Center of small quantum group

and affine Springer fibers

(joint with R. Bezrukavnikov, P. Boixeda-Alvarez and P. Shan)

G connected quasi-simple linear group / $\mathbb{C}$

$\check{G}$ dual group, simply connected

$T \subset B \subset G$ Cartan and Borel subgroups

$\Lambda = X^{\vee}(T) \supset Q = \mathbb{Z}\,\Phi \supset \Phi$ = root system

$\mathfrak{t} = \mathrm{Lie}\, T$, $\mathfrak{g} = \mathrm{Lie}\, G$, $U(\mathfrak{g})$ = enveloping algebra

Define $\check{T}$, $\check{\Lambda}$, $\check{\Phi}$, $\check{\mathfrak{g}}$ similarly

W = Weyl group

$W_{\ell,af} = W \ltimes \ell Q^{\vee} \subset W_{\ell,ex} = W \ltimes \ell \check{\Lambda}$

Ⓐ Positive characteristic :

Assume G defined over $k = \bar{k}$, $\mathrm{char}(k) = p > 0$

$Fr : G \longrightarrow G$ Frobenius map

$G \supset G_1 = \mathrm{Ker}(Fr)$ = Frobenius kernel = a group scheme

$k[G] \twoheadrightarrow k[G_1]$ = finite dimensional Hopf algebra

$\mathfrak{g}$ = restricted Lie algebra with p-operation

$$\mathfrak{g} \to \mathfrak{g} \quad , \quad x \mapsto x^{[p]}$$

$u(\mathfrak{g})$ = restricted enveloping algebra

$$= U(\mathfrak{g}) / (x^p - x^{[p]} \, ; \, x \in \mathfrak{g})$$

$= k[G_1]^*$ as Hopf algebra (finite dimensional)

$\mathrm{Rep}(G_1)$ = finite dimensional rational G_1 - modules

$= \mathrm{Comod}(\Bbbk[G_1])$

$= \mathrm{Rep}(u(G))$

$\mathrm{Dist}(G) \subset \Bbbk[G]^*$ distribution algebra

$\mathrm{Rep}(G)$ = finite dimensional rational G - modules

$= \mathrm{Rep}(\mathrm{Dist}(G))$

= finite dimensional integrable $\mathrm{Dist}(G)$ - modules

There are algebra homomorphisms

$$U(\mathfrak{g}) \twoheadrightarrow u(G) \hookrightarrow \mathrm{Dist}(G)$$

Steinberg tensor product formula $\Rightarrow \mathrm{Irr}(G_1)$ determins $\mathrm{Irr}(G)$

(B) Quantum groups :

$\zeta \in \mathbb{C}^\times$ root of 1 of order $= \ell$

ℓ odd, $\ell > h =$ Coxeter number, prime to 3 in type G_2

We have the following quantum groups attached to $\check{G}$:

(a) $\mathcal{U}_\zeta$ = DeConcini-Kac quantum group (an. of $U(\check{\mathfrak{g}})$)

(b) U_ζ = Lusztig quantum group (an. of $\mathrm{Dist}(\check{G})$)

(c) u_ζ = small quantum group (an. of $u(\check{G})$)

There are algebra homomorphisms

$$\mathcal{U}_\zeta \twoheadrightarrow u_\zeta \hookrightarrow U_\zeta \xrightarrow{\mathrm{Fr}} U_1 = U(\check{\mathfrak{g}})$$

$$\mathrm{Fr}^* : \mathrm{Rep}(\check{G}) = \mathrm{Rep}(U_1) \longrightarrow \mathrm{Rep}(U_\zeta)$$

$\mathrm{Rep}(U_\zeta) =$ $\check{\Lambda}$ graded, integrable, finite dimensional modules

$\mathrm{Rep}(U_1) = \mathrm{Rep}(\check{G})$

$\mathrm{Rep}(u_\zeta) =$ finite dimensional modules

NB: $A =$ abstract group

$$\begin{array}{ccc} \text{Braided tensor} & & \text{Braided tensor} \\ \mathbb{C}\text{-linear categories} & \underset{\text{equivariantization}}{\overset{\text{de-equivariantization}}{\rightleftarrows}} & \mathbb{C}\text{-linear categories} \\ \text{with } \mathrm{Rep}(A)\text{-action} & & \text{with } A\text{-action} \end{array}$$

$$\mathcal{C}^A \longleftrightarrow \mathcal{C}$$

$\mathrm{Obj}(\mathcal{C}^A) = \{ A\text{-equivariant objects in } \mathcal{C} \}$

