satake.

See Nakajima's lectures: conjecture to hold for all types

Remarks

If μ dominant, $\overline{W}^\lambda_\mu \hookrightarrow Gr_G$ transverse slice to $Gr^\mu \subset \overline{Gr^\lambda}$



1) $\overline{W}^\lambda_\mu$ = space of "scattering matrices"

$$= \underbrace{U_1^+[[z^{-1}]]\, T_1[[z^{-1}]]\, z^\mu\, U_1^-[[z^{-1}]]}_{\text{"Gauss decomposition"}} \cap \underbrace{\overline{G[z]\, z^\lambda\, G[z]}}_{\text{Hecke type} \le \lambda}$$

Expected: $\overline{W}^\lambda_\mu$ is a moduli of singular G_{compact}-monopoles on $\mathbb{R}^3$

2) In particular, when $\lambda = 0$ $(\iff \text{all } w_i = 0)$

$$\mathcal{M}_C \simeq \overline{W}^0_\mu \simeq \left\{ \begin{array}{l} \text{Based maps } \varphi : \mathbb{P}^1 \longrightarrow G/B \\ \qquad\qquad\qquad\quad \infty \longmapsto B \\ \qquad \text{of degree } -\mu \end{array} \right\}$$

(see Hurtubise's 1st lecture) (Jarvis, ...)

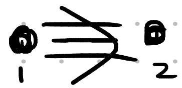
II) Symmetrizable types

* Cartan matrix $A=(a_{ij})_{i,j\in I}$ is <u>symmetrizable</u> if

$$d_i a_{ij} = d_j a_{ji}$$

d_i – "symmetrizers"

for some $d_i > 0$

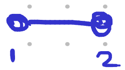
ex: Type G_2 

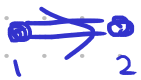
$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}$$

Cartan matrix $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ symmetrizers $d_1 = 3,\ d_2 = 1$

$d_1 = 3c \quad d_2 = c$

* Fix orientation $Q=(I,E)$ of "underlying graph"

A_2 

B_2 

G_2 

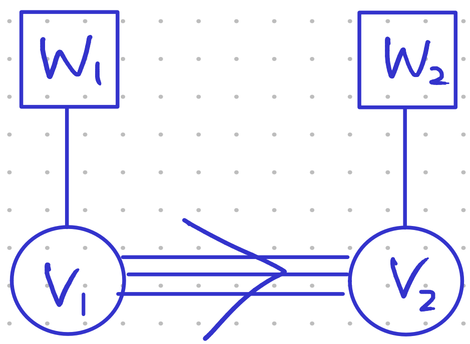
$\rightsquigarrow \quad Q = \overset{}{\underset{1}{\circ}} \rightarrow \underset{2}{\circ}$ or $\underset{1}{\circ} \leftarrow \underset{2}{\circ}$

* Choose $V = \bigoplus_i V_i$, $\quad W = \bigoplus_i W_i$

* Given $(\underline{A}, \underline{\{d_i\}}, \underline{Q}, \underline{V}, \underline{W})$, we'll construct $\mathcal{M}_C$

Independent of $\{d_i\}$ and orientation of Q, up to isomorphism

$\Longrightarrow$



$\mathcal{M}_C$ is well-defined up to isom.

Rmk: If all d_i are equal recover usual Coulomb branch

Thm: (Nakajima–W.) In finite BCFG types,

$$\mathcal{M}_C \simeq \overline{\mathcal{W}}^{\lambda}_{\mu}$$

$\longleftarrow$ slice of same BCFG type

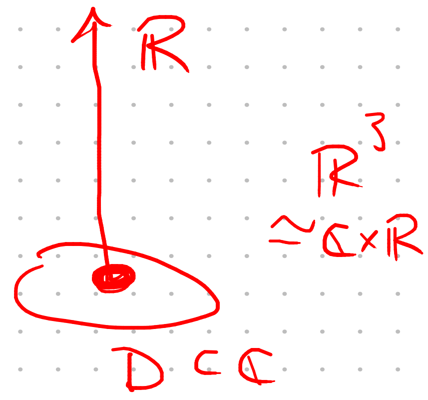
By Satake, $\mathcal{M}_C \simeq \overline{\mathcal{W}}^{\lambda}_{\mu}$ is geometric avatar of

$$V(\lambda)_\mu \subset V(\lambda) \circlearrowleft G^\vee$$

Construction of $\mathcal{M}_C$

* For each $i \in I$, formal discs

$$D_i = \operatorname{Spec} \mathbb{C}[[z_i]]$$
$$D_i^\times = \operatorname{Spec} \mathbb{C}((z_i))$$



But have multiple discs...

