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Time-evolution on the information lattice
&
Ultraslow growth of number entropy in MBL

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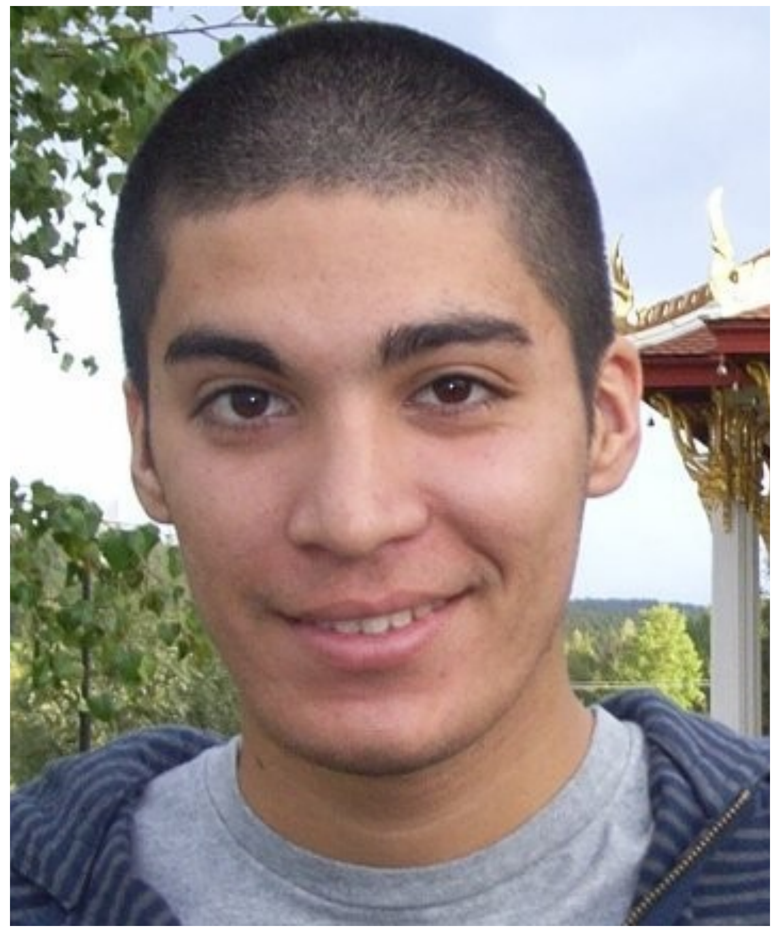

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SciPost Physics **13**, 080 (2022); arXiv:2310.06036; arXiv:2312.13420

Thermalization, entanglement and the ETH

At long times

Quench

$$|\psi_0\rangle \to |\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$$

Volume law entanglement

Local observables thermal

&

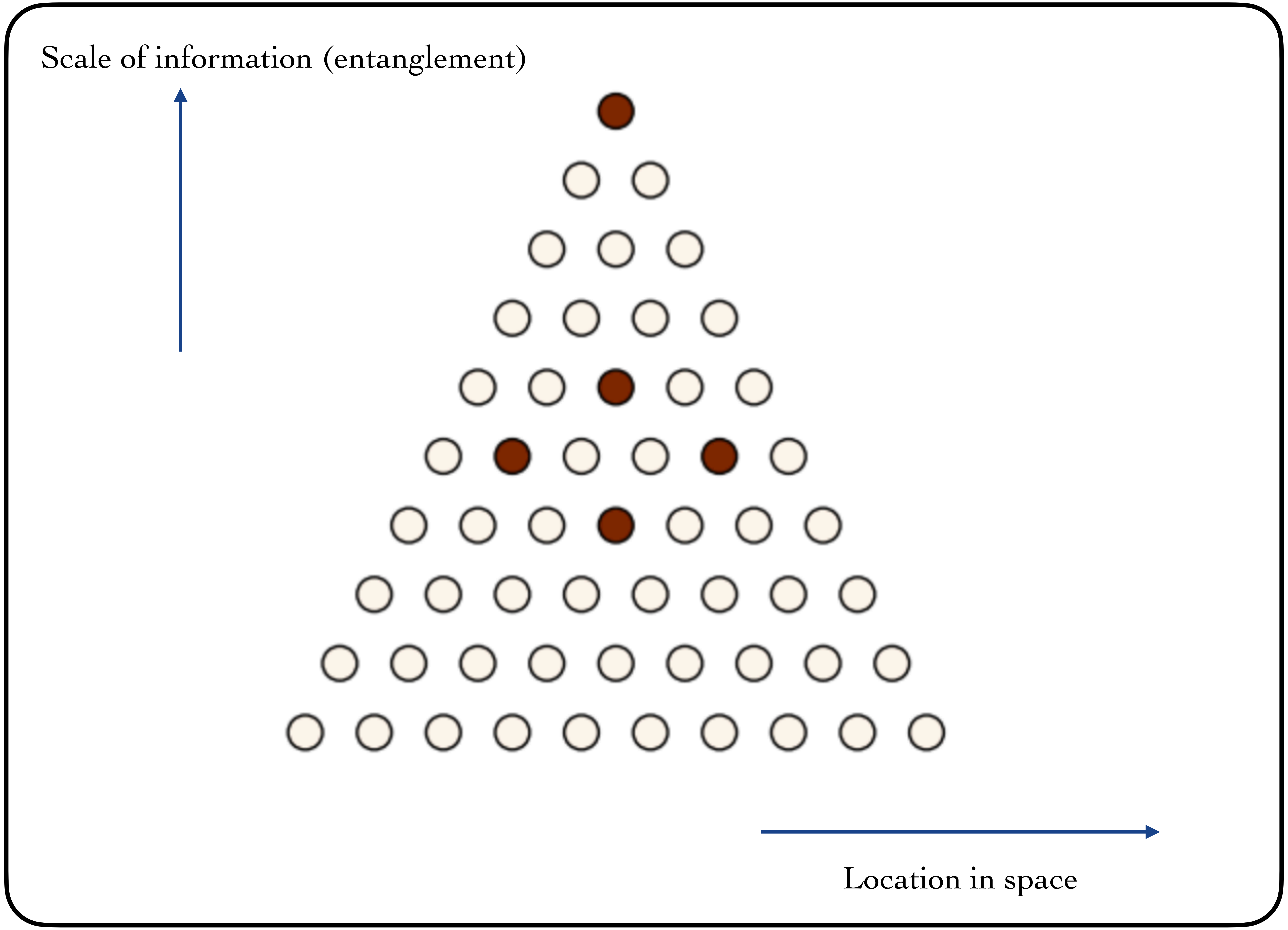
smooth functions of energy

} Details of long range entanglement don't matter (but it matters that it is there)

Motivation/Problem: How to simulate thermalizing dynamics to long times?

The information lattice – organization of entanglement at various scales

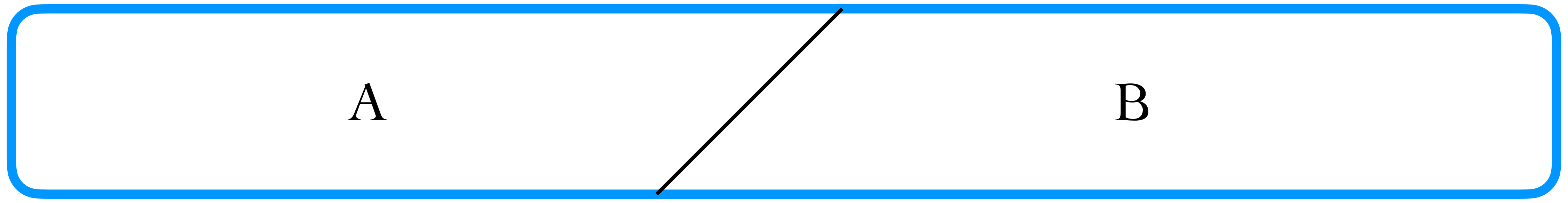
$|\psi\rangle \longrightarrow$



What is entanglement? And how is it related to information?

"Entanglement is […] the characteristic trait of quantum mechanics" — Schrödinger 1935

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



$$\left.\begin{array}{l} \rho \in \mathcal{H} \\ \rho^2 = \rho \end{array}\right\} \quad \rho \neq \rho_A \otimes \rho_B \quad \text{(in any basis)} \quad \Rightarrow \quad \rho \quad \text{has entanglement between A and B}$$

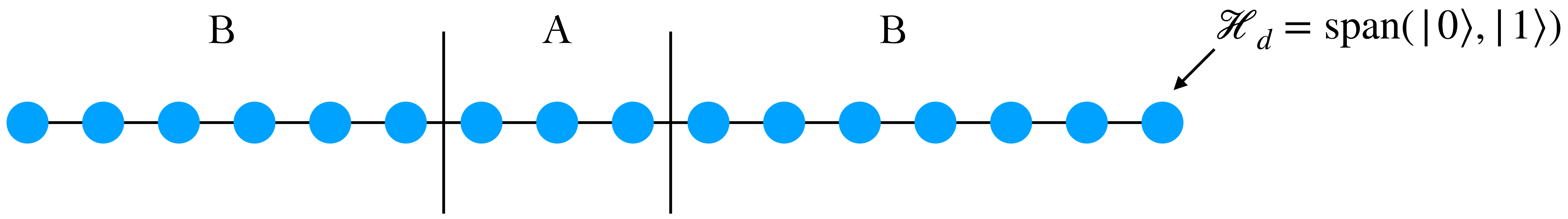
Information of state accessible in A is contained in $\rho_A = \mathrm{Tr}_B\, \rho$ via $\langle O_A \rangle = \mathrm{Tr}_A(\rho_A O_A)$

If ρ entangled, then all information in the state can **not** be accessed locally in A and B!

"Entanglement is nonlocal information"

"Quantum correlations are locally inexplicable" Bell '81

So, how much information is accessible locally?



If $\rho_A^2 = \rho_A$ then in principle we have access to l_A bits of information

If $\rho_A \propto \mathbf{1}_2$ then in principle we have access to 0 bits of information

In between, the accessible information is the Von Neumann (Shannon) information

$$I(\rho_A) = \log_2 2^{l_A} - S(\rho_A) = l_A - S(\rho_A)$$

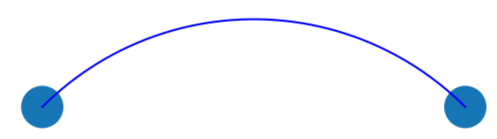
$$S = -\,\mathrm{Tr}_A(\rho_A \log_2 \rho_A)$$

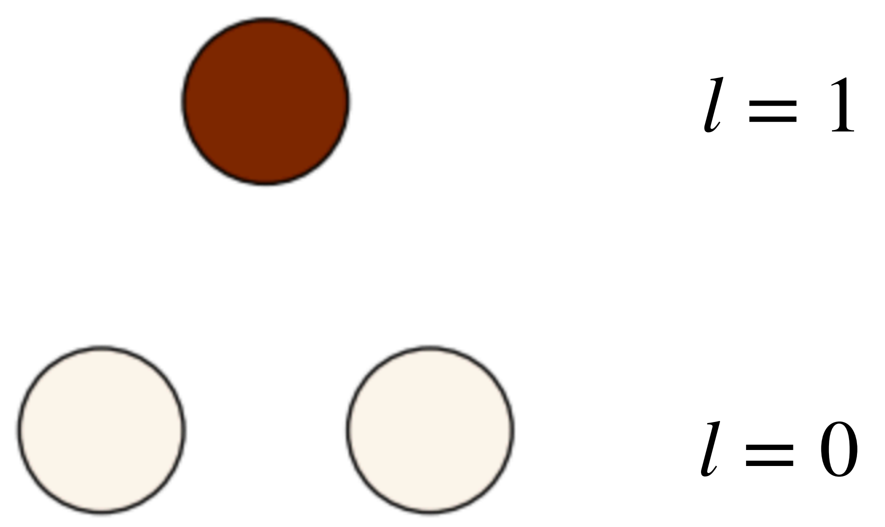
Q: what is the distribution of information in a given state?

Example: information in a singlet

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$\rho = \frac{1}{4}\mathbf{1}_4$$





There is **no** information in the single sites!

$$\rho_A = \frac{1}{2}\mathbf{1}_2 = \rho_B$$

$$S_A = \log_2 2 = 1$$

$$I_A = 0 = I_B$$

Information on single site says nothing about information on two sites

$$\rho_A = \frac{1}{2}\mathbf{1}_2 = \rho_B$$

$$S_A = \log_2 2 = 1 \qquad I_A = 0 = I_B$$

All information is on the two sites together

$$S_{A\cup B} = 0$$

$$I_{A\cup B} = \log_2 2^2 = 2$$

Fully mixed two site state has no information

$$S_{A\cup B} = \log_2 2^2 = 2$$

$$I_{A\cup B} = 0$$

(Mutual) information on scale $l = 1$

$$I_{l=1} = I_{A\cup B} - I_A - I_B$$

Example: information in the Greenberger-Horne-Zeilinger state

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

There is **no** information in the single sites!

$$\rho_1 = \rho_2 = \rho_3 = \frac{1}{2}\mathbf{1}_2$$

$$S_1 = \log_2 2 = 1$$

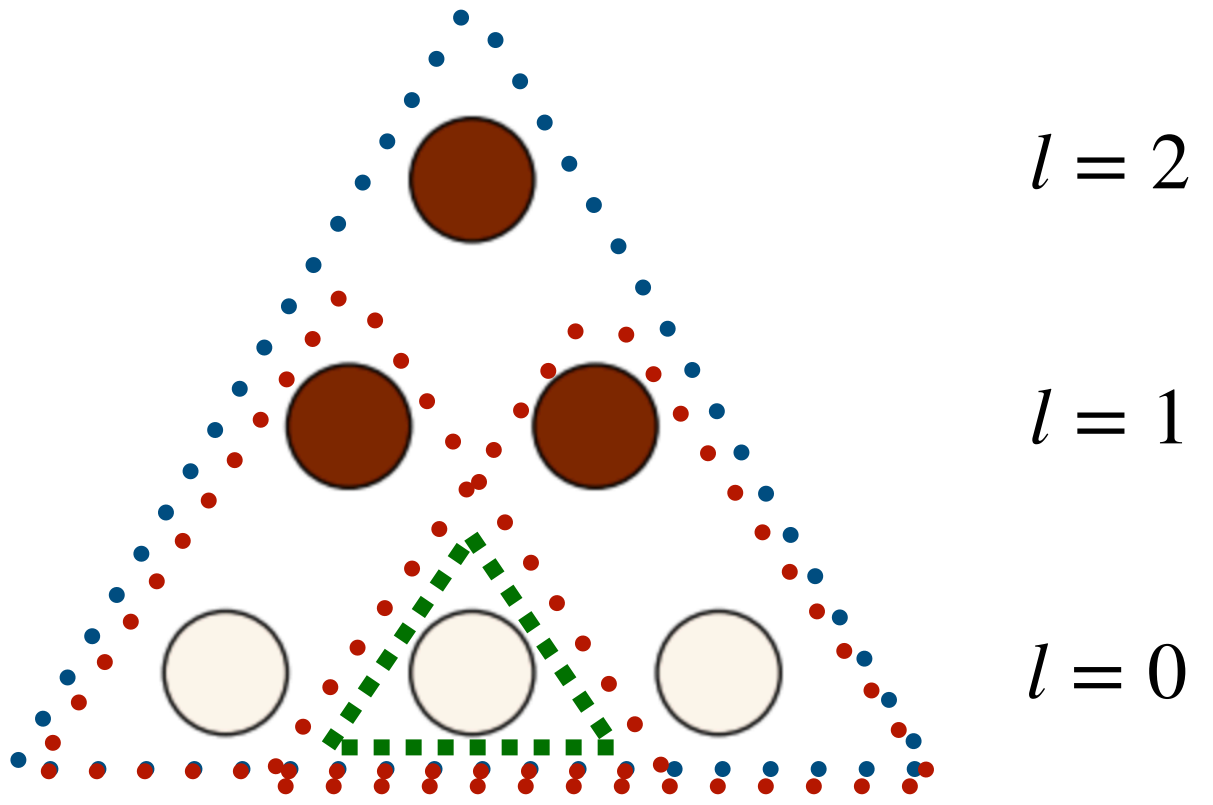
$$I_1 = I_2 = I_3 = 0$$

There **is** information on three sites since

$$\rho_{12}^2 \neq \rho_{12}$$

$$S_{123} = 0$$

$$I_{123} = 3 - I_{12} - I_{23} = 1$$



There is **some** information in two sites

$$\rho_{12} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$S_{12} = \log_2 2 = 1$$

$$I_{12} = 2 - S_{12} = 1$$

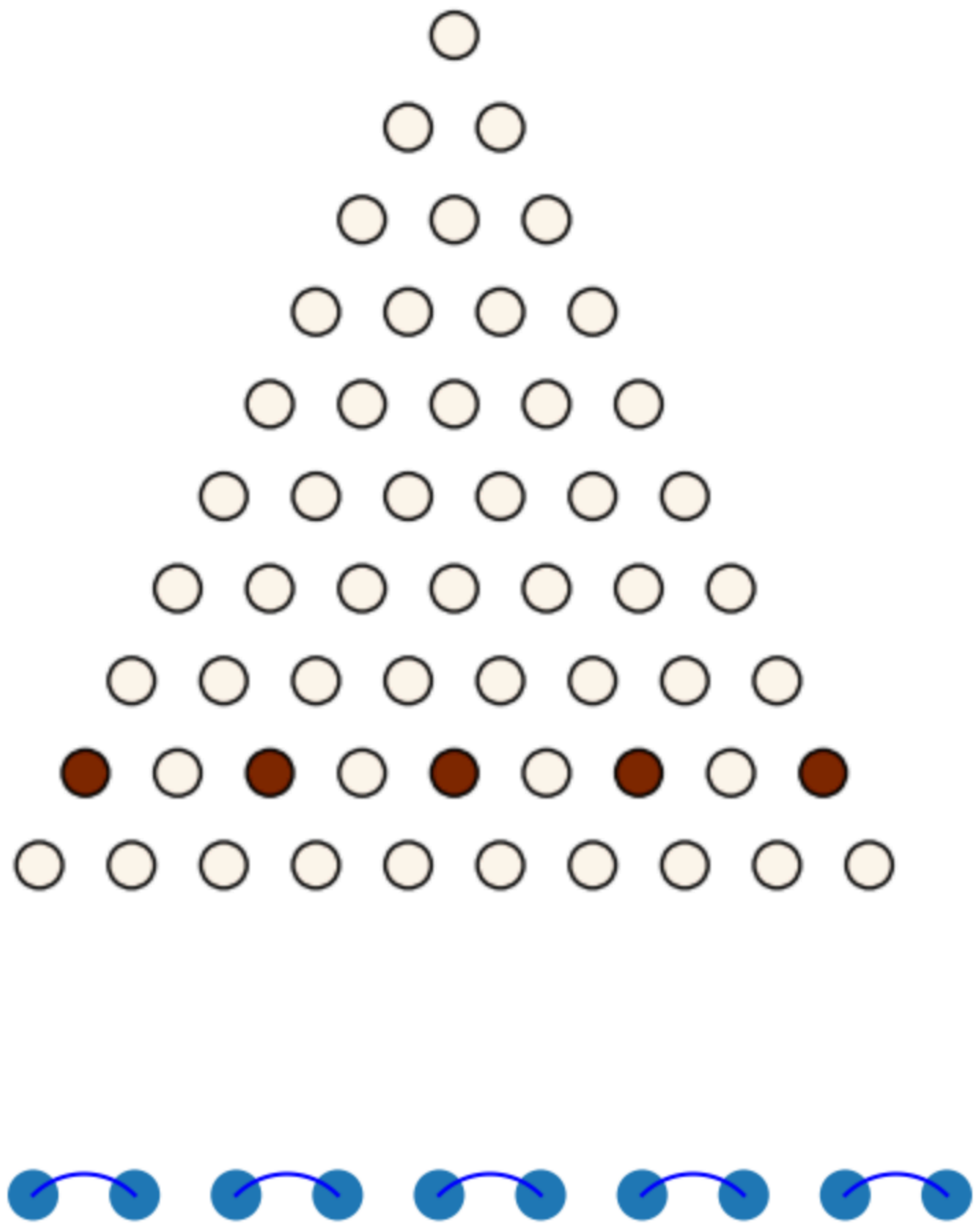
More generally information on scale 3

$$I_{l=3} = I_{123} - I_{12} - I_{23} + I_2$$

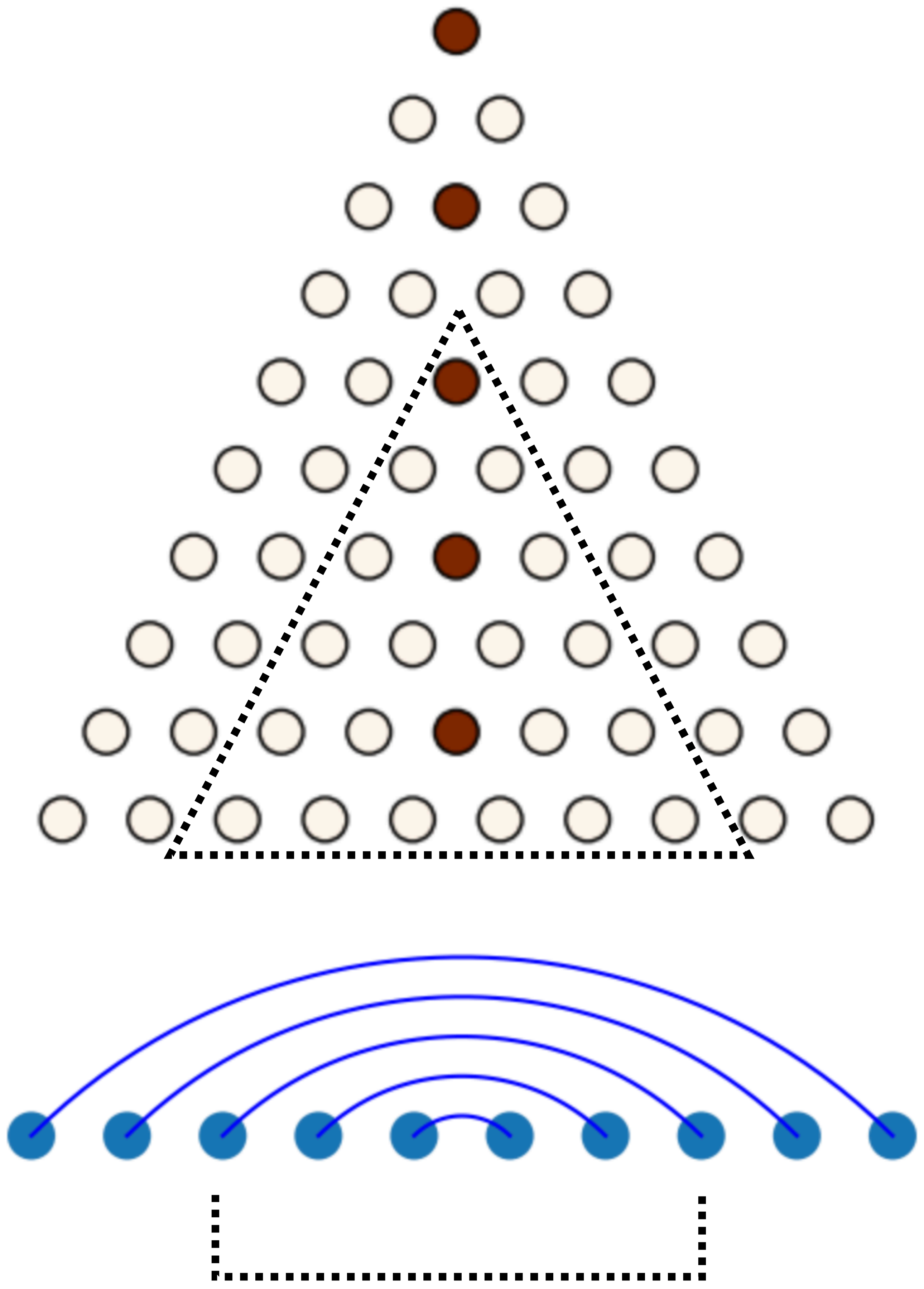
$$I_l = I_{A\cup B} - I_A - I_B + I_{A\cap B}$$