Idem if A is an affine algebraic group

EX:

(a) G-variety $X \Rightarrow \mathrm{Coh}(X)^G = \mathrm{Coh}_G(X)$

(b) (Arkhipov - Gaitsgory)

$$\mathrm{Rep}(\check{G}) \overset{Fr^*(-)\otimes -}{\circlearrowright} \mathrm{Rep}(U_\zeta) = \mathrm{Rep}(u_\zeta)^{\check{G}} \underset{eq.}{\overset{de\text{-}eq.}{\rightleftarrows}} \mathrm{Rep}(u_\zeta) \circlearrowleft \check{G}$$

$$\Rightarrow \quad Z(u_\zeta)^{\check{G}} = Z(U_\zeta) \cap u_\zeta$$

Block decomposition:

$$u_\zeta = \bigoplus_{\omega \in \check{\Lambda}/W_{\ell,af}} u_\zeta^\omega$$

u_ζ^0 = principal block

CONJ (Lachowska – Qi): Type A

(a) $\dim Z(u_\zeta) = \frac{1}{(h+1)\ell}\binom{(h+1)\ell}{h}$

(b) $\dim Z(u_\zeta^\circ) = (h+1)^{h-1}$

(c) $\check{G}$ acts trivially on $Z(u_\zeta)$

Relation with Haiman's work:

$$\{\text{diagonal coinvariants}\} = \mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^\vee] / (\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^\vee]^W_+)$$

$$= W \ltimes \mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^\vee]\text{-module}$$

$$W \ltimes \mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^\vee] = \mathrm{gr}(\text{R DAHA})$$

In type A we have (Haiman)

$\dim = (h+1)^{h-1}$

$$\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^\vee]/(\mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^\vee]^W_+) = \mathbb{C}[Q/(h+1)Q]$$

$$= \mathrm{gr}(\text{simple R DAHA-module})$$

Ⓒ Geometrization:

Affine flag manifolds:

$$\check{\Lambda} \subset \check{\Lambda} \underset{\mathbb{Z}}{\otimes} \mathbb{R} = t_{\mathbb{R}}$$

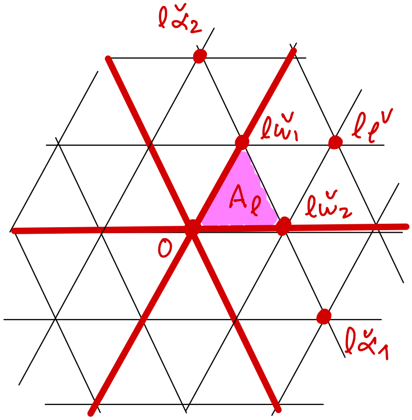
$t_{\mathbb{R}} \supset \bar{A}_\ell =$ closed fundamental alcove

$$= \{ \lambda \in t_{\mathbb{R}} \; ; \; 0 \leq (\mu, \check{\alpha}) \leq \ell \; , \; \forall \check{\alpha} \in \check{\Phi} \}$$

$=$ fundamental domain for $W_{\ell,af} \curvearrowright t_{\mathbb{R}}$

$$\check{\Lambda} / W_{\ell,af} \simeq \check{\Lambda} \cap \bar{A}_\ell$$

EX: $G = PSL_3$



$K = \mathbb{C}((\pi)) \supset \mathcal{O} = \mathbb{C}[\![\pi]\!]$

To $\omega \in \bar{A}_1$ corresponds a parahoric subgroup $P^\omega \subset G(K)$

$Fl^\omega = G(K)/P^\omega$ partial affine flag manifold $/G$

EX:

(a) $\omega = 0 \Rightarrow Fl^\omega = Gr = G(K)/G(\mathcal{O})$

with $G(\mathcal{O})$ maximal compact in $G(K)$

(b) $\omega \in A \Rightarrow Fl^\omega = Fl = G(K)/I$

with $I \subset G(\mathcal{O})$ Iwahori

$$K^{(\ell)} = \mathbb{C}((\pi^\ell))$$

$$G(K^{(\ell)}) \subset G(K)$$

$P^\omega \subset G(K^{(\ell)})$ parahoric for $\omega \in \overline{A}_\ell$

$Fl^{\omega,(\ell)} = G(K^{(\ell)})/P^\omega$ partial affine flag manifold

$$Fl^\omega = Fl^{\omega,(1)}$$

$\mathbb{C}^\times \circlearrowright Fl^\omega$ by loop rotation

$$Gr^\zeta = \bigsqcup_{\omega \in \check{\Lambda}/W_{\ell,ex}} Fl^{\omega,(\ell)} \quad , \quad \zeta \in \mathbb{C}^\times \; , \; \zeta^\ell = 1$$

Affine Springer fibers:

Choose $\gamma \in \mathfrak{g} \otimes K$ regular semi-simple

$Fl_\gamma^\omega \subset Fl^\omega$ affine Springer fiber

$$= \{ Ad(g)(P^\omega) \; ; \; g \in G(K), \; Ad(g^{-1})(\gamma) \in rad(Lie\, P^\omega) \}$$

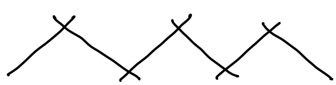
$\gamma_\ell = s \otimes \pi^\ell$, $\gamma = \gamma_1$, $s \in \mathfrak{g}^{rs}$ regular semisimple

Fl_γ^ω ind-scheme, pure, equidimensional, finite dimensional

LEM:

$$\boxed{(Gr_{\gamma_\ell})^{\zeta} = \bigsqcup_{\omega \in \check{\Lambda}/W_{\ell,ex}} Fl_{\gamma_\ell}^{\omega,(\ell)} \overset{(\pi^\ell \leftrightarrow \pi)}{\simeq} \bigsqcup_{\omega \in \check{\Lambda}/W_{\ell,ex}} Fl_\gamma^\omega}$$

EX: $G = SL_2$, $\gamma = \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix}$

$Fl_\gamma = \ldots$  $\ldots$ chain of $\mathbb{P}^1$'s

Computation of $H^\bullet(Gr_{\gamma_e}^{\xi})$:

(a) $T \circlearrowright Fl_\gamma^\omega$

$$(Fl_\gamma^\omega)^T = (Fl^\omega)^T = W_{ex}/W_\omega$$

W_ω = stabilizer of $\omega \in \Lambda^\vee$ in W_{ex}

(b) Fl_γ^ω has a paving by affine cells given by intersections with Iwahori orbits (= Bruhat cells)

$\Rightarrow$ GKM applies to $H_T^\bullet(Fl_\gamma^\omega, \mathbb{Q})$

$\Rightarrow$ Restriction to $(Fl_\gamma^\omega)^T$ yields explicit embedding

$$H_T^\bullet(Fl_\gamma^\omega, \mathbb{Q}) \subset \mathrm{Fun}(W_{ex}/W_\omega, H_T^\bullet)$$

$\Rightarrow$ Sum over all ω's

$$H_T^\bullet(Gr_{\gamma_e}^{\xi}, \mathbb{Q}) = \{ (a_x)_{x\in\check{\lambda}} \in \mathrm{Fun}(\Lambda^\vee, H_T^\bullet) ;$$

$$a_x \equiv a_{s_{\alpha,m}\cdot x} \mod \alpha, \ \forall (\alpha, m) \in \Phi\times\mathbb{Z} \}$$

$s_{\alpha,m}$ = reflexion in $\mathfrak{t}$ / affine hyperplane

Symmetries of affine Springer fibers :

$T(k) \circlearrowright Fl_\gamma^\omega$

(a) Left W_{ex} - action on $H_T^\bullet(Fl_\gamma^\omega, \mathbb{Q})$, $\forall \omega$