* Consider moduli space $\underline{\mathcal{R}}$ of data:

1) Vector bundle $\mathcal{E}_i$ on D_i of rank v_i

2) Trivialization $\varphi_i : \mathcal{E}_i|_{D_i^\times} \xrightarrow{\sim} \mathcal{O}_{D_i^\times}^{\oplus v_i}$

   over $D_i^\times$

3) Morphisms

$$s_i : \mathcal{O}_{D_i}^{\oplus w_i} \longrightarrow \mathcal{E}_i$$

$$s_{i\to j} : \underset{D_i}{\rho_i^* \mathcal{E}_i} \longrightarrow \underset{D_j}{\rho_j^* \mathcal{E}_j}$$

4) $s_i, s_{i\to j}$ remain regular under φ's

ex: $\varphi_i \circ s_i$ is a priori defined only on $D_i^\times$

For each $i \to j$

$$\begin{tikzcd} & D_{ij} \arrow[dl, "\rho_i"'] \arrow[dr, "\rho_j"] & \\ D_i & & D_j \end{tikzcd}$$

$$\deg(\rho_i) = \frac{d_i}{\gcd(d_i, d_j)}$$

$$\deg(\rho_j) = \frac{d_j}{\gcd(d_i, d_j)}$$

* R has natural action of

$$G(\mathcal{O}) = \prod_i GL(v_i)[[z_i]] = \prod_i \mathrm{Aut}\left(\mathcal{O}_{D_i}^{\oplus v_i}\right)$$

<u>Thm/Def:</u> (Nakajima – W.)

There exists a convolution product on $H_*^{G(\mathcal{O})}(R)$ making it a commutative algebra,

$$\mathcal{M}_C \overset{\text{def}}{=\!=} \mathrm{Spec}\, H_*^{G(\mathcal{O})}(R)$$

* Proof by embedding into commutative ring!

Properties

1) $\mathcal{M}_C$ is an irreducible normal affine variety over $\mathbb{C}$

$$\dim_{\mathbb{C}} \mathcal{M}_C = 2 \sum_{i \in I} v_i$$

loop rotation

$\mathbb{C}^\times \circlearrowright D_i$

$s: z_i \mapsto s^{d_i} z_i$

2) Deformation quantization $\mathcal{A}_\hbar = H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(R)$

$\Rightarrow$ Poisson structure on $\mathcal{M}_C$

Smooth locus $\mathcal{M}_C^{reg} \subset \mathcal{M}_C$ is holomorphic symplectic

$T^*\mathbb{C}^\times = \mathbb{C} \times \mathbb{C}^\times = \mathbb{R}^2 \times \mathbb{R} \times S^1$

3) Birational map

$$\mathcal{M}_C \dashrightarrow \prod_i (T^*\mathbb{C}^\times)^{v_i} /\!/ S_{v_i} \simeq \prod_i (\underbrace{\mathbb{R}^3 \times S^1}_{})^{v_i} /\!/ S_{v_i}$$

4) Satisfies <u>twisted monopole formula</u>

of Cremonesi-Ferlito-Hanany-Mekareeya:

graded dimension of $H_*^{G(\mathcal{O})}(R) = \sum_{\gamma} t^{2\Delta(\gamma)} P_\gamma(t)$

$\overline{\mathcal{W}}^{\lambda}_{\mu} \simeq \mathcal{M}_C$

good $\equiv$ μ dominant

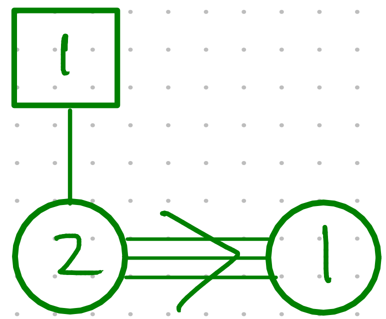
bad $\equiv$ $\langle \mu, \alpha_i \rangle \geq -1 \ \forall i$