The information lattice for larger singlet states

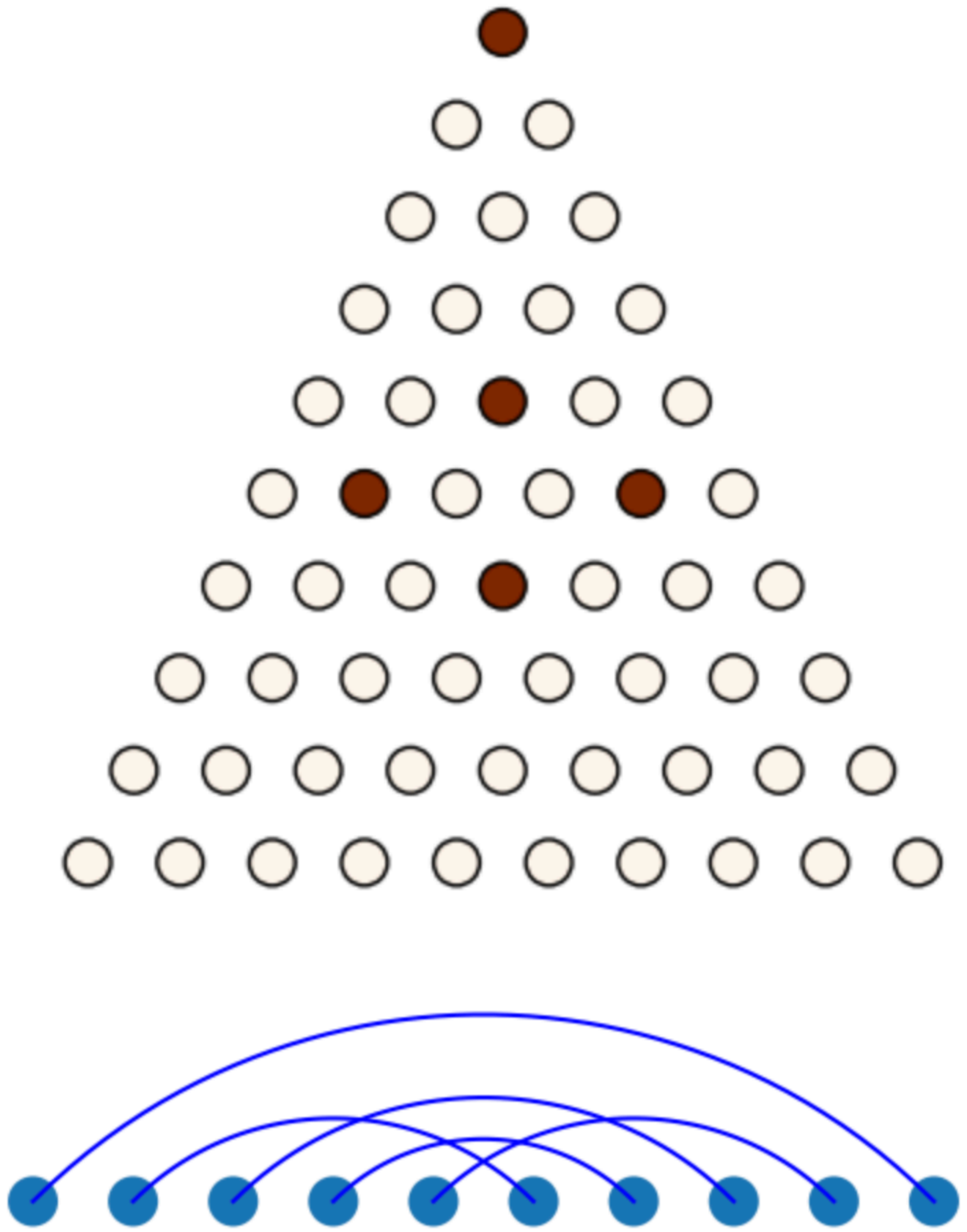
Product state of neighbouring singlets



Rainbow scar state



Random singlet

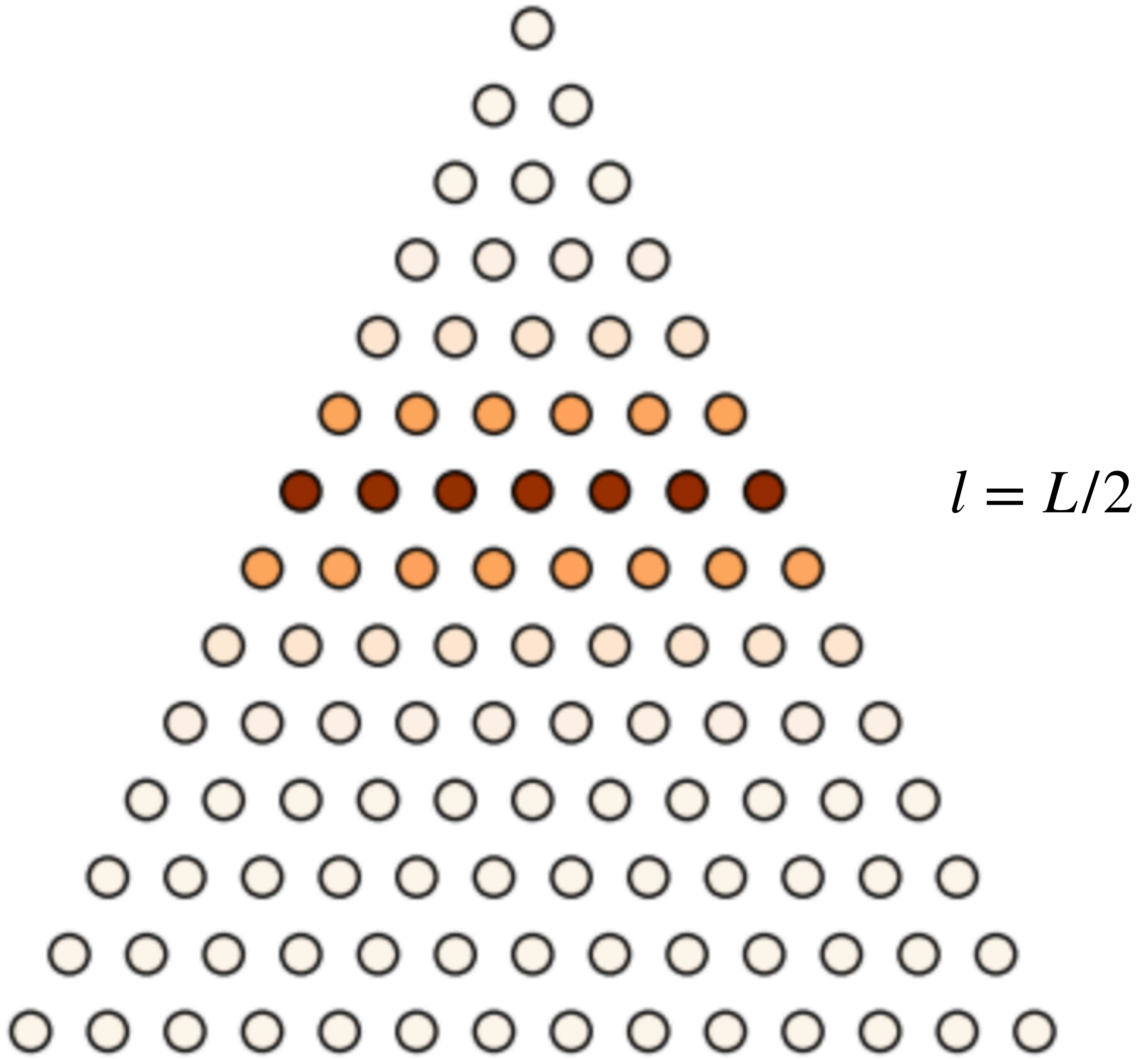


A random state in Hilbert space has maximum entropy at system size scale

$$|\mathrm{RMT}\rangle = \sum_{\boldsymbol{\sigma}} \psi_{\sigma_1,\ldots,\sigma_L} |\boldsymbol{\sigma}\rangle$$

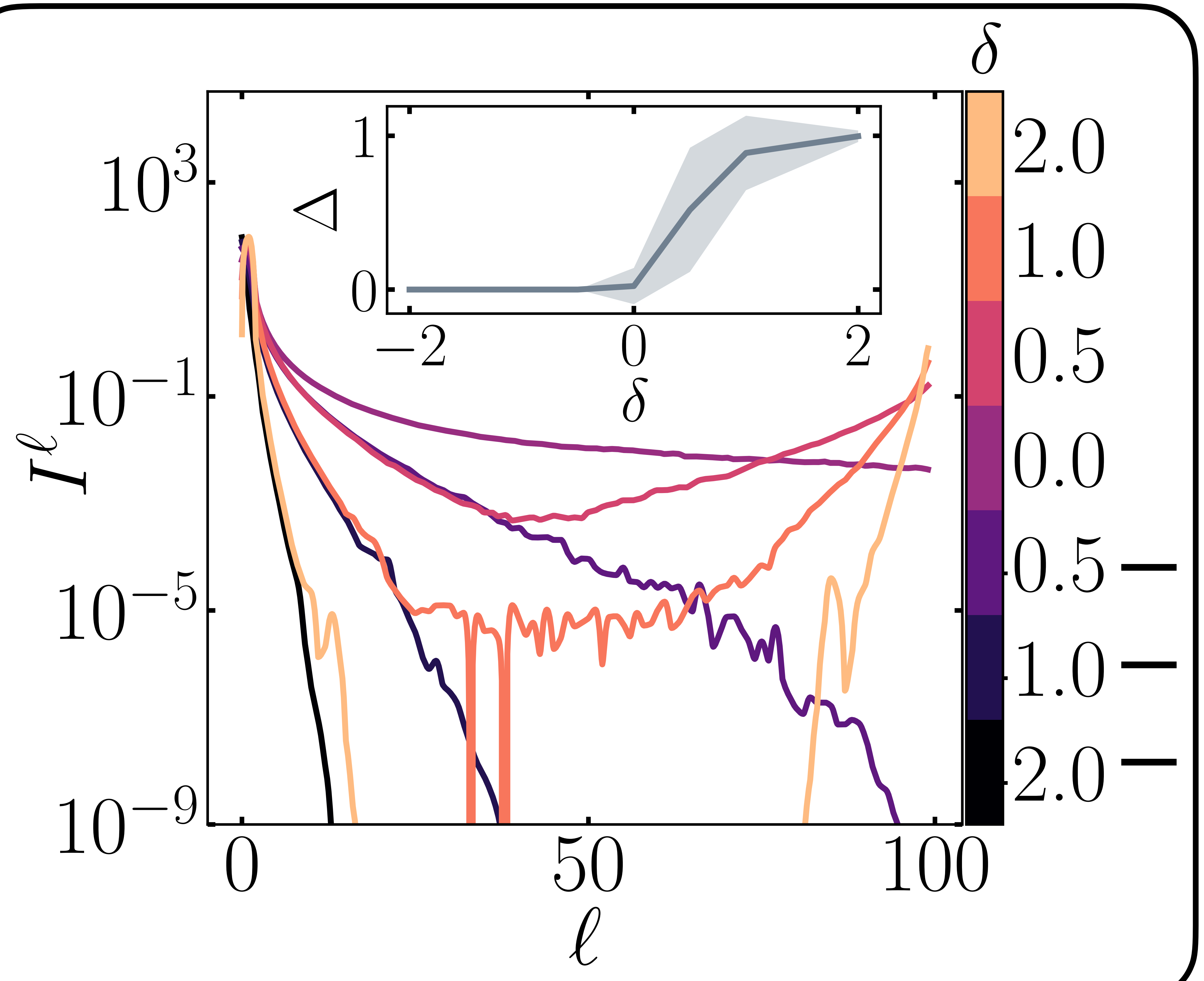
$$P(\psi_{\boldsymbol{\sigma}}) = \delta\left(\sum_{\boldsymbol{\sigma}} |\psi_{\boldsymbol{\sigma}}|^2 - 1\right)$$

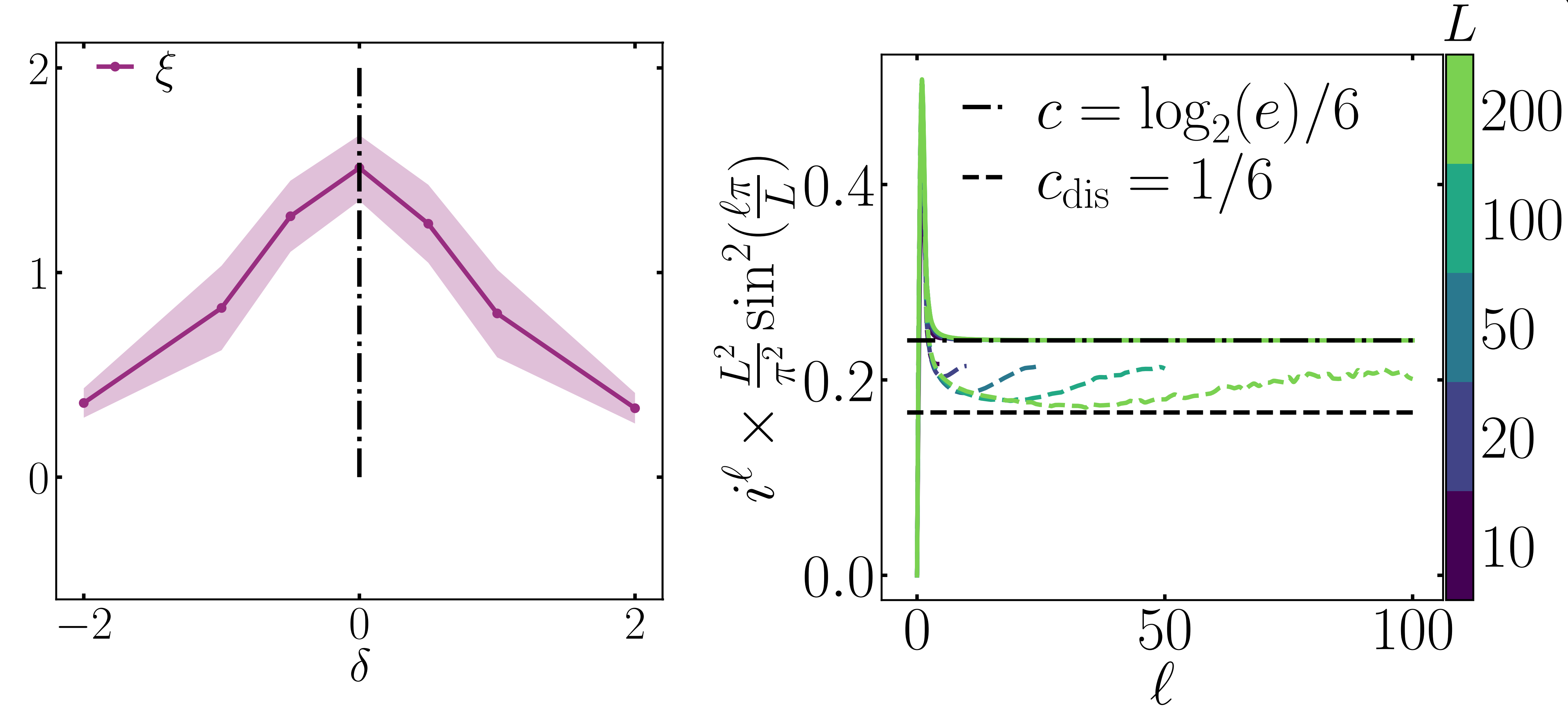
$$S_{\mathrm{Page}} = l_A - \frac{1}{2}\frac{2^{l_A}}{2^{L-l_A}}$$



Equivalent to infinite temperature thermal state according to the eigenstate thermalisation hypothesis

Localisation on the information lattice – topological superconductor





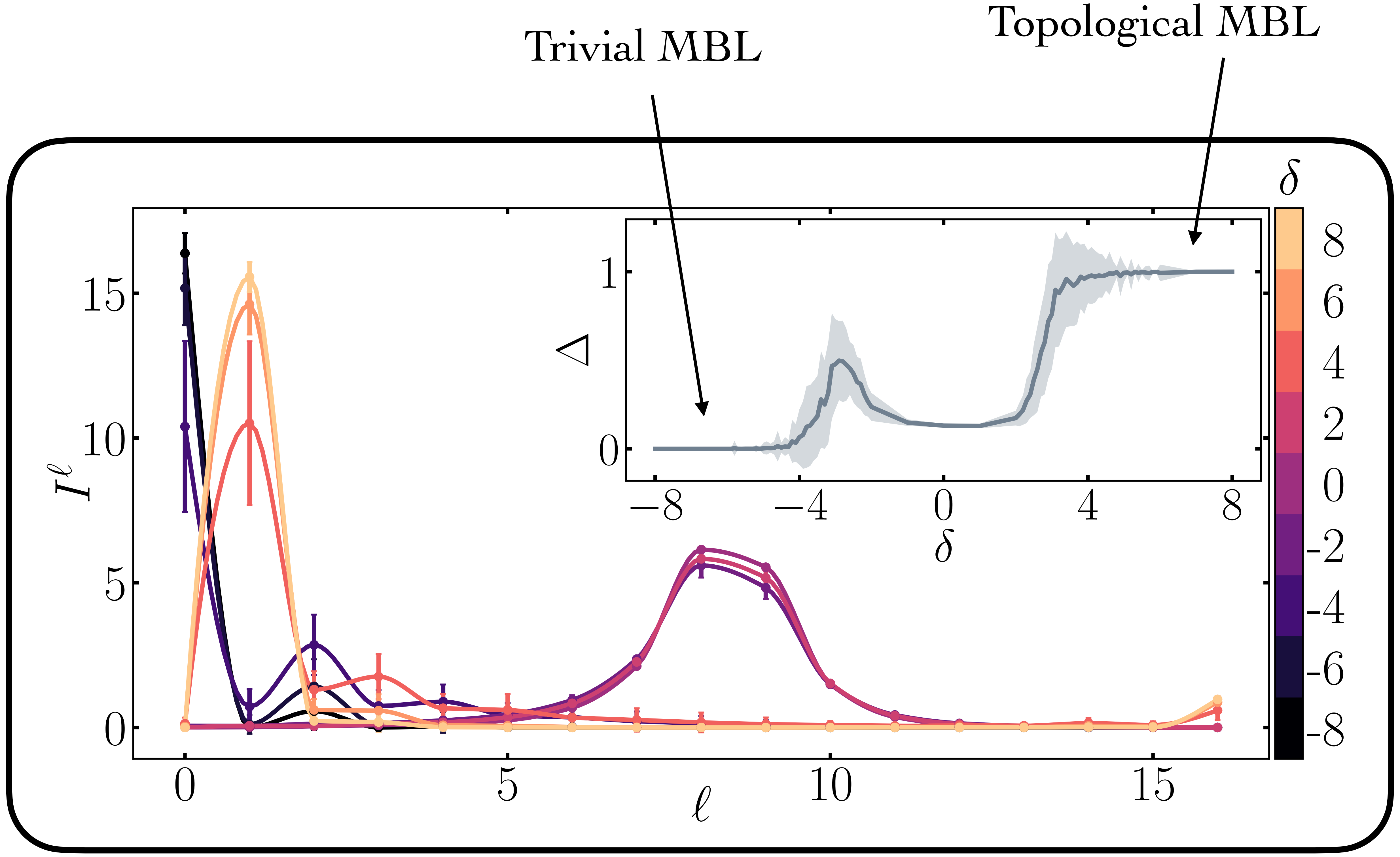
δ Disorder strength

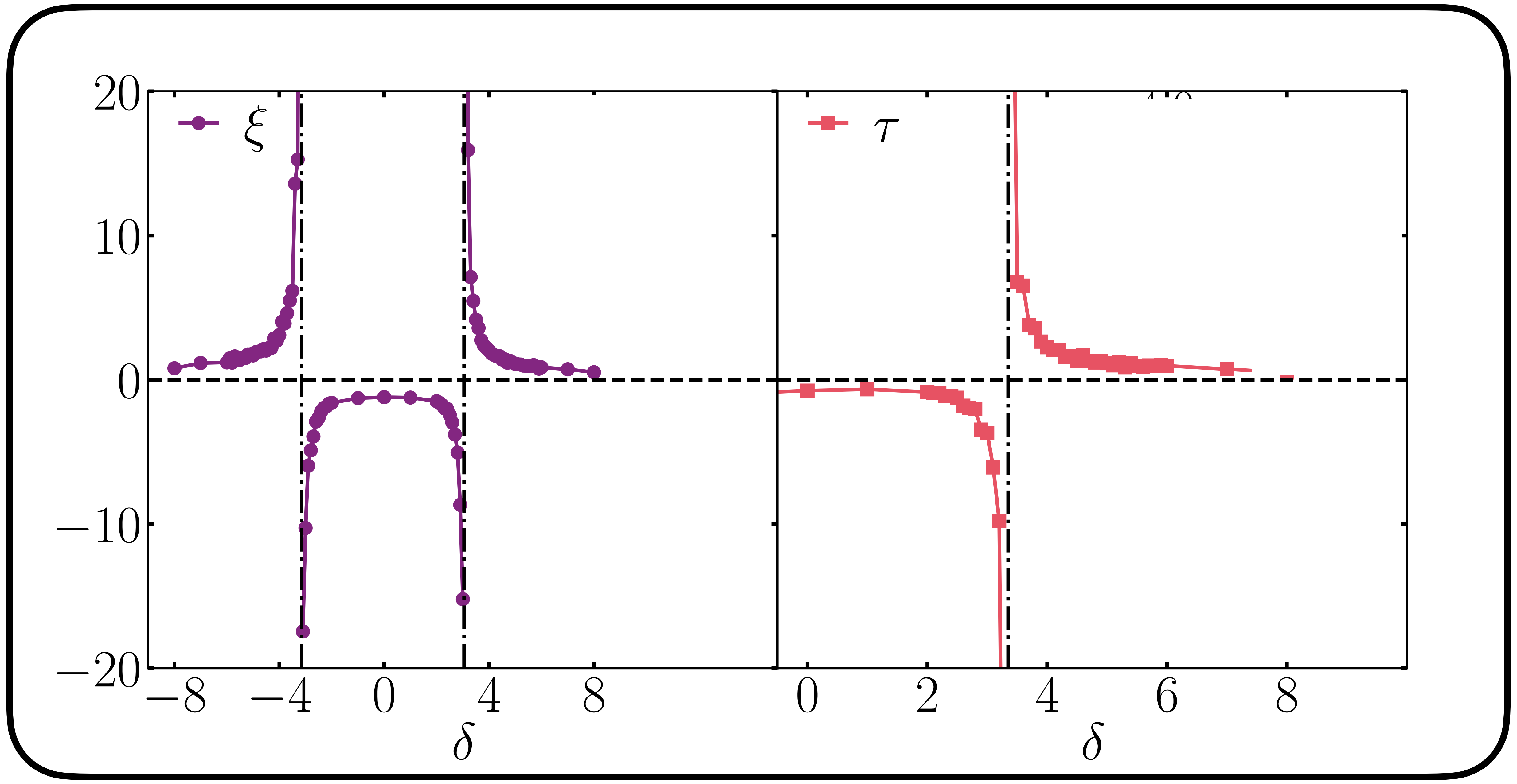
Δ Local topological marker
(see Hannukainen, Martinez, JHB, Klein-Kvorning PRL '23)