(b) Right W_{ex} - action on $H_T^\bullet(Fl_\gamma, \mathbb{Q})$ = Springer

Both use GKM description $\mathrm{Fun}(W_{ex}/W_\omega, H_T^\bullet)$

left action

PROP:

(a) $\dim H^\bullet(Fl_\gamma, \mathbb{Q})^{W_{ex}} = (h+1)^{rk}$

(b) $\dim H^\bullet(Gr_{\gamma_\ell}^{\mathfrak{Z}}, \mathbb{Q})^{W_{\ell,ex}} = \frac{1}{\# W} \prod_{i=1}^{rk\, G} ((h+1)\ell - h + e_i)$

left action exponent of W

NB: In type $\neq A$ part (a) is due to Boixeda Alvarez - Losev

CONJ: We have a commutative diagram

$$\begin{tikzcd} H^\bullet(Gr^{\xi}_{\gamma_\ell}, \mathbb{Q})^{W_{\ell,ex}} \arrow[r, "\sim"] \arrow[d, hook] & Z(u_\xi)^{\check{G}} \arrow[d, hook] \\ H^\bullet(Gr^{\xi}_{\gamma_\ell}, \mathbb{Q})^{\ell\check{\Lambda}} \arrow[r, "\sim"] & Z(u_\xi)^{\check{T}} \end{tikzcd}$$

compatible with blocks decomposition.

The lower map is W-invariant

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THM:

(a) We have a commutative diagram as above

with injective horizontal maps. The lower map is

W-invariant

(b) In type A the upper map is invertible

NB: Restricting to principal block we get

$$\begin{tikzcd}
H^\bullet(\mathcal{F}\ell_\gamma, \mathbb{Q})^{W_{ex}} \arrow[r, hook, "A"] \arrow[d, hook] & Z(u_\zeta^\circ)^{\check{G}} \arrow[d, hook] \\
H^\bullet(\mathcal{F}\ell_\gamma, \mathbb{Q})^{\check{\Lambda}} \arrow[r, hook, "B"] & Z(u_\zeta^\circ)^{\check{T}}
\end{tikzcd}$$

In type A we get

$$\begin{tikzcd}
H^\bullet(\mathcal{F}\ell_\gamma, \mathbb{Q})^{W_{ex}} \arrow[r, equal, "A"] \arrow[d, equal] & Z(u_\zeta^\circ)^{\check{G}} \arrow[d, hook] \\
H^\bullet(\mathcal{F}\ell_\gamma, \mathbb{Q})^{\check{\Lambda}} \arrow[r, hook, "B"] & Z(u_\zeta^\circ)^{\check{T}}
\end{tikzcd}$$

Thus B surjective $\Rightarrow$ $Z(u_\zeta^\circ)^{\check{G}} = Z(u_\zeta^\circ)^{T}$

$$\overset{[BL]}{\Rightarrow} \quad Z(u_\zeta^\circ)^{\check{G}} = Z(u_\zeta^\circ)$$

$(Z(u_\zeta^\circ) = \check{G}$-module with trivial W-action on 0-weight space$)$

NB: Compare with the Soergel Theorem:

$\mathcal{O}(\mathfrak{g})$ = BGG category $\mathcal{O}$ of $\mathfrak{g}$

= finitely generated B-integrable $U(\mathfrak{g})$-modules

$Z(\mathcal{O}(\mathfrak{g})) = H^\bullet(G/B, \mathbb{Q})$

$= \mathbb{C}[t]/(\mathbb{C}[t]^W_+)$

Ⓓ Definition of the maps:

* Definition of B:

Take weights in $H_T^\bullet = \mathrm{Sym}(\mathfrak{t}^*)$

$Z(\check{T}u_\zeta^\circ, H_T^\bullet)$ = center of $H_T^\bullet$-deformed category of $\check{T}u_\zeta^\circ$-modules

GKM description of $H_T^\bullet(Fl_\gamma, \mathbb{Q})$

+

Similar description of $Z(\check{T}u_\zeta^\circ, H_T^\bullet)$ using localization of $H_T^\bullet$

$$\Rightarrow \quad H_T^\bullet(Fl_\gamma, \mathbb{Q}) = Z(\check{T}u_\zeta^\circ, H_T^\bullet)$$