"almost dominant"

5) <u>Symmetries:</u>

T_C = Pontryagin dual of $\pi_1\left(\prod_{i\in I} GL(V_i)\right) \simeq (\mathbb{C}^\times)^N$

Then $T_C \circlearrowright \mathcal{M}_C$



Sometimes extends to larger group G_C

(<u>e.g.</u> balanced nodes)

← $\mu_i = 0$, when writing $\mu = \sum \mu_i \varpi_i^\vee$

$G_2 \circlearrowright \mathcal{M}_C$

6) Mass parameters:

$T(W_i) \subseteq GL(W_i)$ maximal torus

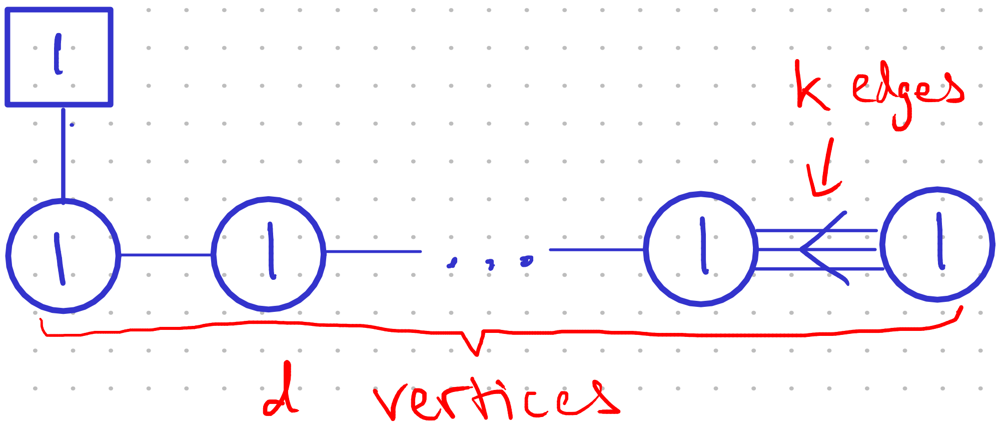
$$T_F = \prod_i T(W_i) \times (\mathbb{C}^\times)^{b_1(Q)}$$

$\rightsquigarrow$ partial resolutions $\mathcal{M}_C^m \twoheadrightarrow \mathcal{M}_C$

An example:

Minimal singularities $h_{d,k}$

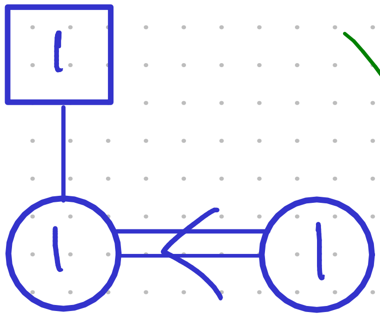
(studied by Bourget – Grimminger – Hanany – Sperling – Zafrir – Zhong)



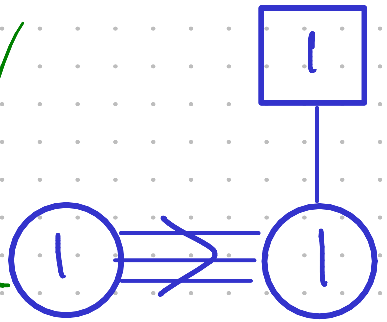
$$\mathcal{M}_C \simeq \mathbb{C}^{2d} /\!\!/ \mathbb{Z}_k$$

- $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z} \circlearrowright \mathbb{C}^d_1 \oplus \mathbb{C}^d_{-1}$
- equivariant for $GL(d)$ actions

In particular,

$\mathcal{M}_C\Big($  $\Big) \simeq \mathbb{C}^2/\!/\mathbb{Z}_2 \simeq$ affine Grass. slice $\overline{\mathcal{W}}^{\varpi_1^\vee}_{0}$ of type C_2