[Artiaco, Aceituno, Klein-Kvorning, JHB '24]

Localisation on the information lattice – Many-body localisation (Ising Z_2)

$$H_{\mathrm{IM}} = \sum_i J_i \sigma_i^x \sigma_{i+1}^x + h_i \sigma_i^z + g(\sigma_i^z \sigma_{i+1}^z + \sigma_i^x \sigma_{i+1}^x)$$





δ Disorder strength $\delta = \overline{\ln J / \ln h}$

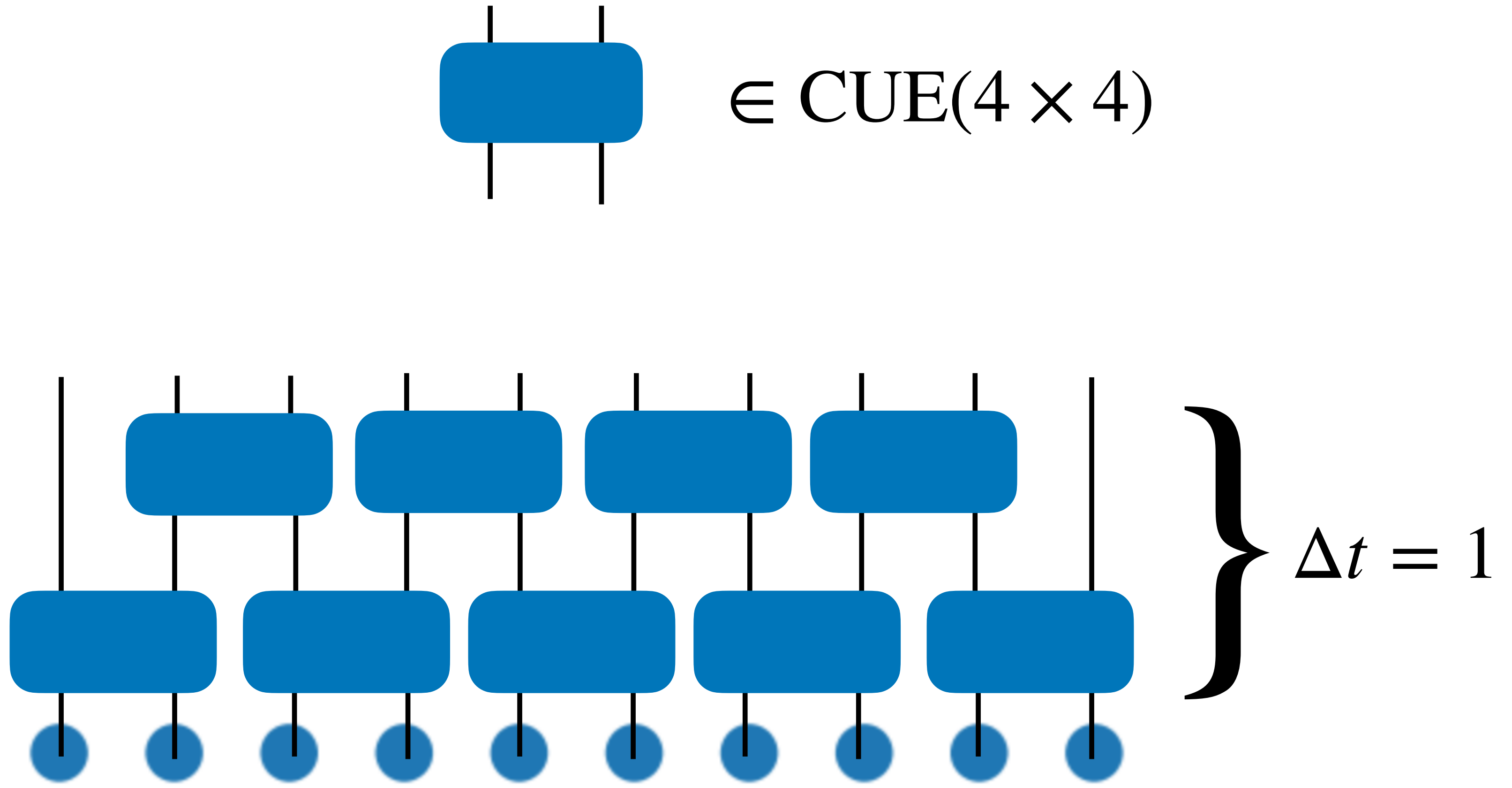
Δ Local topological marker

(see Hannukainen, Martinez, JHB, Klein-Kvorning PRL '23; arXiv:2307.06447)

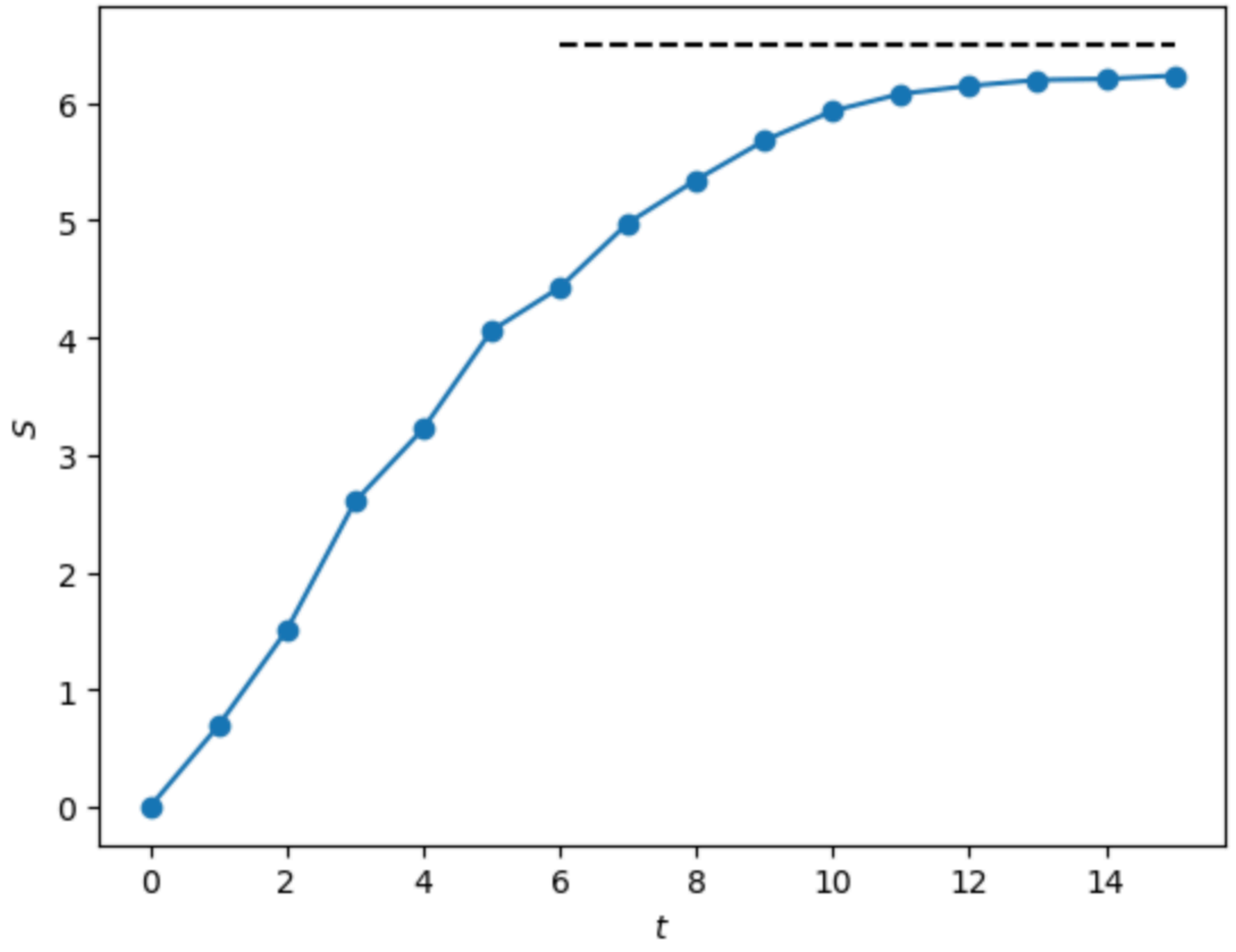
[Artiaco, Aceituno, Klein-Kvorning, JHB '24]

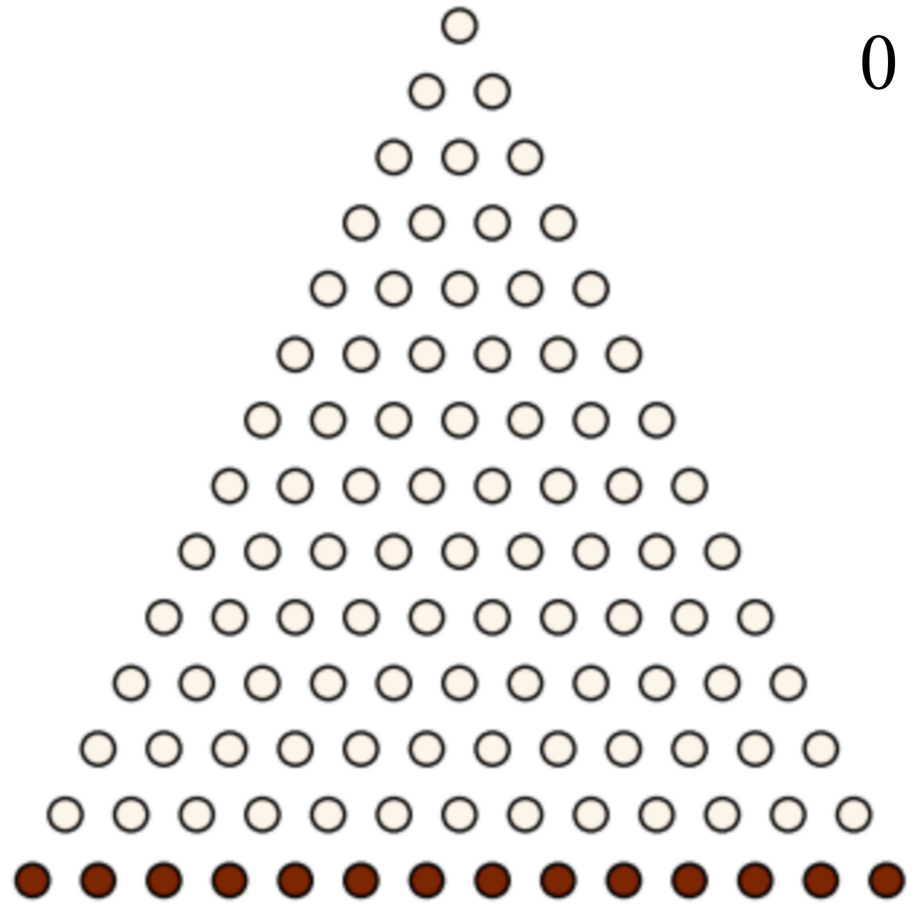
Dynamics on the information lattice – ballistic growth of entanglement in a random unitary circuit

Random brickwork of independent random unitaries



Entanglement entropy grows ballistically before saturating at the Page value





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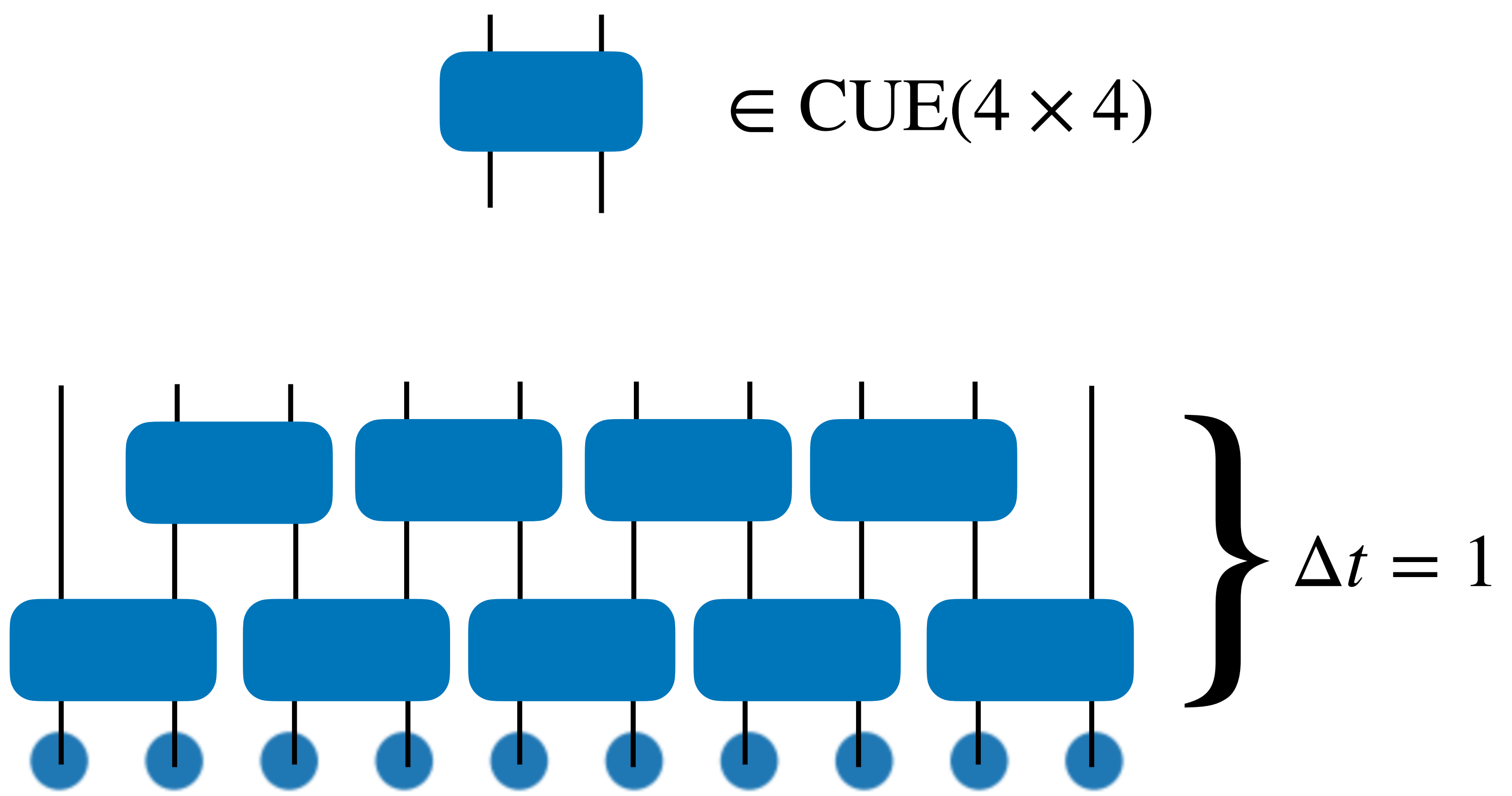

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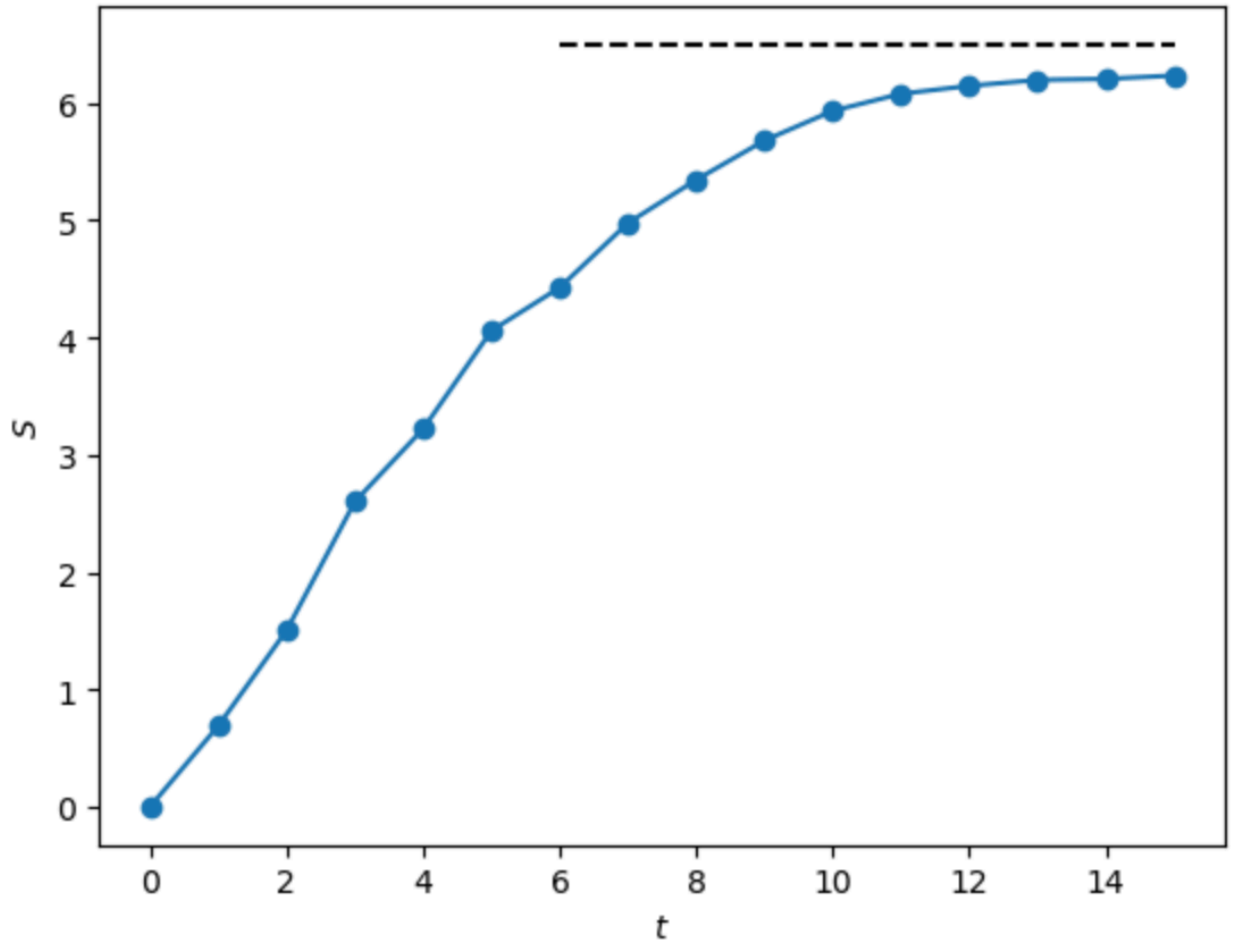

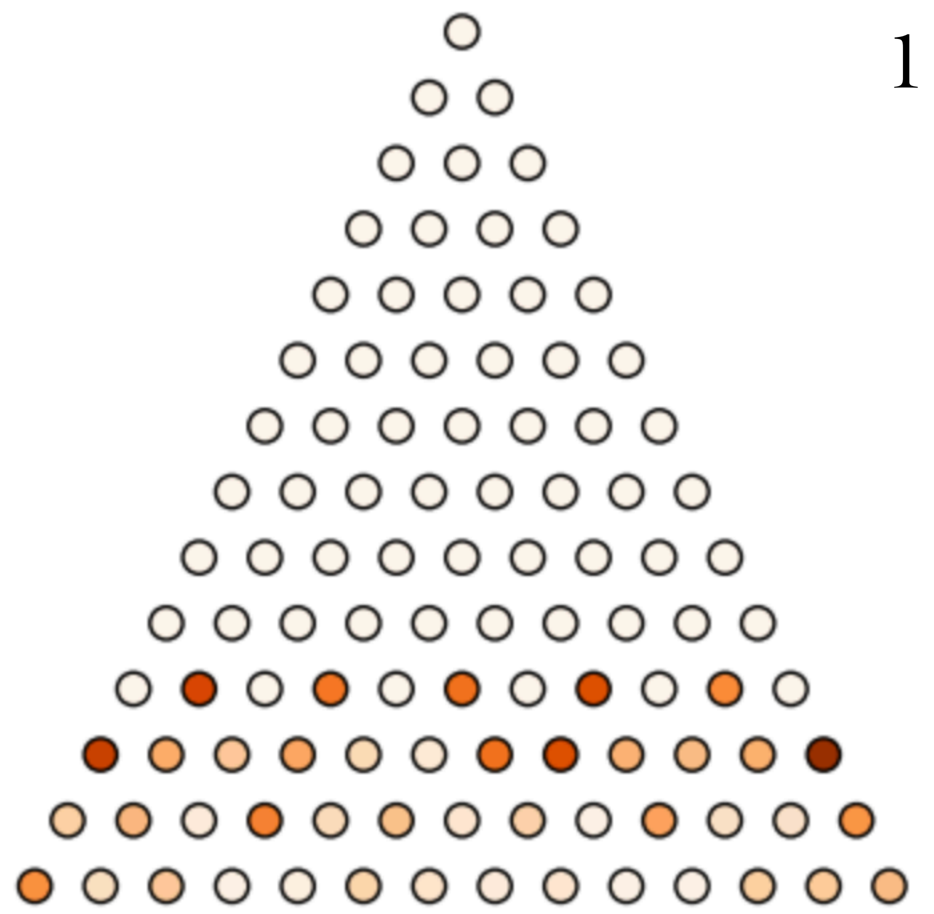

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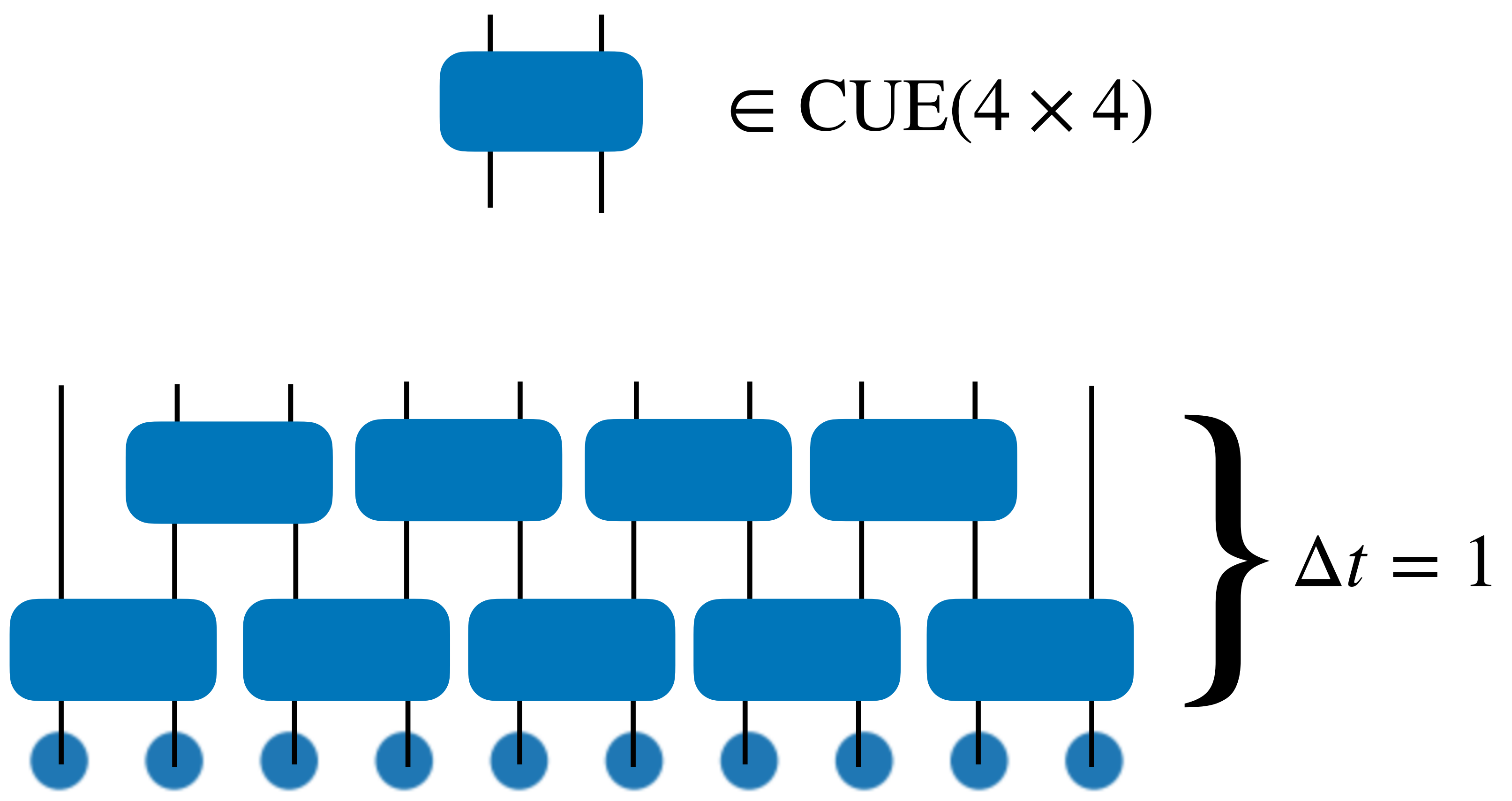