Equivariant formality of $H_T^\bullet(Fl_\gamma, \mathbb{Q})$ gives a map

$$H^\bullet(Fl_\gamma, \mathbb{Q}) = H^\bullet_T(Fl_\gamma, \mathbb{Q}) \otimes_{H^\bullet_T} \mathbb{C}$$

$$= Z(\check{T}\, u^\circ_\xi, H^\bullet_T)$$

$$\subset Z(u^\circ_\xi)$$

* Definition of $\ddot{A}$:

* $Y = \mathrm{Spec}(A/I) \subset X = \mathrm{Spec}(A)$

$\tilde{N}_Y(X)$ = deformation to normal cone

$= \mathrm{Spec}(A[\hbar, \hbar^{-1} I])$ (Rees algebra)

$N_Y(X)$ = normal cone of Y into X

$$= \mathrm{Spec}\left(\bigoplus_{n \geq 0} I^n / I^{n+1}\right)$$

$$\tilde{N}_Y(X) \longrightarrow \mathbb{C} = \mathrm{Spec}(\mathbb{C}[\hbar])$$

$N_Y(X)$ = 0-fiber

$* \ \mathcal{L}(\lambda) \in \mathrm{Pic}(\mathcal{F}\ell)$, $\mathcal{L}(\lambda) = G \times_B \mathbb{C}_\lambda$, $\lambda \in \Lambda$

T-equivariance $\qquad$ cup product by $c_1(\mathcal{L}(\lambda))$'s

$$\begin{tikzcd} H_T^\bullet \otimes H_T^\bullet \arrow[r] \arrow[d, equal] & H_T^\bullet(\mathcal{F}\ell, \mathbb{Q}) \arrow[d, equal] \\ \mathbb{C}[\mathfrak{t} \times \mathfrak{t}] \arrow[r] & \mathbb{C}[N_\Delta(\mathfrak{t} \times \mathfrak{t})] \end{tikzcd}$$

Borel construction

$$\Delta = \mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t} \subset \mathfrak{t} \times \mathfrak{t}$$

Ω = fiber at $1 \in T/W$ of the map

$$T/W \to T/W, \quad W.t \mapsto W.t^\ell$$

$$\Rightarrow H^\bullet(Gr^3, \mathbb{Q}) = \mathbb{C}[N_\Omega(T/W)]$$

* U_q = Lusztig quantum group = $\mathbb{C}[q,q^{-1}]$-algebra

$U_\zeta = U_q / (q-\zeta)$

$U_{\hat\zeta}$ = completion at ζ = $\mathbb{C}[\![\hbar]\!]$-algebra, $\hbar = q-\zeta$

LEM: The Harish Chandra isomorphism gives maps

$$\begin{tikzcd} \mathbb{C}[\![\hbar]\!][T/W] \arrow[d] \arrow[dr, "HC"] & \\ \mathbb{C}[\widetilde{N}_\Omega(T/W)]\widehat{\ }_{\hbar=0} \arrow[r, "(*)"'] & Z(U_{\hat\zeta}) \end{tikzcd}$$

Proof uses De Concini-Kac's Theorem on $Z(\mathcal{U}_\zeta)$

and the map $\mathcal{U}_{\hat\zeta} \longrightarrow U_{\hat\zeta}$

The fiber at $\hbar=0$ of $(*)$ gives a map

$$H^\bullet(Gr^\zeta, \mathbb{Q}) \longrightarrow Z(U_\zeta)$$

PROP: This map factors through

$$\begin{tikzcd}
H^\bullet(Gr^{\mathfrak{Z}}, \mathbb{Q}) \arrow[rr] \arrow[dr, "res"'] & & Z(u_{\mathfrak{Z}})^{\check{G}} \subset Z(U_{\mathfrak{Z}}) \\
& H^\bullet(Gr^{\mathfrak{Z}}_{\gamma_\ell}, \mathbb{Q})^{W_{\ell,ex}} \arrow[ur, "A"'] &
\end{tikzcd}$$

Proof uses GKM description of $H^\bullet(Gr^{\mathfrak{Z}}_{\gamma_\ell}, \mathbb{Q})$