Numerical Relativity: Mathematical Formulation

A brief review of tensor properties

A tensor is a physical object and has an intrinsic meaning independently of coordinates or basis vectors. An example is a rank-1 tensor $\mathbf{A}$ that has a certain length and points in a certain direction independently of the coordinate system in which we express this tensor.

We can expand any tensor in terms of either basis vectors $\mathbf{e}_a$ or basis 1-forms $\tilde{\omega}^a$. Expanding a rank-1 tensor in terms of basis vectors, for example, yields

$$\mathbf{A} = A^a \mathbf{e}_a. \tag{1}$$

This expression deserves several comments. For starters, we have used the Einstein summation rule, meaning that we sum over repeated indices. The "up-stairs" index a on A^a refers to a *contravariant* component of $\mathbf{A}$ – meaning one that is used in an expansion of $\mathbf{A}$ in terms of basis vectors. The index a on the basis vector $\mathbf{e}_a$, on the other hand, does *not* refer to a component of the basis vector – instead it denotes the name of the basis vector (e.g. the basis vector pointing in the x direction). If we wanted to refer to the b-th component of the basis vector $\mathbf{e}_a$, say, we would write $(\mathbf{e}_a)^b$.

We now write the dot product between two vectors as

$$\mathbf{A} \cdot \mathbf{B} = (A^a \mathbf{e}_a) \cdot (B^b \mathbf{e}_b) = A^a B^b \, \mathbf{e}_a \cdot \mathbf{e}_b. \tag{2}$$

Defining the metric as

$$g_{ab} \equiv \mathbf{e}_a \cdot \mathbf{e}_b \tag{3}$$

we obtain

$$\mathbf{A} \cdot \mathbf{B} = A^a B^b g_{ab}. \tag{4}$$

Expanding a rank-1 tensor in terms of 1-forms $\tilde{\omega}^a$ yields

$$\mathbf{B} = B_a \tilde{\omega}^a, \tag{5}$$

where the "down-stairs" index a refers to a *covariant* component. We call the basis 1-forms $\tilde{\omega}^a$ dual to the basis vectors $\mathbf{e}_b$ if

$$\tilde{\omega}^a \cdot \mathbf{e}_b = \delta^a{}_b \tag{6}$$

where

$$\delta^a_{\ b} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

is the *Kronecker delta*. This is what we will assume throughout. We can then compute the contraction between $\mathbf{B}$ and $\mathbf{A}$ as

$$\mathbf{A} \cdot \mathbf{B} = (A^a \mathbf{e}_a) \cdot (B_b \tilde{\omega}^b) = A^a B_b \, \mathbf{e}_a \cdot \tilde{\omega}^b = A^a B_b \, \delta_a{}^b = A^a B_a. \tag{8}$$

Since both this expression and (4) have to hold for any tensor $\mathbf{A}$, we can compare the two and identify

$$B_a = g_{ab} B^b. \tag{9}$$

We refer to this operation as "lowering the index of B^a".

We define the inverse metric g^{ab} so that

$$g^{ac} g_{cb} = \delta^a{}_b. \tag{10}$$

We then "raise the index of B_a" using

$$B^a = g^{ab} B_b. \tag{11}$$

We can also show that

$$g^{ab} = \tilde{\omega}^a \cdot \tilde{\omega}^b. \tag{12}$$

Note that we can find the contravariant component of a rank-1 tensor $\mathbf{A}$ by computing the dot product with the corresponding 1-form,

$$A^a = \mathbf{A} \cdot \tilde{\omega}^a. \tag{13}$$

We can verify this expression by inserting the expansion (1) for $\mathbf{A}$, and then using the duality relation (6).

Under a change of basis, i.e. when we transform from one coordinate system x^a to another, say $x^{b'}$, basis vectors and basis 1-forms transform according to

$$\begin{aligned} \mathbf{e}_{b'} &= M^a_{\ b'}\mathbf{e}_a &(14)\\ \tilde{\omega}^{b'} &= M^{b'}_{\ a}\tilde{\omega}^a &(15) \end{aligned}$$

where $M^{b'}_{\ a}$ is the transformation matrix and $M^a_{\ b'}$ its inverse, so that

$$M^{a'}_{\ c}M^c_{\ b'} = \delta^{a'}_{\ b'}. \tag{16}$$

Note that vectors and 1-forms transform in "inverse ways". This guarantees that the duality relation (6) also holds in the new coordinate system,

$$\tilde{\omega}^{a'} \cdot \mathbf{e}_{b'} = (M^{a'}_{\ c}\tilde{\omega}^c) \cdot (M^d_{\ b'}\mathbf{e}_d) = M^{a'}_{\ c}M^d_{\ b'}(\tilde{\omega}^c \cdot \mathbf{e}_d) = M^{a'}_{\ c}M^d_{\ b'}\delta^c_{\ d} = M^{a'}_{\ c}M^c_{\ b'} = \delta^{a'}_{\ b'}. \tag{17}$$

The components of a vector then transform according to

$$A^{b'} = \mathbf{A} \cdot \tilde{\omega}^{b'} = \mathbf{A} \cdot (M^{b'}_{\ a}\tilde{\omega}^a) = M^{b'}_{\ a}A^a \tag{18}$$

and similarly

$$B_{b'} = M^a_{\ b'}B_a. \tag{19}$$

The fact that contravariant and covariant components transform in "inverse" ways guarantees that the dot product (8) is invariant under coordinate transformations,

$$A^{b'}B_{b'} = M^{b'}_{\ a}A^a M^c_{\ b'}B_c = \delta^c_{\ a}A^aB_c = A^aB_a, \tag{20}$$

as it is supposed to be.

We can generalize all the above concepts to higher-rank tensors. A rank-n tensor can be expanded into n basis vectors or 1-forms, and we transform the components of a rank-n tensor with n copies of the transformation matrix or its inverse.

For transformations between *coordinate bases*, for which the basis vectors are tangent to coordinate lines, we have

$$M^{b'}_{\ a} \equiv \frac{\partial x^{b'}}{\partial x^a} = \partial_a x^{b'}. \tag{21}$$

As an illustration of the above concepts, consider the components of a displacement vector dx^a, which measures the displacement between two points expressed in a coordinate system x^a. To compute the components of this vector in a different coordinate system, say a primed coordinate system $x^{b'}$, we use the chain rule to obtain

$$dx^{b'} = \frac{\partial x^{b'}}{\partial x^a}dx^a = M^{b'}_{\ a}dx^a \tag{22}$$

where we have used (21) in the last step. As expected, the components of dx^a transform like the vector components in (18).

As an example of a 1-form, consider the components of the gradient $\partial f/\partial x^a$ of a function f, again expressed in some coordinate system x^a. To transform to a new coordinate system $x^{b'}$ we again use the chain rule, but this time we obtain

$$\frac{\partial f}{\partial x^{b'}} = \frac{\partial x^a}{\partial x^{b'}}\frac{\partial f}{\partial x^a} = M^a_{\ b'}\frac{\partial f}{\partial x^a} \tag{23}$$

as in (19). We see that the "inverse" transformation of the components of a gradient are a result of the chain rule.

Finally, consider the difference df in the function values f at two (close) points. Clearly, this difference is an invariant, i.e. independent of coordinate choice. We can express this difference as the dot product between the vector displacement vector dx^a between the two points and the 1-form $\partial f/\partial x^a$,

$$df = \frac{\partial f}{\partial x^a}dx^a. \tag{24}$$

As an exercise, apply the above transformation rules for the components of vectors and 1-form to show that df is indeed invariant under a coordinate transformation.