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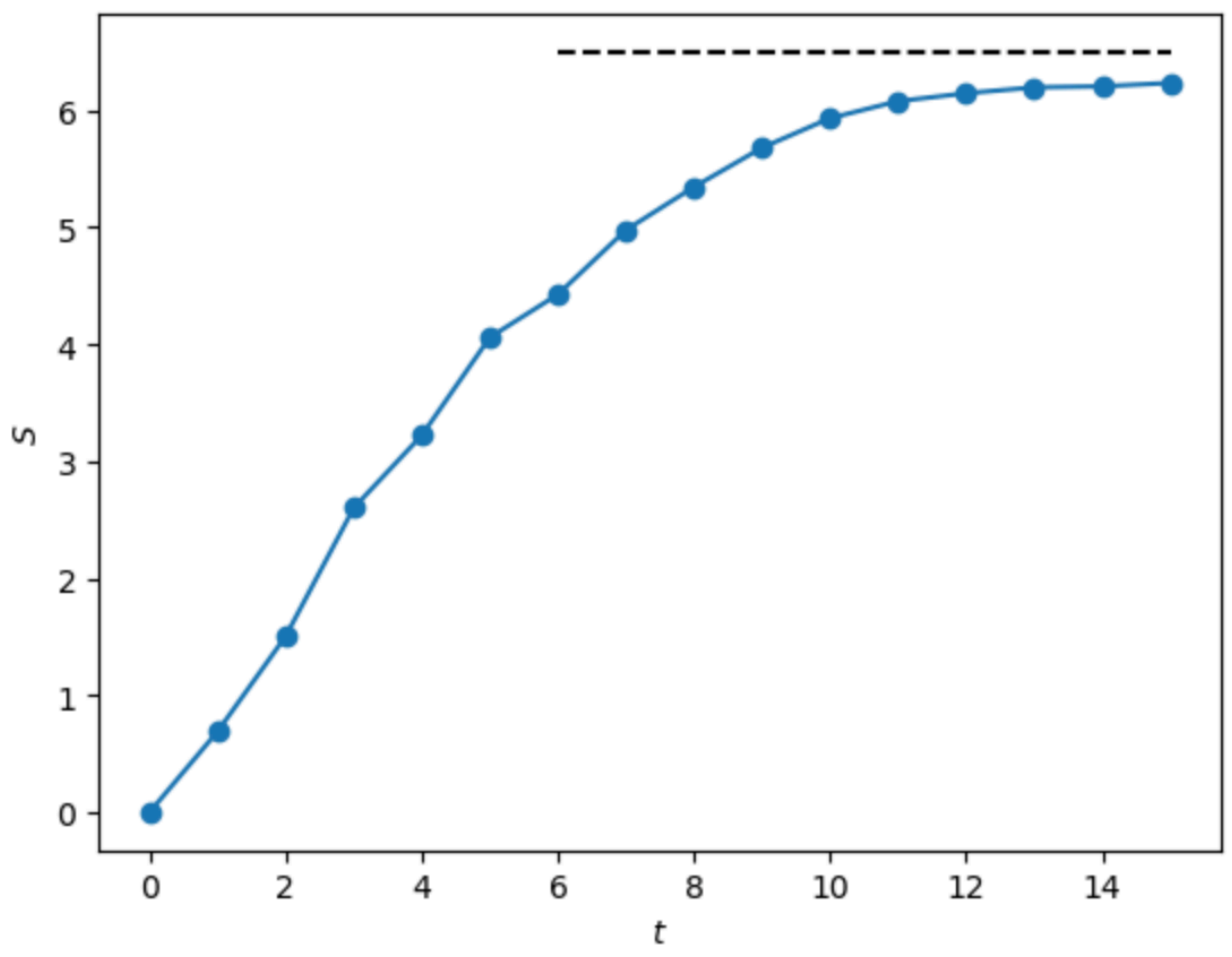

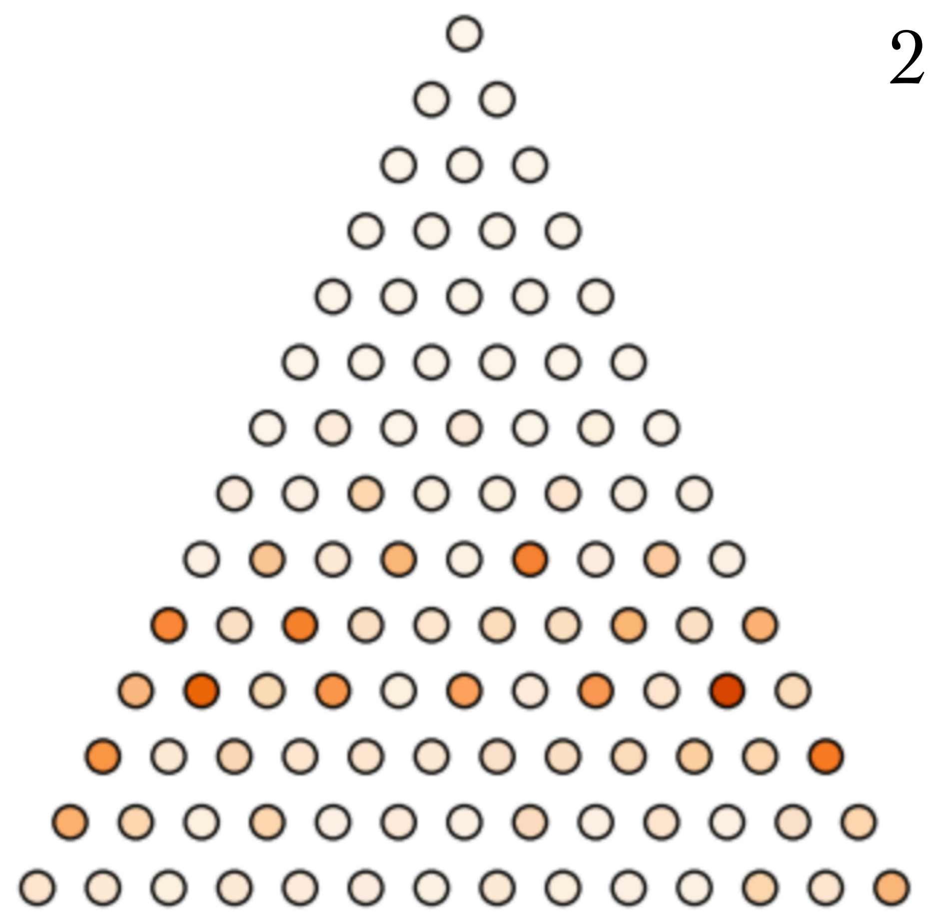

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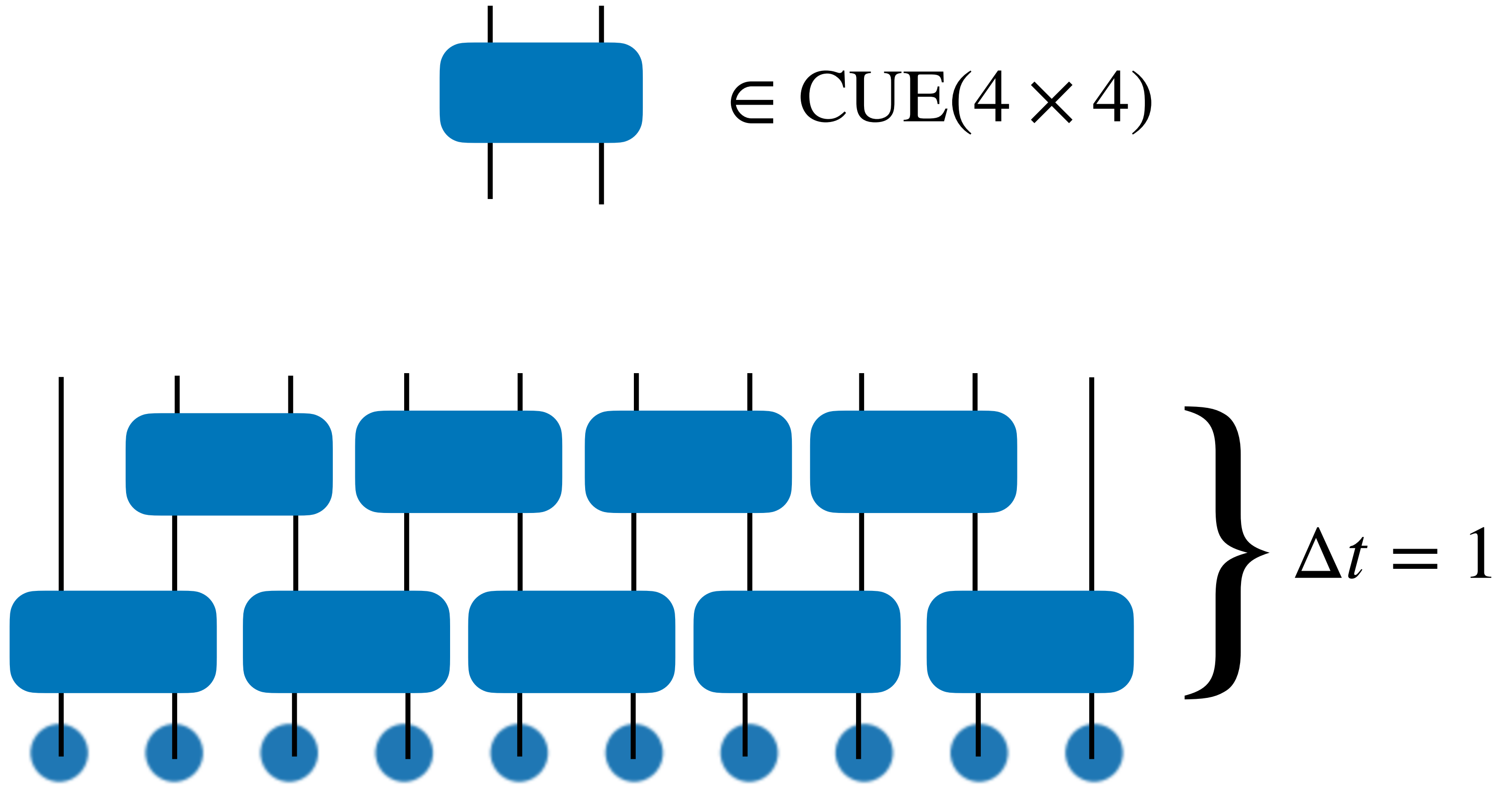

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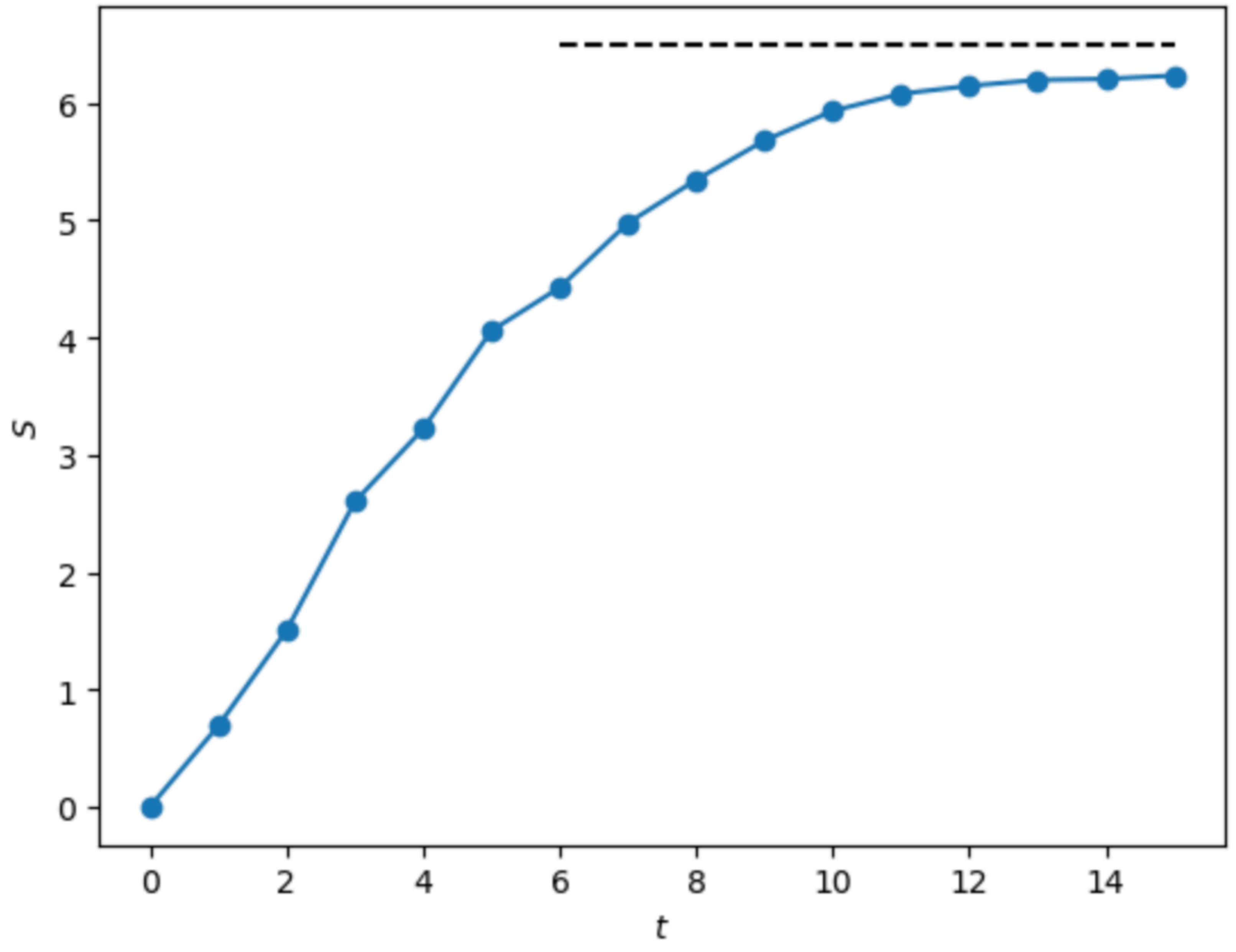

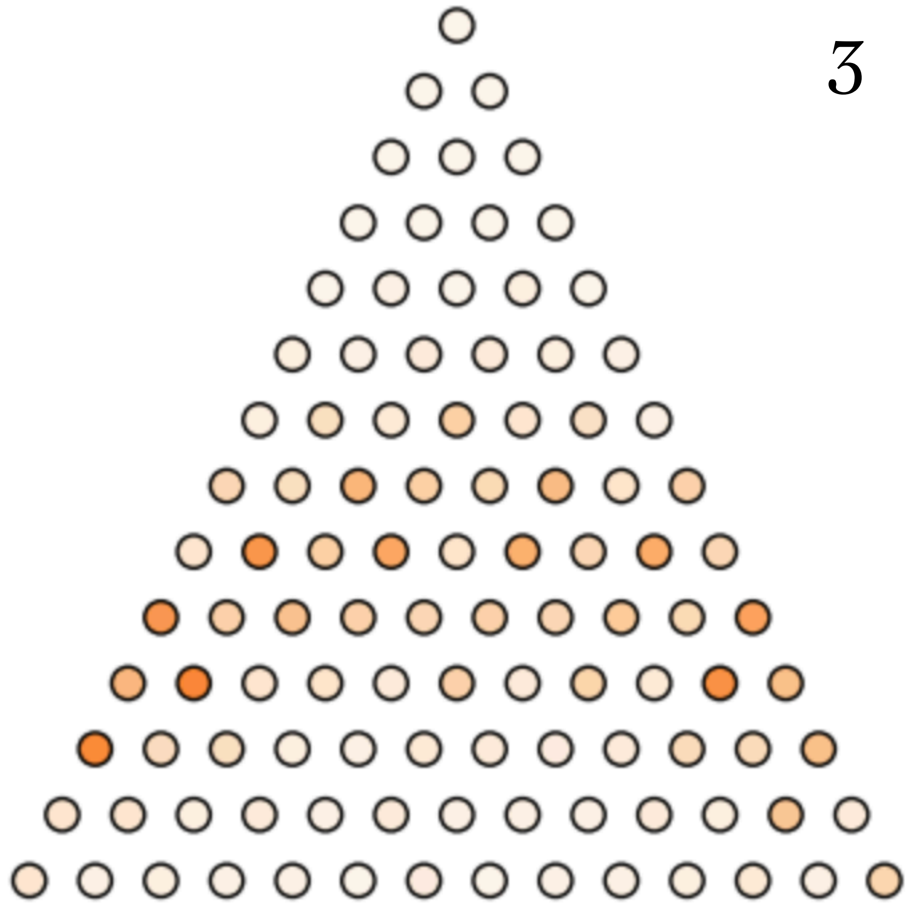

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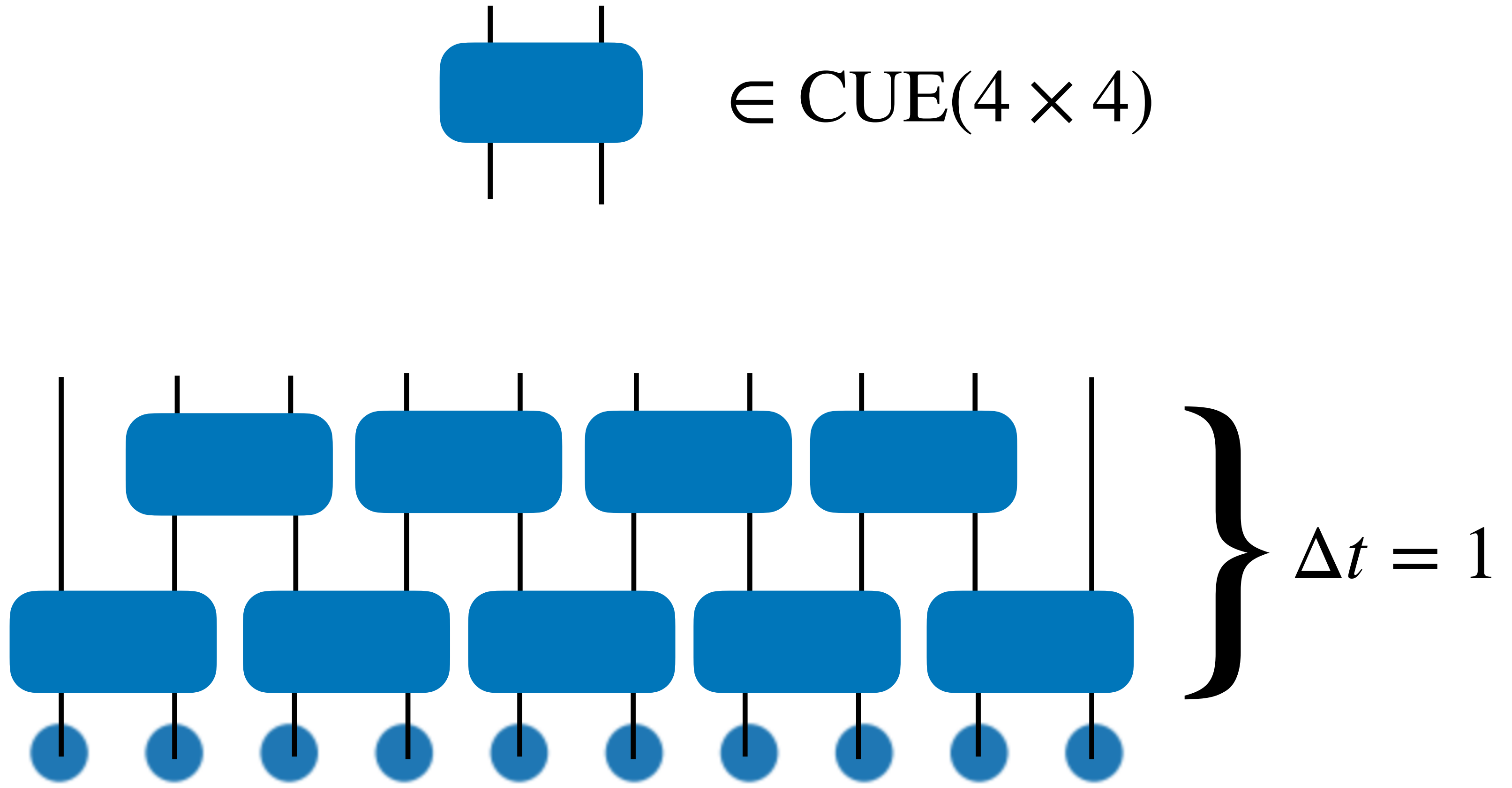