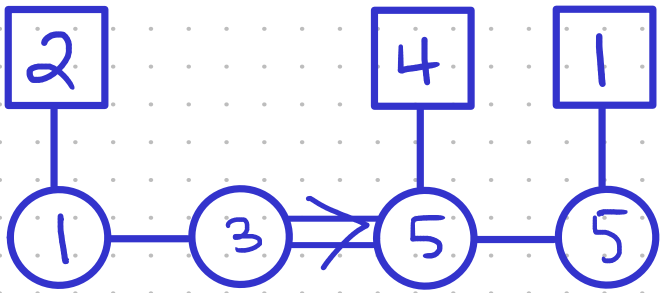
and

$\mathcal{M}_C\Big($  $\Big) \simeq \mathbb{C}^2/\!/\mathbb{Z}_3 \simeq$ affine Grass. slice $\overline{\mathcal{W}}^{\varpi_2^\vee}_{\varpi_1^\vee}$ of type G_2

These are minimal degenerations in affine Grassmannians of types C_2, resp. G_2 (Malkin-Ostrik-Vybornov)

Higgs branches?

* By analogy with usual quiver gauge theories, Higgs branches would be "Nakajima quiver varieties"

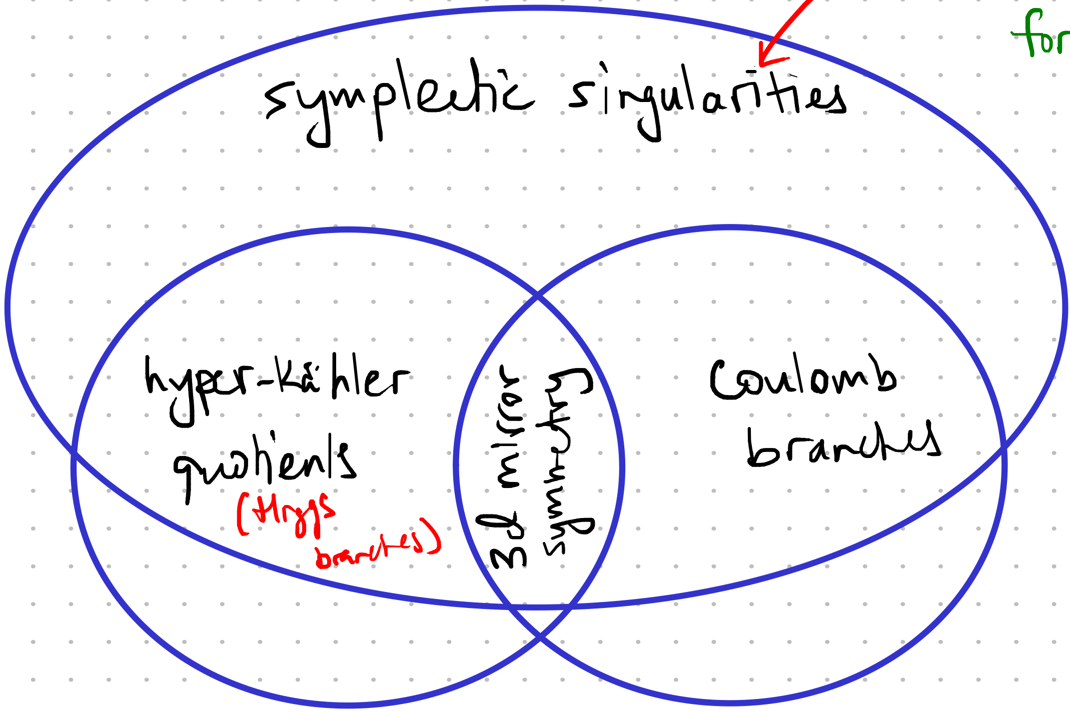
$\mathcal{M}_H\Bigg($  $\Bigg) = ?$

* Several constructions of representations of quivers:
  - modulated graphs (Dlab-Ringel, Tingley-Nandakumar)
  - work of Geiss-Leclerc-Schröer

Unclear how these relate to $\mathcal{M}_C$ or $\mathcal{M}_H$

From Hanany's lectures:

(for me!)
in the sense of Beauville:



for any resolution of singularities $\pi : X \twoheadrightarrow \mathcal{M}_C$

$\pi^*(\omega_{\mathcal{M}_C^{reg}})$ extends to regular 2-form on all of X

$\Longrightarrow$ $\mathcal{M}_C$ has fin. many symplectic leaves

<u>Q</u>: Where in this diagram do $\mathcal{M}_C$ for symmetrizable types live?