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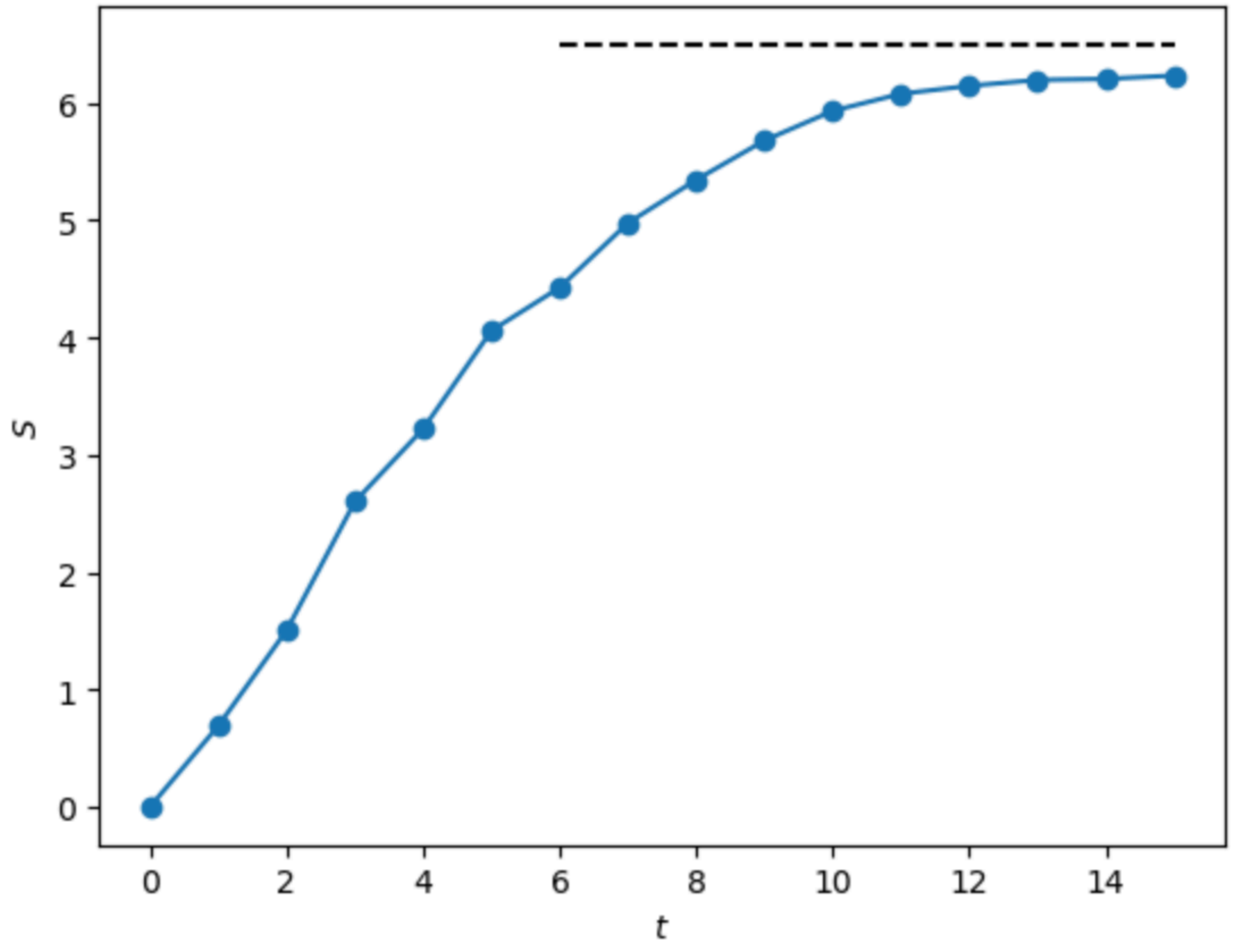
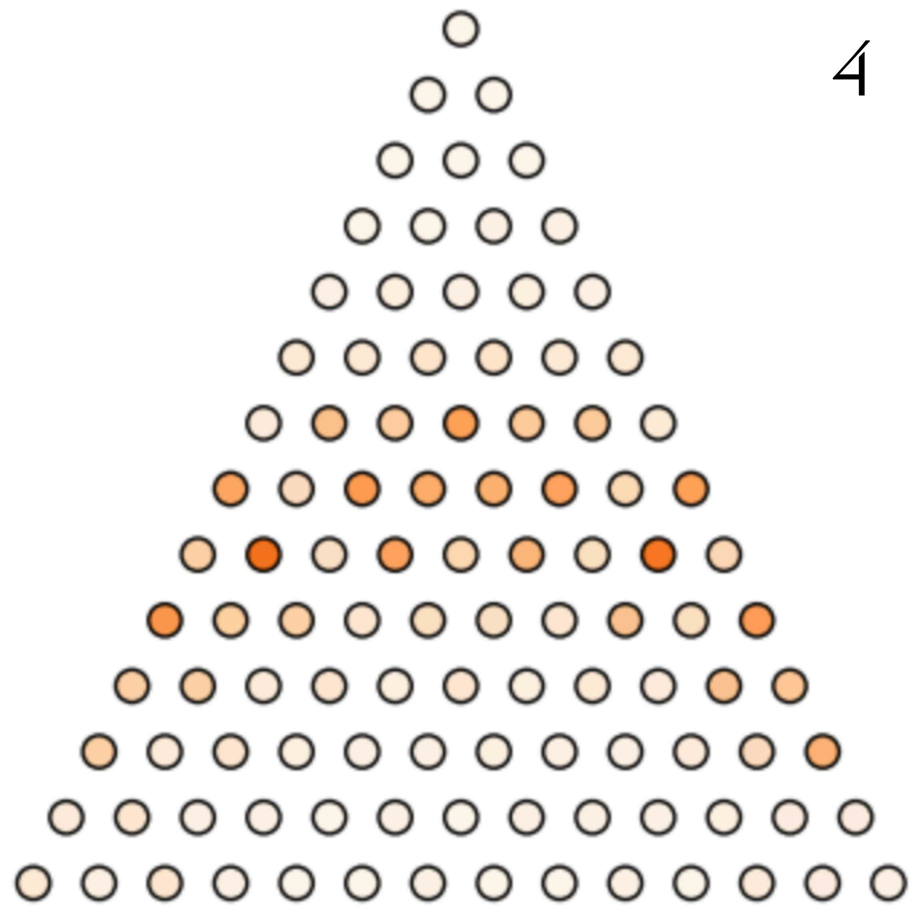
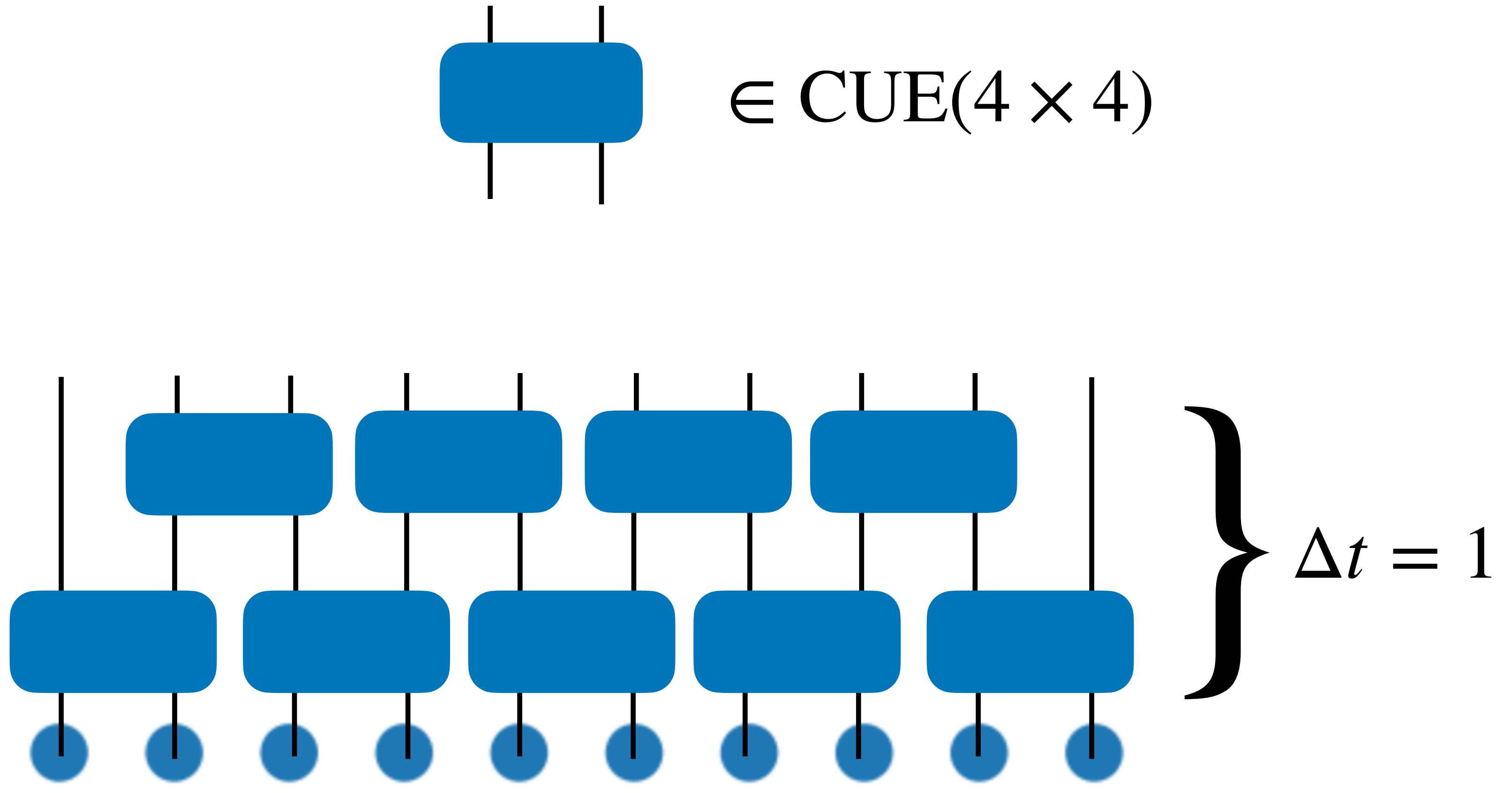




Dynamics on the information lattice – ballistic growth of entanglement in a random unitary circuit

Random brickwork of independent random unitaries



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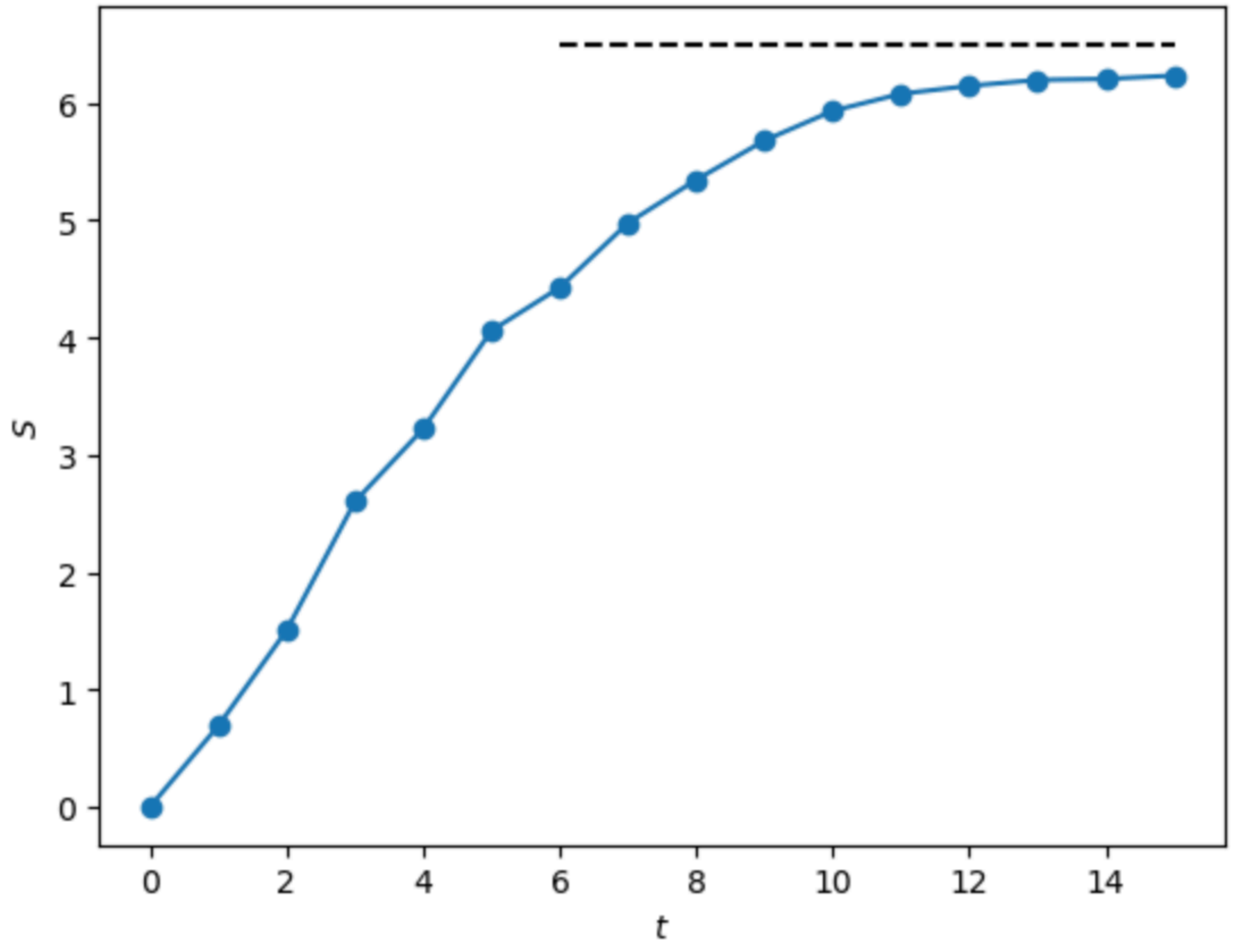

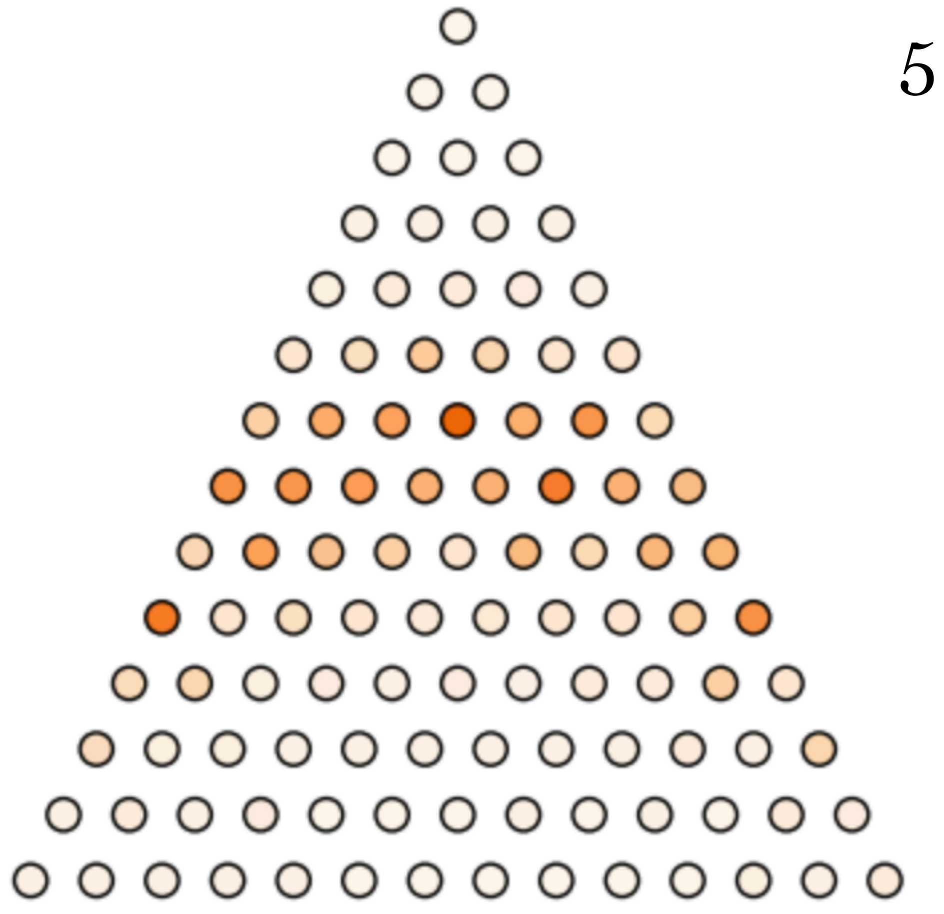

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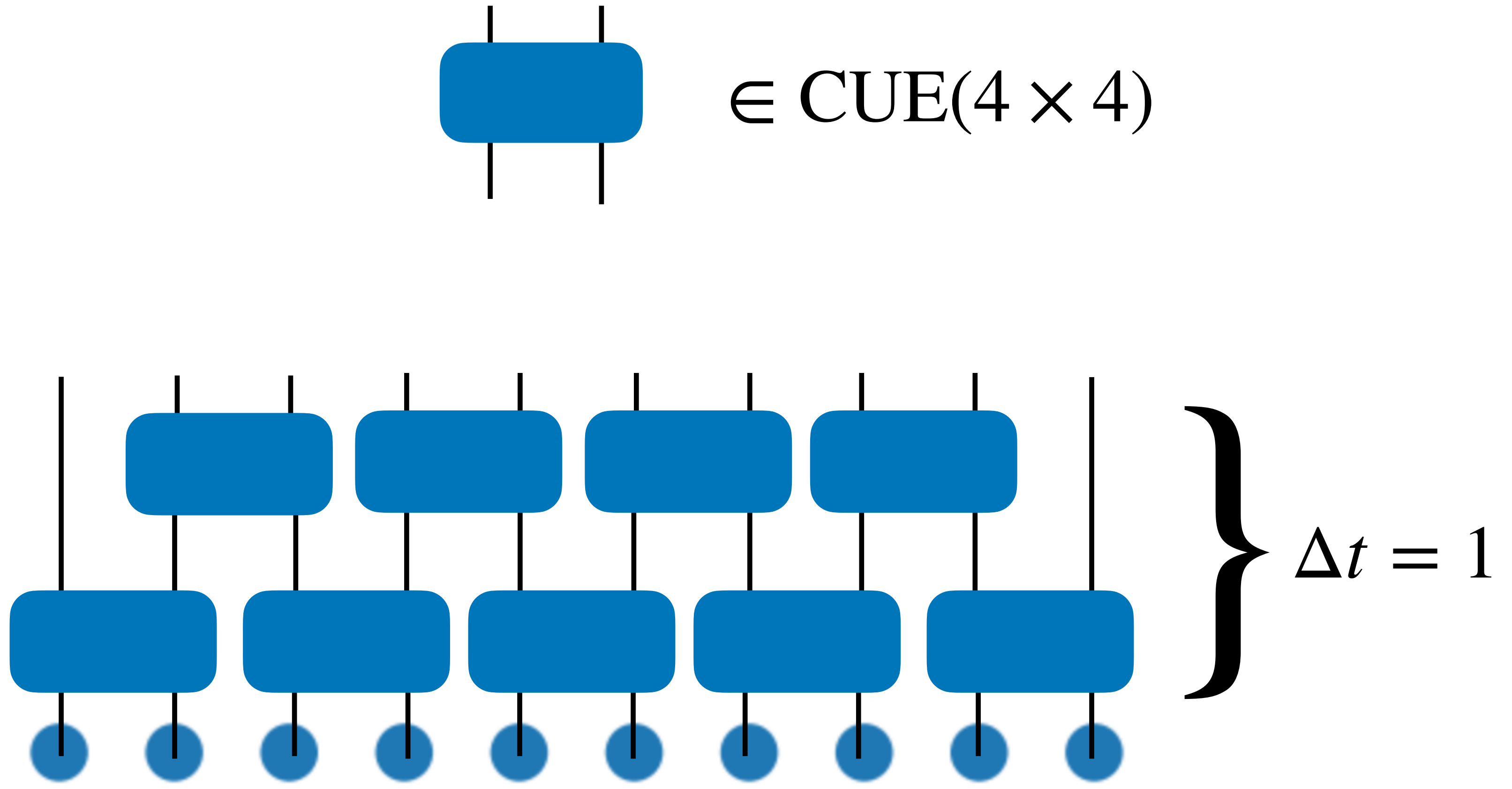

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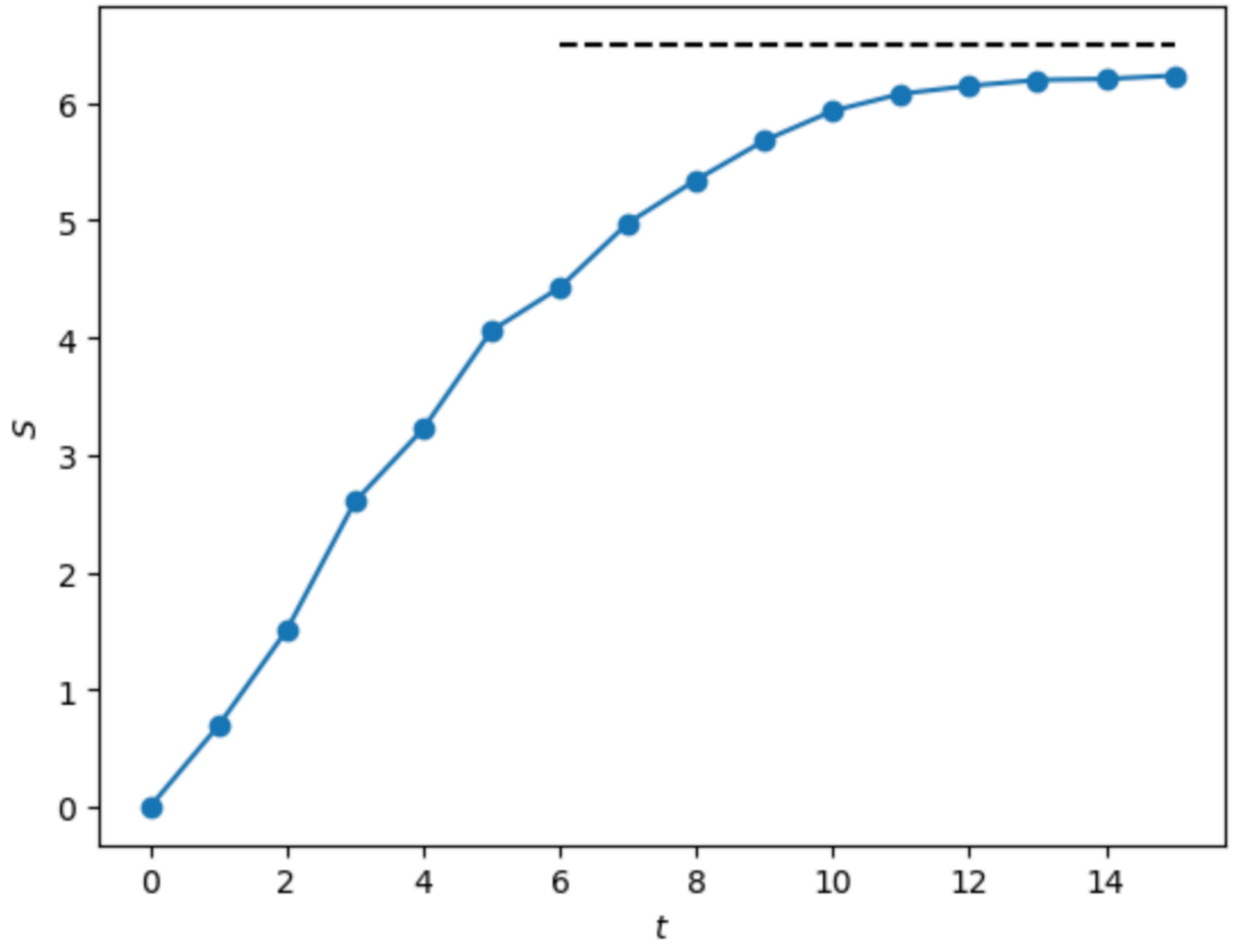

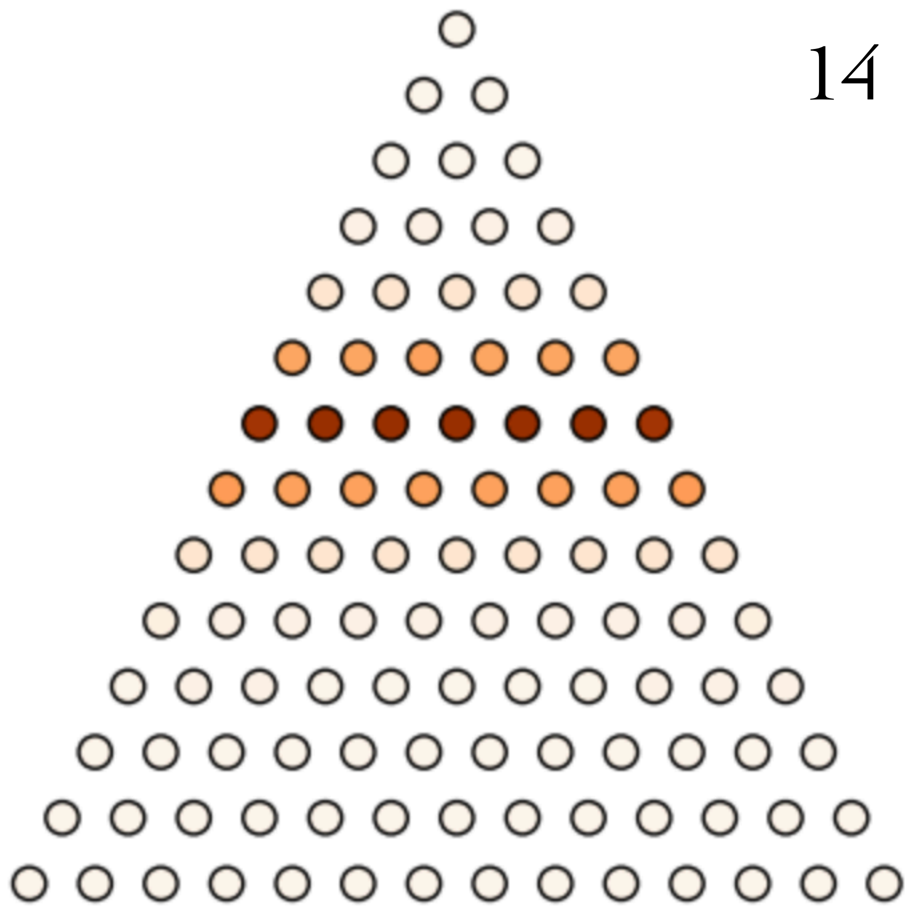

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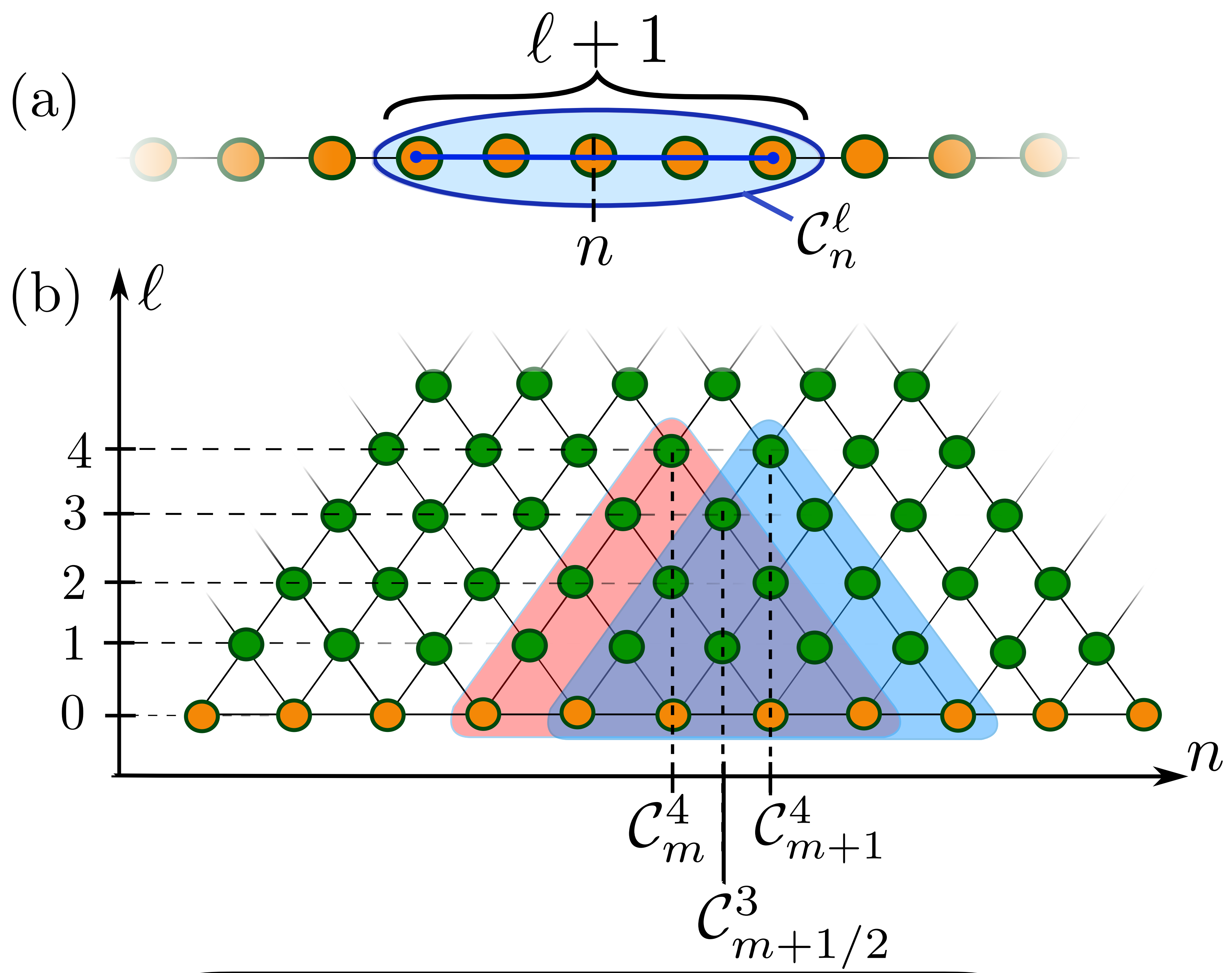
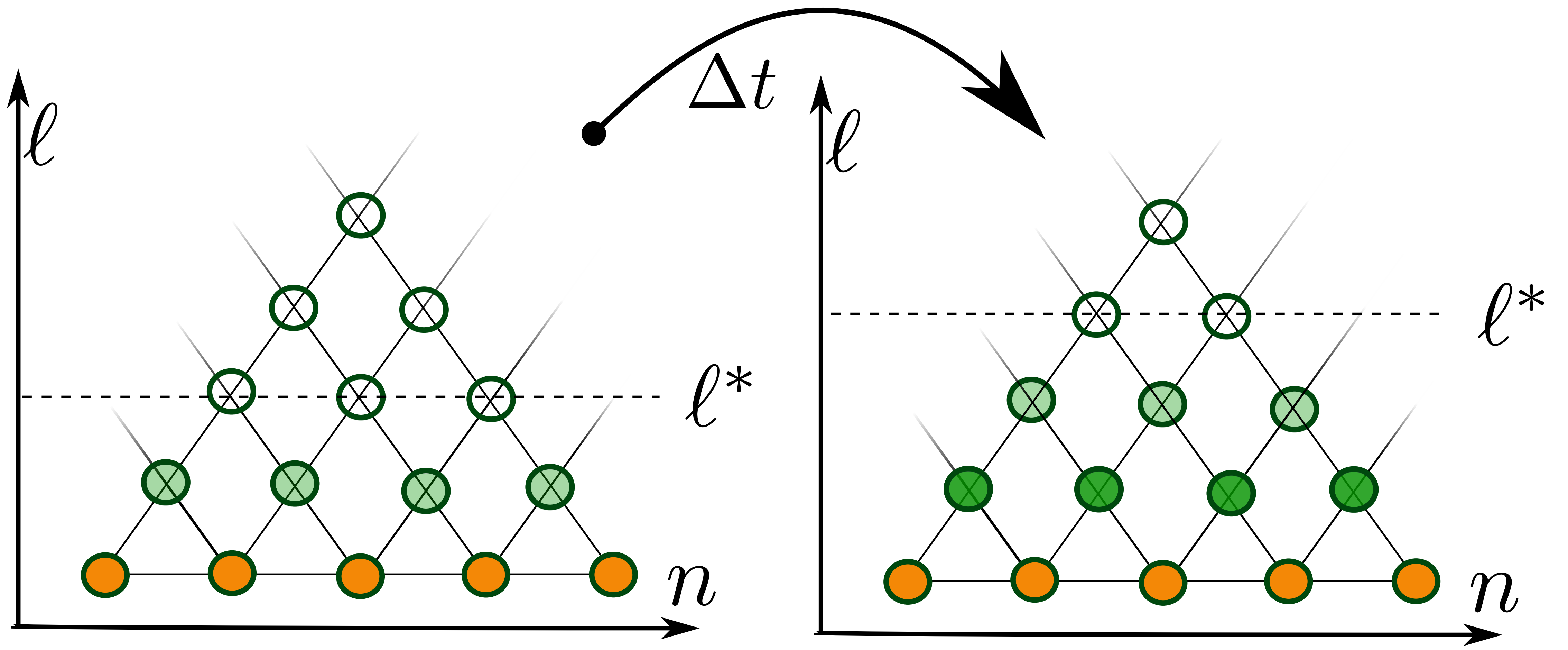
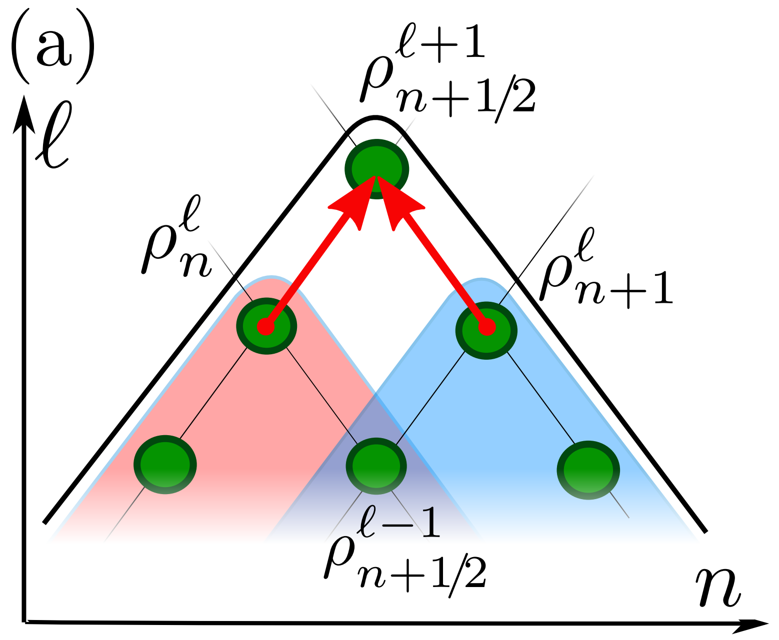
Time evolution on the information lattice



$$\partial_t \rho_n^\ell = -i\left[H_n^\ell, \rho_n^\ell\right] - i\mathrm{Tr}_r^L\left(\left[H_{n-r/2}^{\ell+r} - H_n^\ell, \rho_{n-r/2}^{\ell+r}\right]\right) - i\mathrm{Tr}_r^R\left(\left[H_{n+r/2}^{\ell+r} - H_n^\ell, \rho_{n+r/2}^{\ell+r}\right]\right)$$





If no information on scale $\ell + 1$ can construct $\rho^{\ell+1}$

$$\rho_{n+1/2}^{\ell+1} = \exp\left[\ln(\rho_n^\ell) + \ln(\rho_{n+1}^\ell) + \ln(\rho_{n+1/2}^{l-1})\right]$$

Petz recovery map

But ℓ^* increases with time (growth of entanglement)!

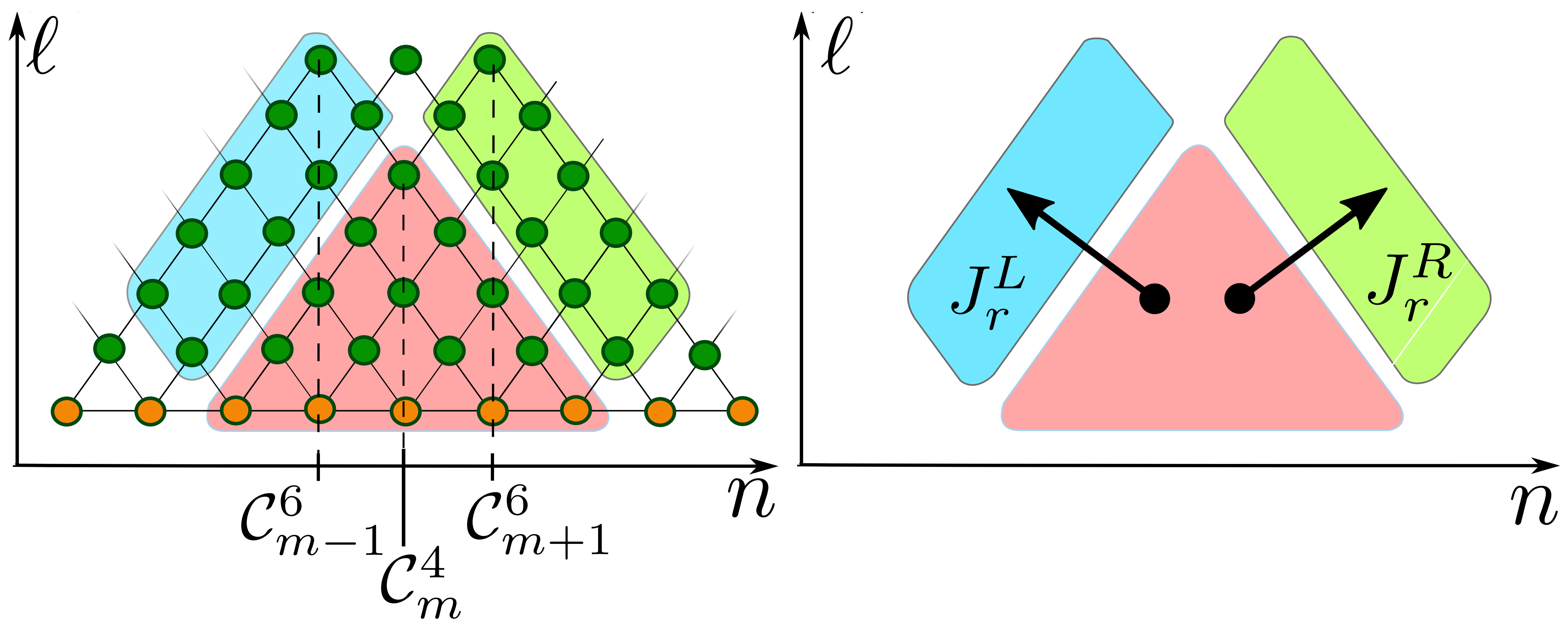
(Projected) Petz recovery map at fixed scale fails

Need to keep track of information flow

$$\partial_t I_n^\ell = \mathrm{Tr}\left(\nabla S_n^\ell \; \partial_t \rho_n^\ell \right)$$

$$\nabla S_n^\ell = -\log\left(\rho_n^\ell\right) - 1$$

$$\begin{aligned}\partial_t I(\rho_n^\ell) = & -i \; \mathrm{Tr}\left(\left[\nabla S_n^\ell , H_n^\ell \right] \; \rho_n^\ell \right) \\ & -i \; \mathrm{Tr}\left(\left[1_{2r} \otimes \nabla S_n^\ell , H_{n-r/2}^{\ell+r} - H_n^\ell \right] \; \rho_{n-r/2}^{\ell+r} \right) \\ & -i \; \mathrm{Tr}\left(\left[\nabla S_n^\ell \otimes 1_{2r} , H_{n+r/2}^{\ell+r} - H_n^\ell \right] \; \rho_{n+r/2}^{\ell+r} \right) \\ & = J_r^R(\rho_{n-r/2}^{\ell+r}) + J_r^L(\rho_{n+r/2}^{\ell+r})\end{aligned}$$



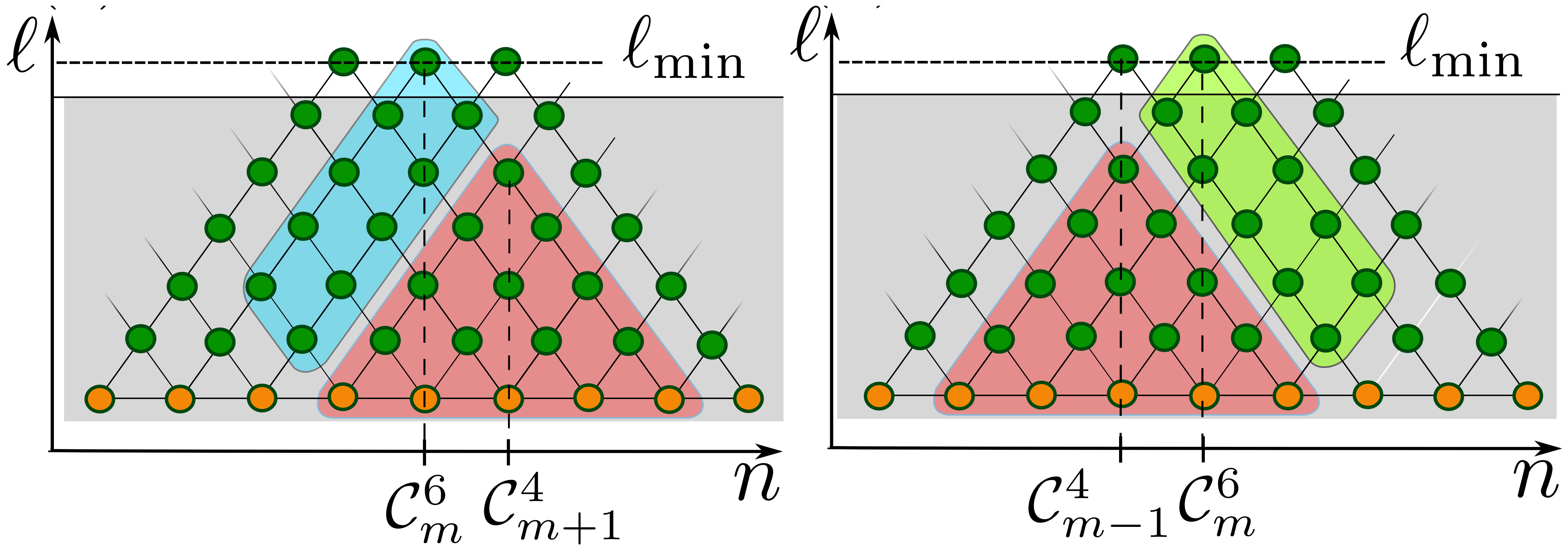
Minimise information on a large scale, keeping information and information flow on smaller scales fixed

$$\bar{\rho}_n^\ell = \rho_n^\ell + \chi_n^\ell$$

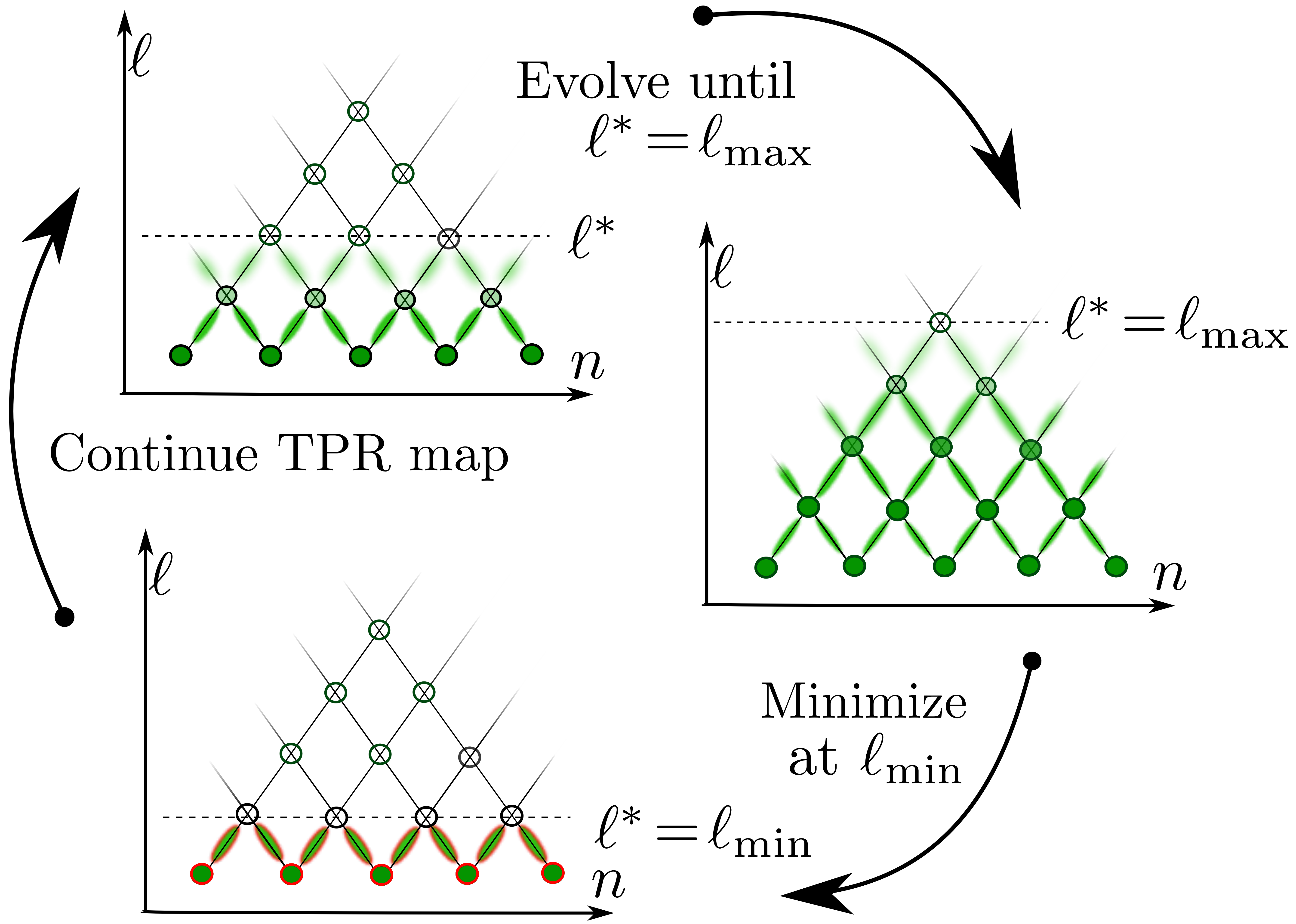
$$\left.\begin{aligned} \mathrm{Tr}_1^R(\chi_n^\ell) &= 0 \\ \mathrm{Tr}_1^L(\chi_n^\ell) &= 0 \\ J_r^R(\chi_n^\ell) &= 0 \\ J_r^L(\chi_n^\ell) &= 0 \end{aligned}\right\}$$

Defines subspace from which to obtain $\xi = \mathbf{P}\chi$

$\longrightarrow$ Maximize $S(\rho+\xi) = S(\rho) + \mathrm{Tr}\left(\mathrm{P}\,\nabla_\rho S\chi\right) + \frac{1}{2}\mathrm{Tr}\left(\chi\ \mathrm{P}\mathscr{H}_\rho \mathrm{P}\ \chi\right) + \mathcal{O}(\xi^3)$



To get reliable information current, evolve information to larger scale and allow to build up before minimising



Benchmark: energy diffusion in the mixed field Ising model

$$H = \sum_i J\sigma_i^z\sigma_{i+1}^z + h_T\sigma_i^x + h_L\sigma_i^z$$

$$J = 1,\ h_T = 1.4,\ h_L = 0.9045$$

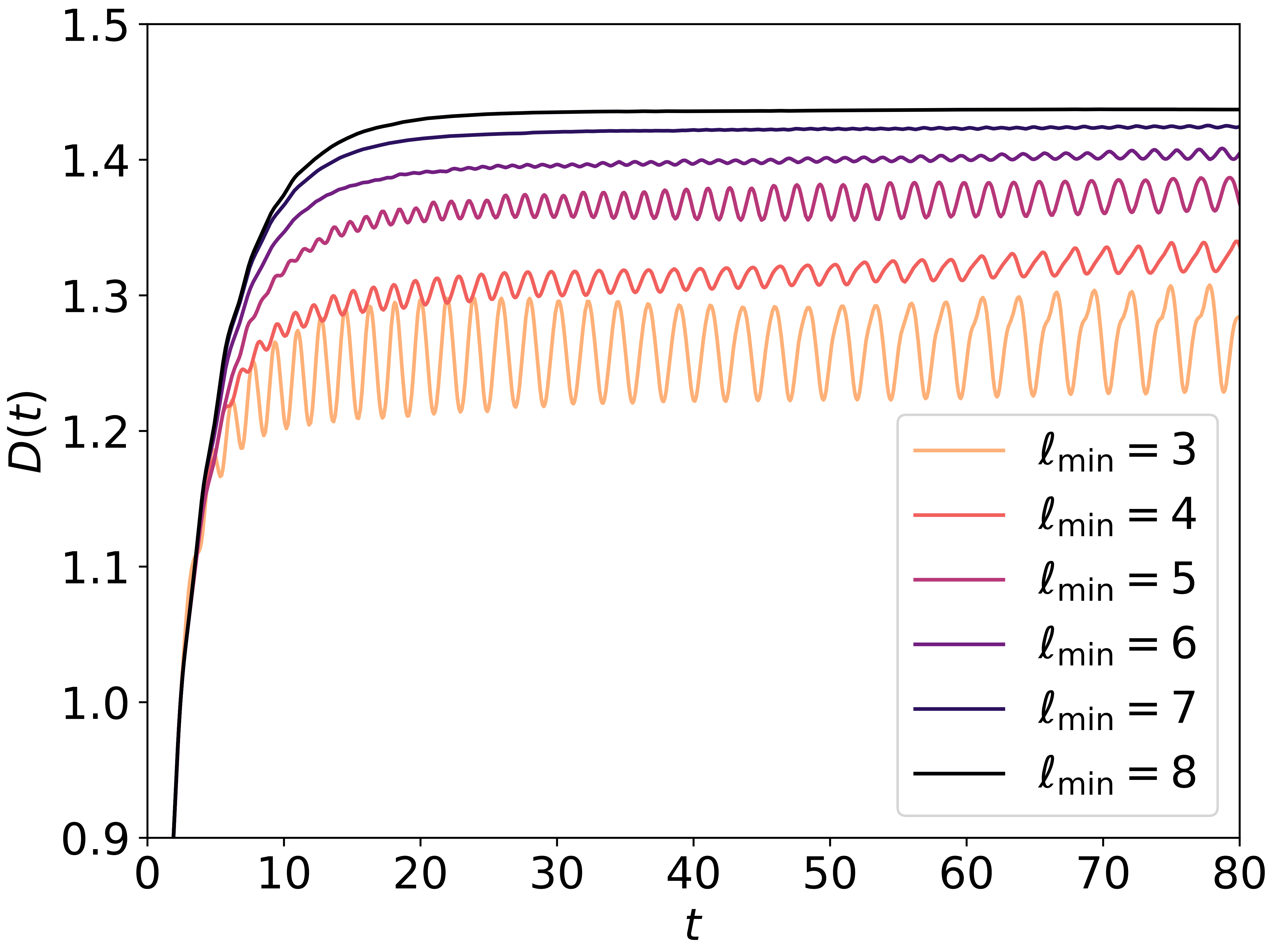
$$\rho_{\text{init}} = \left(\bigotimes_{m<n-\ell/2} \rho_{m,\infty}\right) \otimes \rho_{n,\text{init}}^{\ell} \otimes \left(\bigotimes_{m>n+\ell/2} \rho_{m,\infty}\right)$$

$$\sigma_E^2 = \sum_n (n-\overline{n})^2 \frac{E_n^r}{\langle H\rangle} - \left(\sum_n (n-\overline{n})\frac{E_n^r}{\langle H\rangle}\right)^2$$

$$D = \frac{1}{2}\partial_t \sigma_E^2$$

Compare [Rakovszky, von Keyserlingk, and Pollmann PRB 2022]

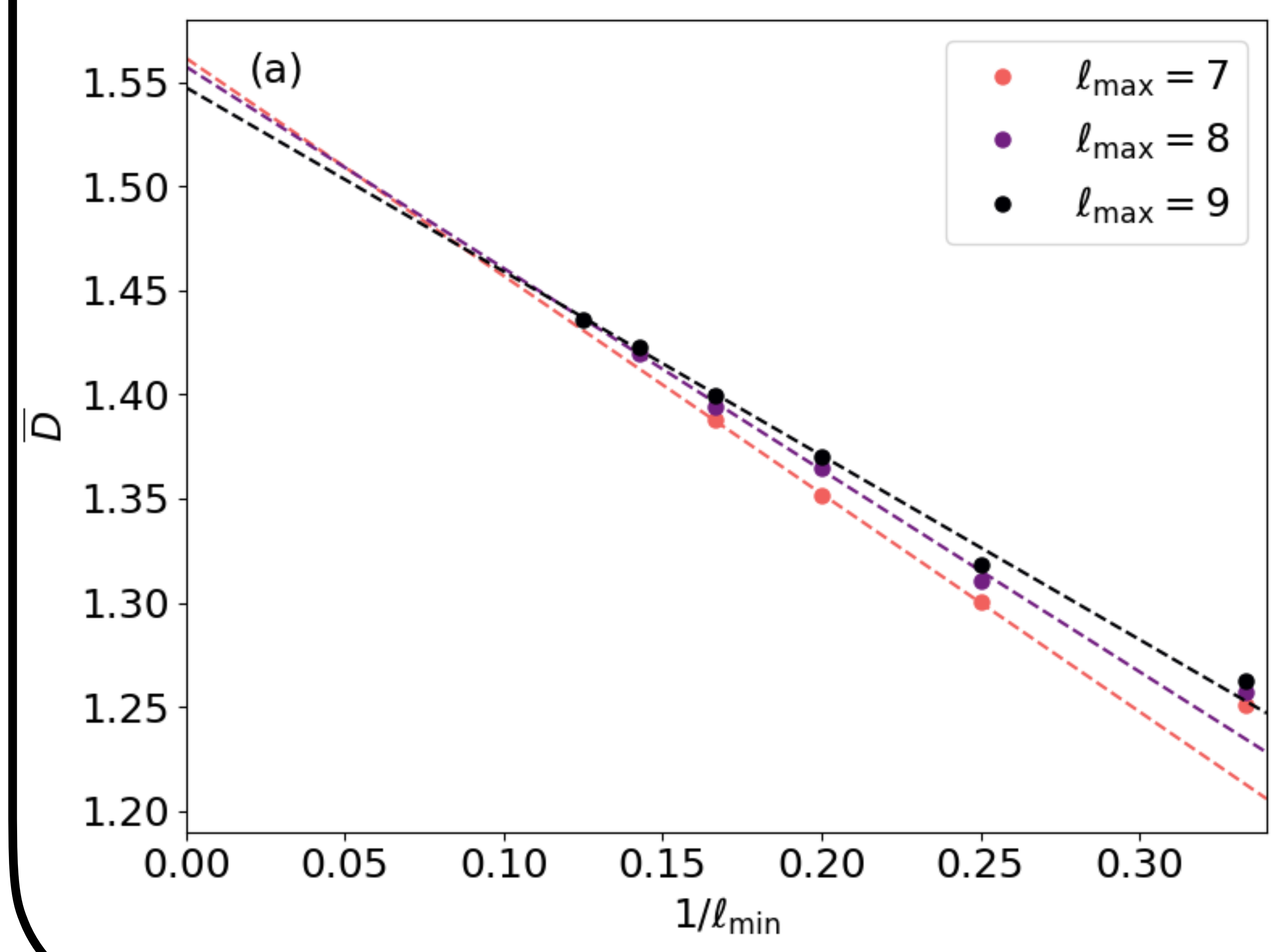
$D \approx 1.40$ at $t \sim 20$ using MPO methods



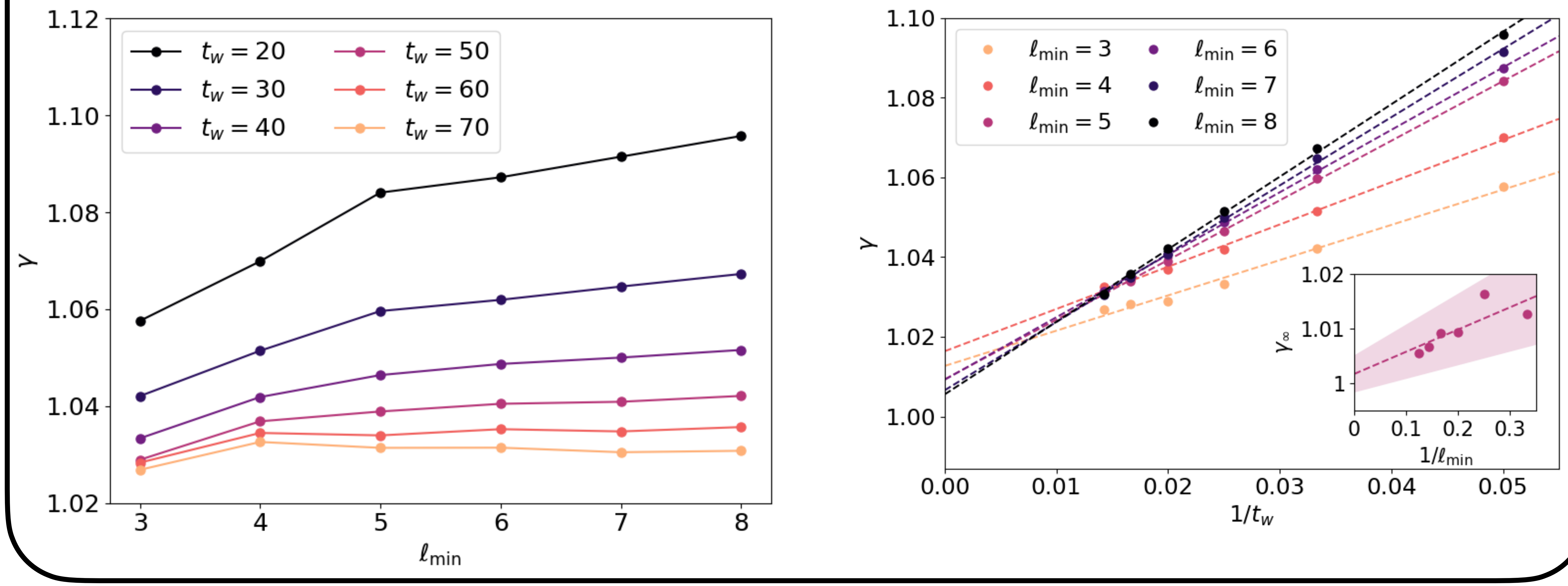
[Artiaco, Fleckenstein, Aceituno, Klein-Kvorning, JHB, arXiv:2310.06036]

Reasonable convergence with increasing scale l_{min} and l_{max}. Diffusion obtained at long time when doing a window fit starting at t_w

Convergence and extrapolation with scale



Exponent with window fit approaches unity in the limit of infinite scale and time



[Artiaco, Fleckenstein, Aceituno, Klein-Kvorning, JHB, arXiv:2310.06036]

Ultraslow growth of number entropy in an l-bit model of MBL

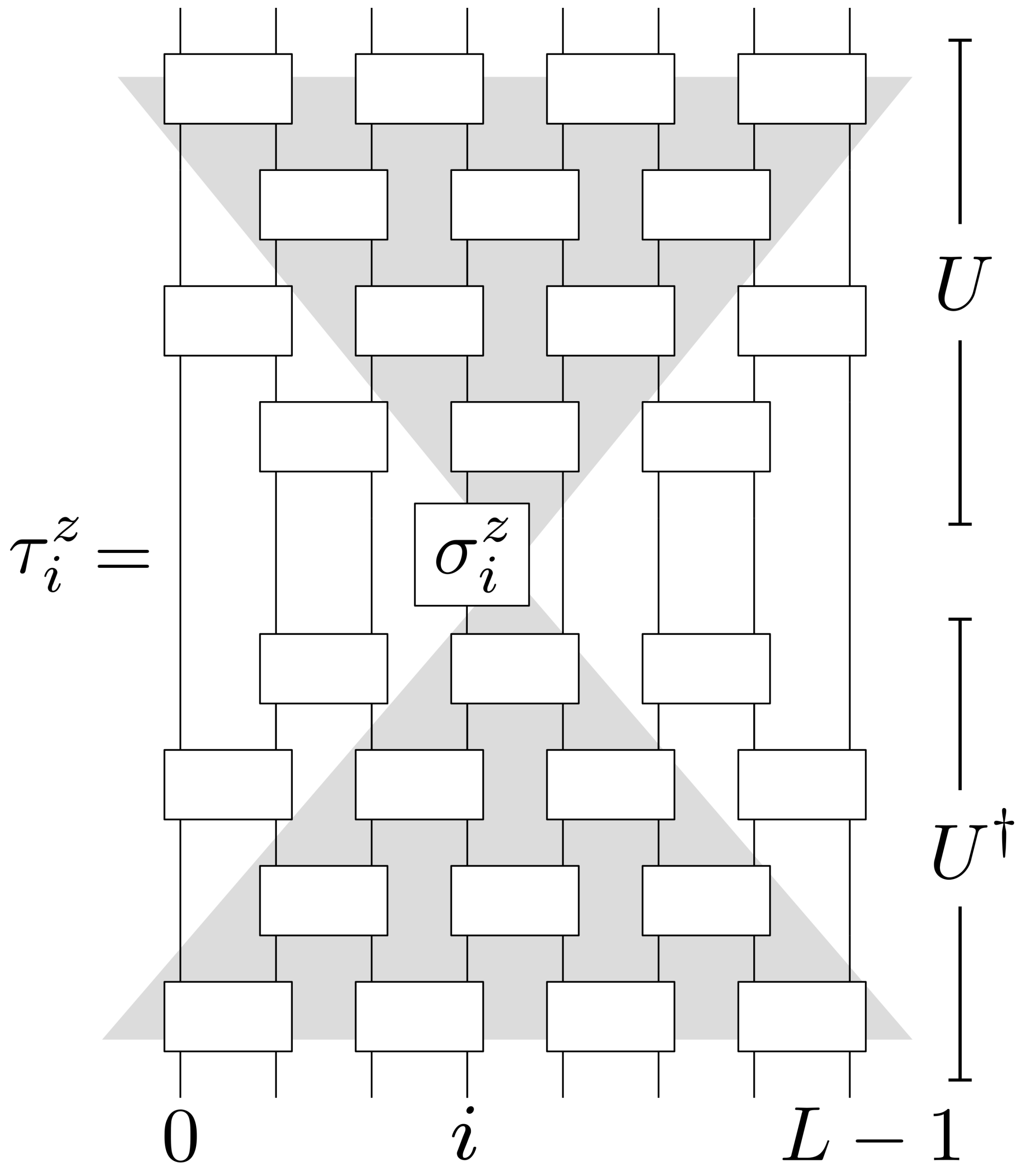
[Aceituno, Artiaco, Klein-Kvorning, Herviou, JHB, arXiv:2312.13420]

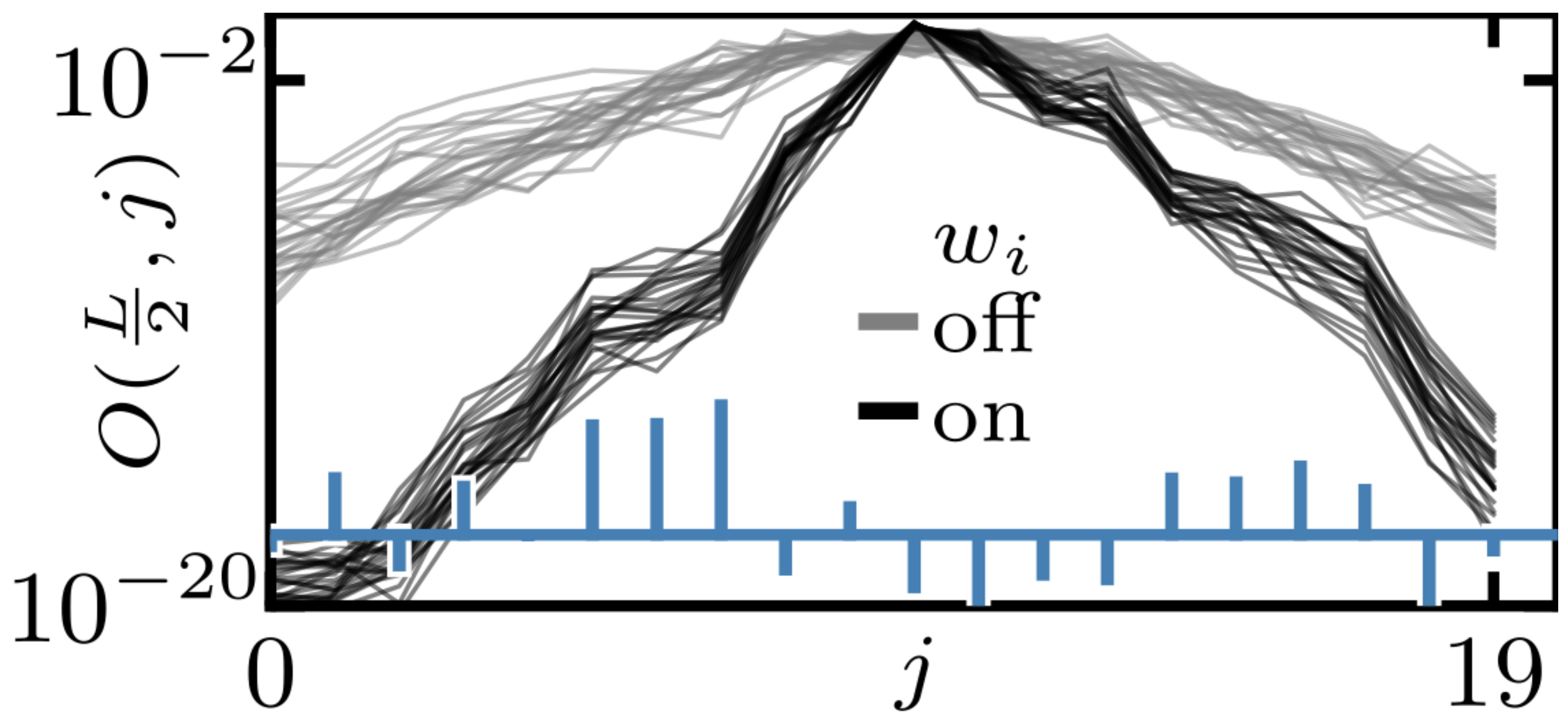
Motivation: Microscopic models show ultraslow growth of number entropy (Sirker group), challenging MBL, though maybe explained by resonances (Gosh and Žnidarič). What do l-bits do?

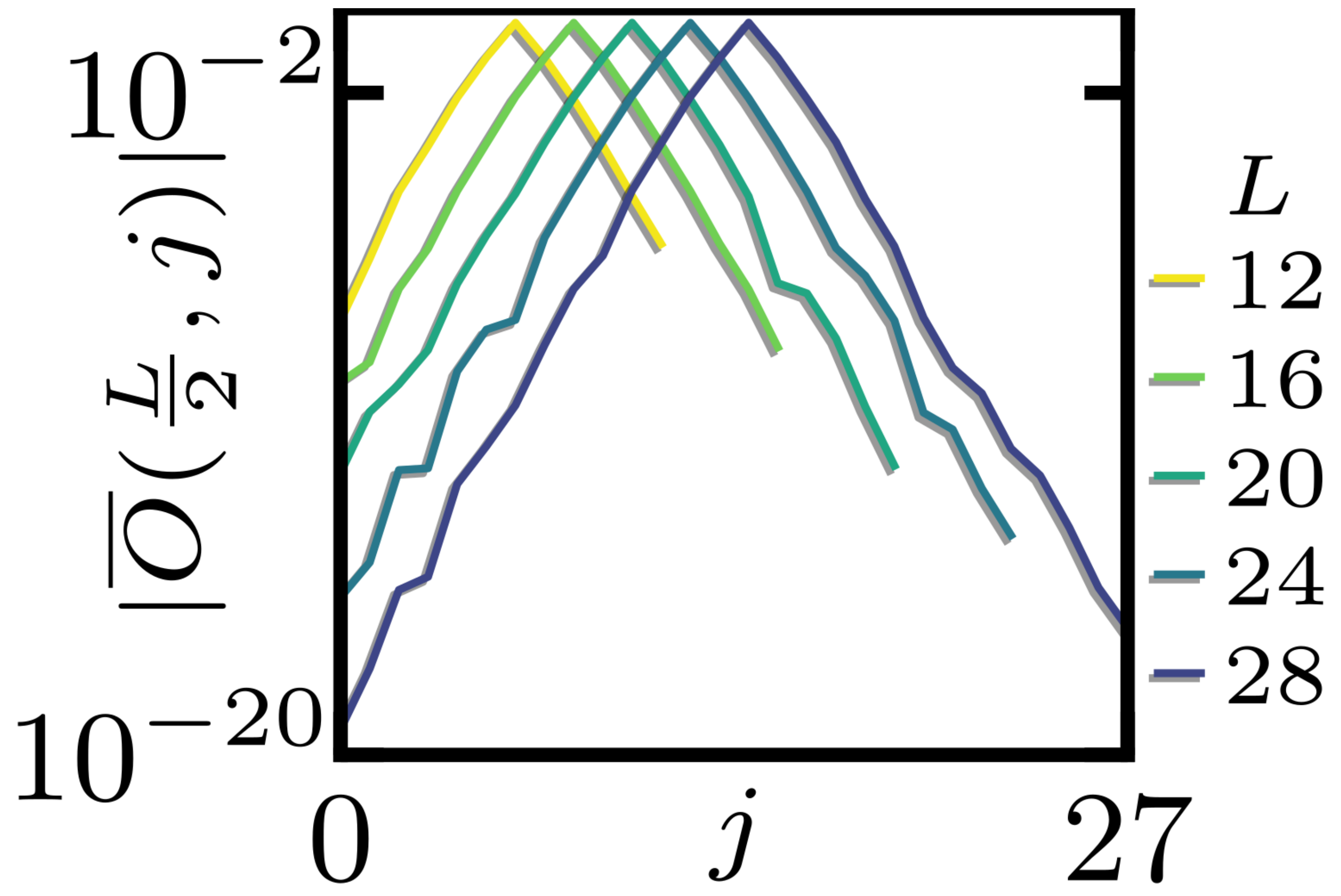
$$S = S_N + S_C$$

$$S_N = -\sum_n p(n) \ln p(n)$$

l-bits from random unitary circuits and a random l-bit Hamiltonian







$$u_i = e^{-\mathrm{i} f w_i M_i}$$

$$w_i = e^{-2|h_i - h_{i+1}|}$$

$$M_i = \sum_{p,q=\pm 1} \frac{\theta_{pq}}{4}(1 + p\sigma^z) \otimes (1 + q\sigma^z) + c\sigma^+ \otimes \sigma^- + c^* \sigma^- \otimes \sigma^+$$

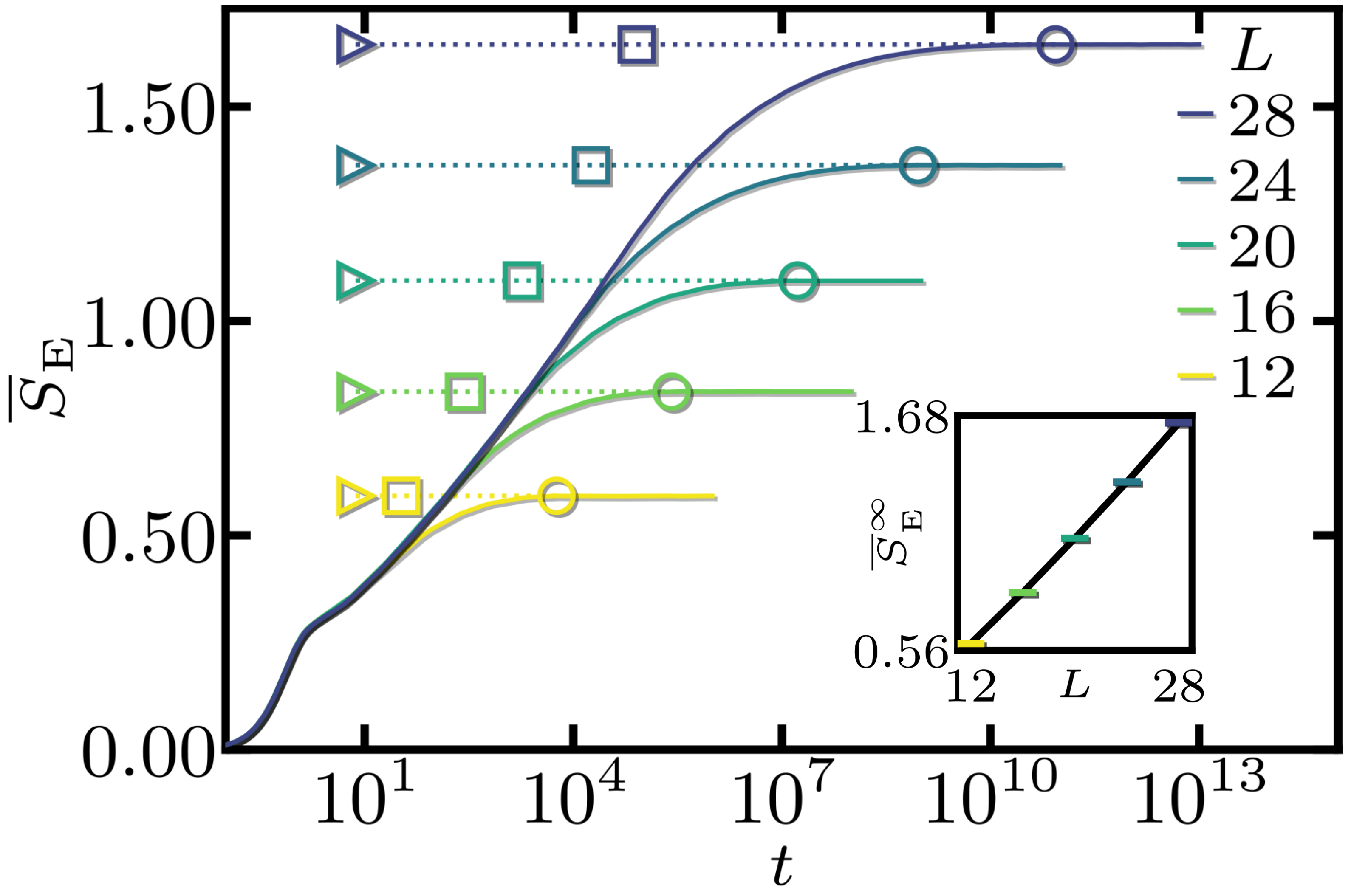
$$H = \sum_i h_i \tau_i^z + \sum_{i<j} J_{ij} \tau_i^z \tau_j^z + \sum_{i<j<k} J_{ijk} \tau_i^z \tau_j^z \tau_k^z + \cdots$$

$$J_{ij} = R_{ij} e^{-|i-j|/\xi_J}$$

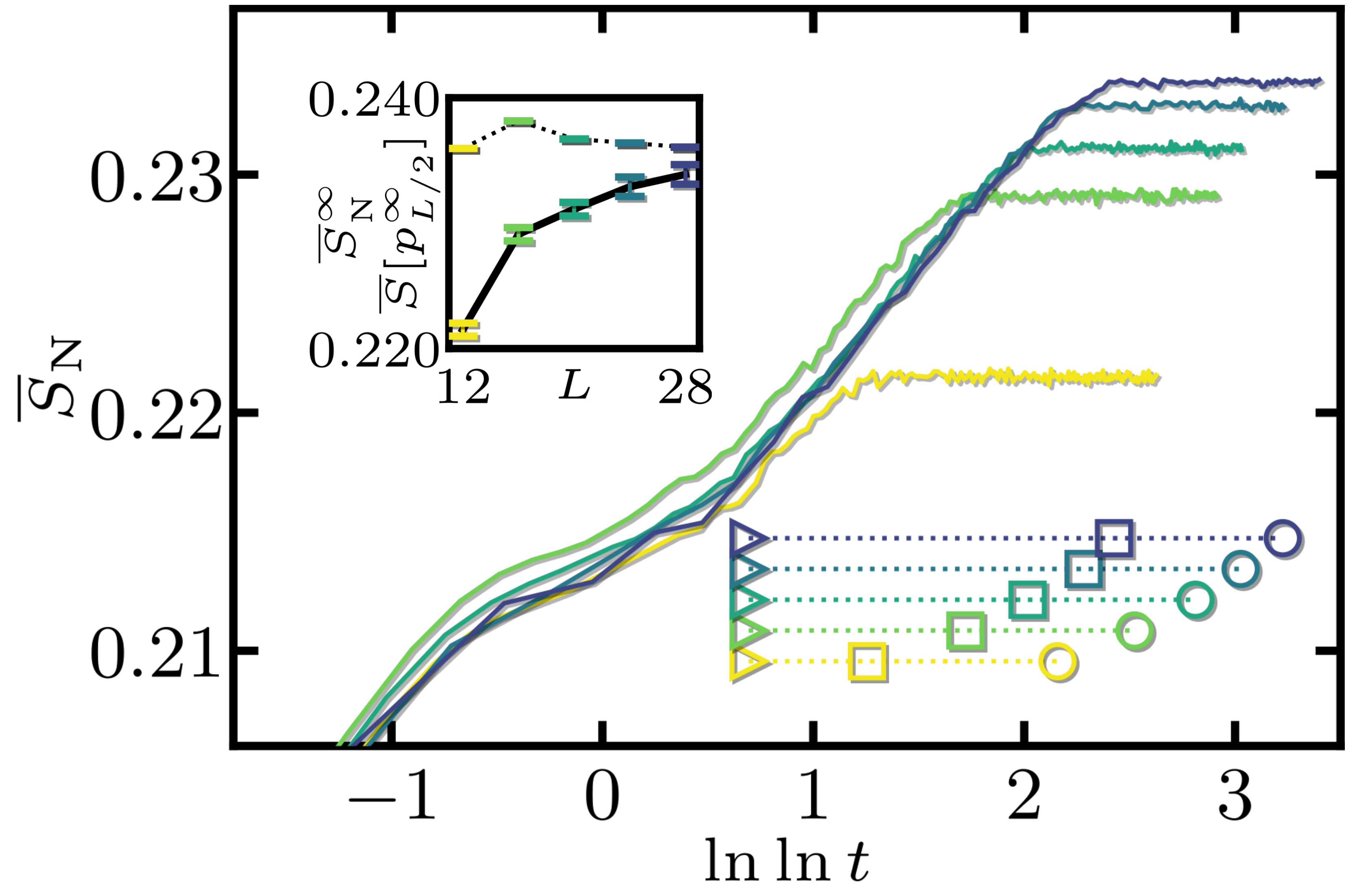
[Aceituno, Artiaco, Klein-Kvorning, Herviou, JHB, arXiv:2312.13420]

Number entropy grows like lnln t for exponentially long time

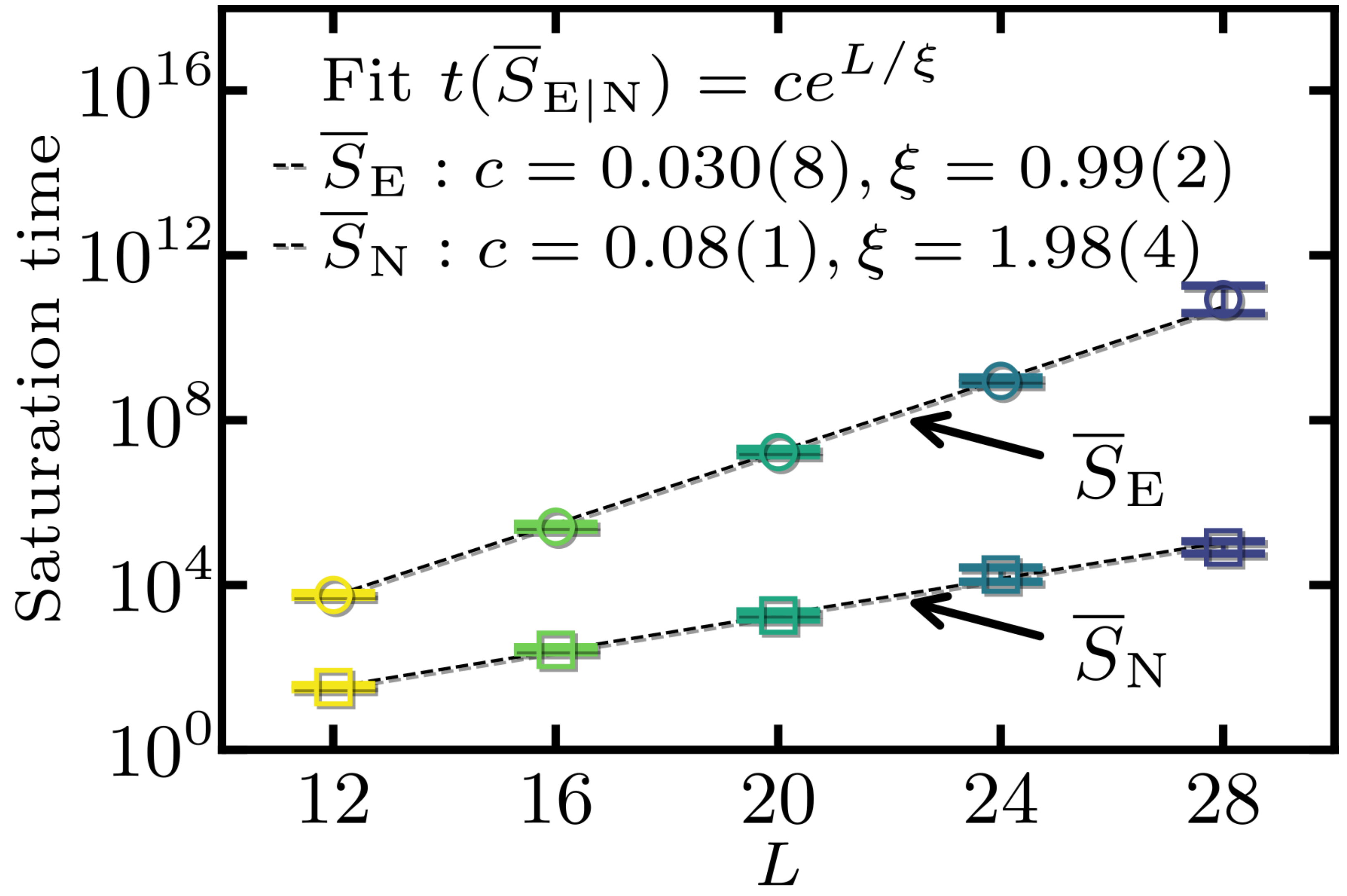
Entanglement entropy



Number entropy

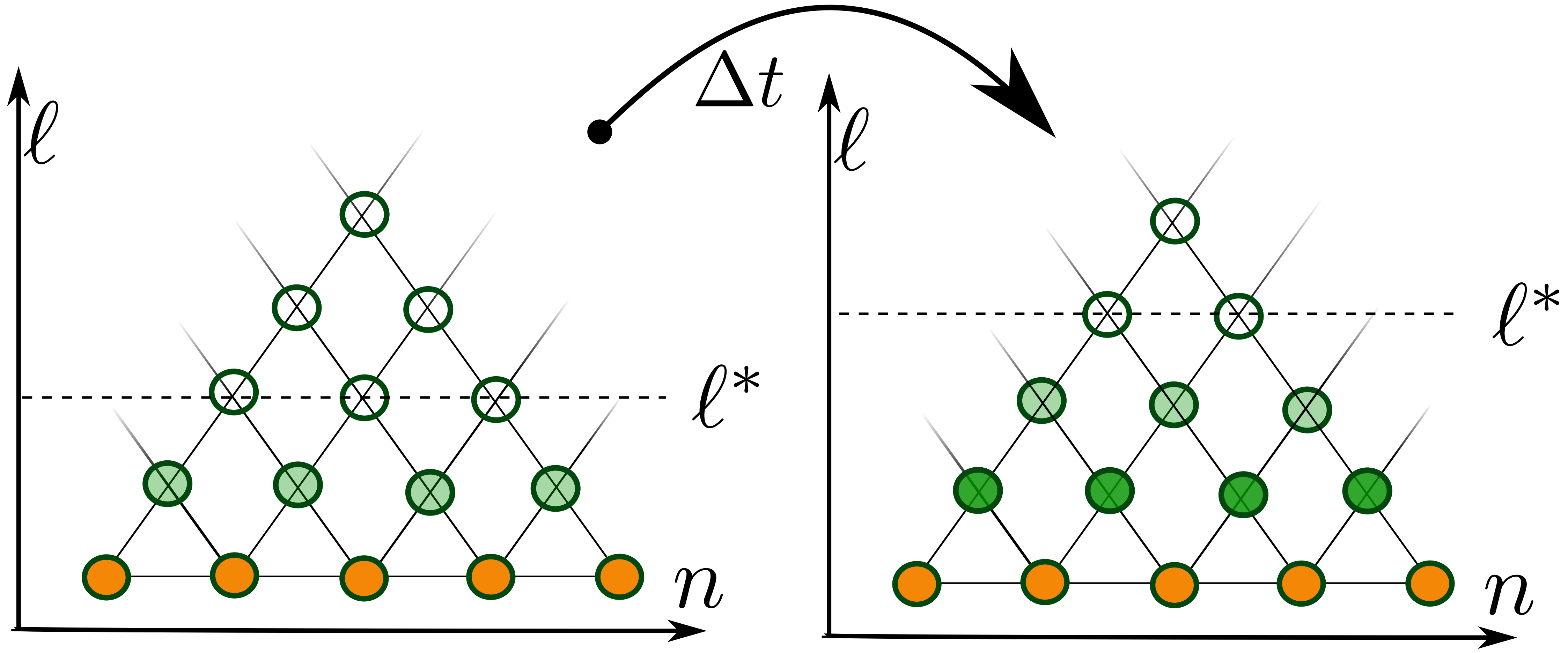


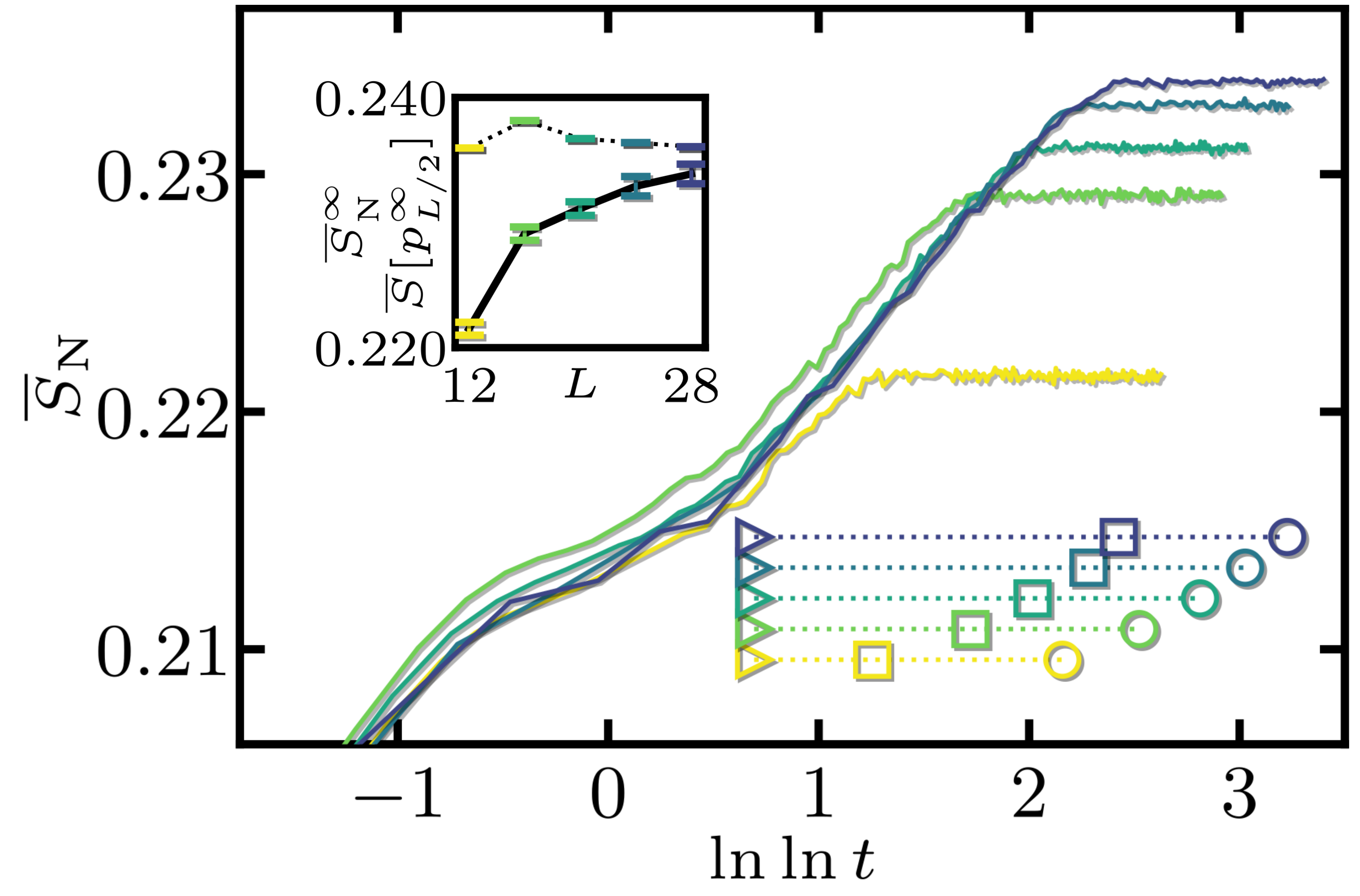
Saturation time

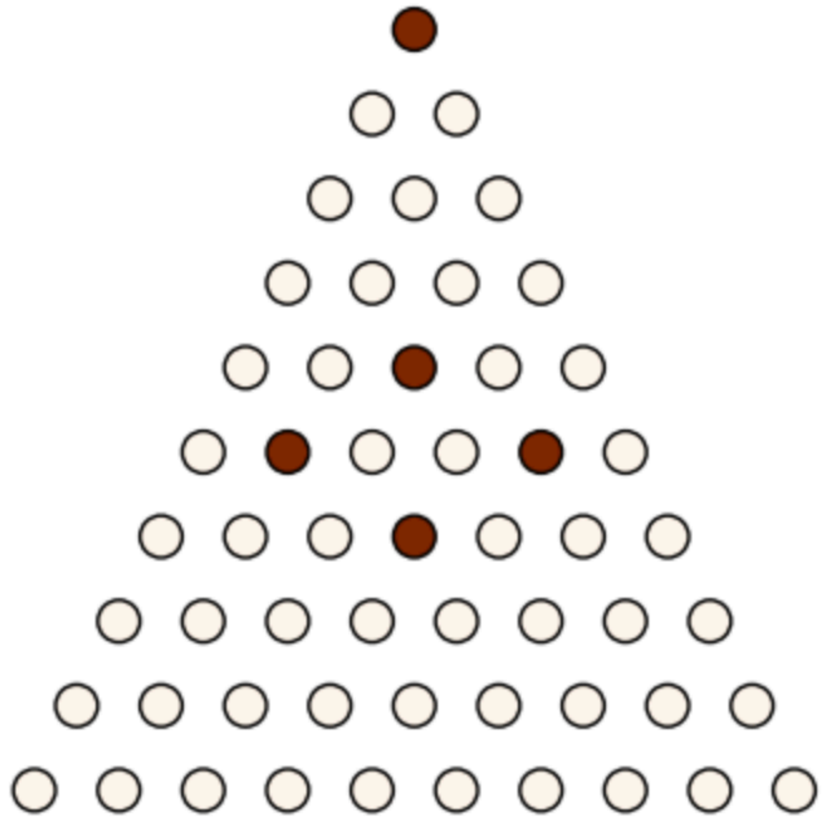


[Aceituno, Artiaco, Klein-Kvorning, Herviou, JHB, arXiv:2312.13420]

Summary, conclusion, and outlook







Information lattice a nice way to visualise the structure of quantum information in many body states

Time evolution of local information and efficient and accurate method for quantum time evolution to large times and in large systems

l-bit model a rich model with many-body l-bits that can lead to slow dynamics.

Need to carefully understand the consequences of his slow dynamics when comparing with microscopic models

Refs: SciPost Physics **13**, 080 (2022); arXiv:2310.06036; arXiv:2312.13420