

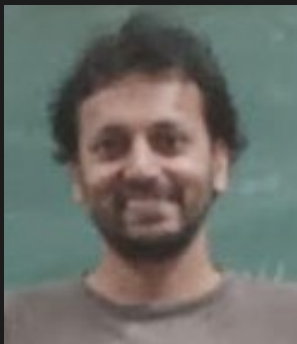
Radio astronomical images, (an)holonomy and gauge theory

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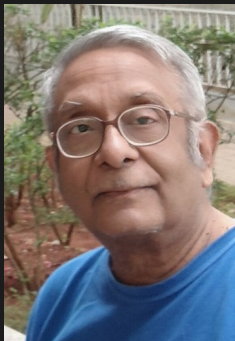
Plan

- How radio astronomers use complex two point electric field correlations to make an image
- Our main motivation – Event Horizon telescope
- Quantities cancelling amplitude and phase errors should be viewed as invariants in a gauge theory using products of correlations around loops
- This generalises to measured correlations from sources with polarisation, both are matrices
- Showing that this works in the real world

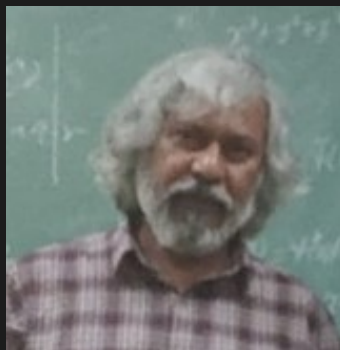
“We”



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arXiv 2108.399 NT, RN, JS Phys Rev D 105 (4) 043019 (2022)
“Invariants in co-polar interferometry: An Abelian gauge theory “

arXiv 2108.400 JS, RN NT Phys Rev Lett 128 (9) 091101 (2022)
“Invariants in polarised interferometry: A non-Abelian gauge theory”

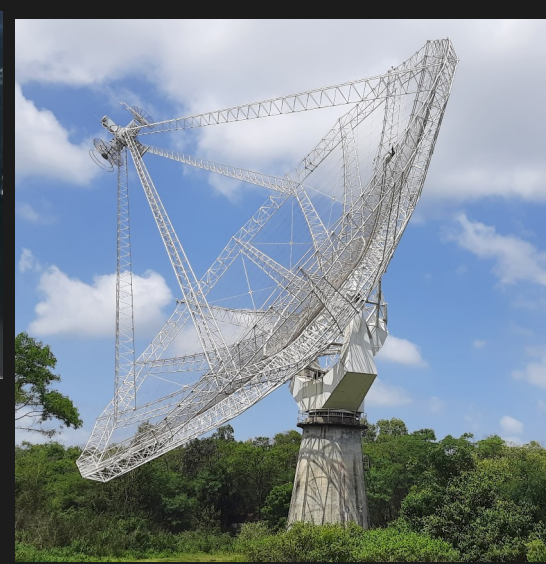
arXiv 2407.00583 VK, RN, JS J.Ap. Astr 46 (2) 56 (2025)
“Closure invariants for interferometry: A graph theoretical approach”

Radio interferometry in a nutshell

- The 'image' is $I(l,m)$ intensity as a function of direction in the sky (different l,m incoherent)
- An interferometer measures $\langle (E_1 + E_2) (E_1 + E_2)^* \rangle$
- Radio arrays measure electric field correlations at the earth.
- $\langle E(x,y) E^*(x+u,y+v) \rangle = V(u,v)$ (Michelson's 'visibility')
- $V(u,v) = \sum_{l,m} I(l,m) \exp(2\pi i (lu+mv) / \lambda)$ (vC- Z theorem)
- Angular resolution λ / D ($D \sim |u|, |v|$)
- Historical rivalry in 1950's – Australia vs. Cambridge
- Ruby Payne-Scott, Martin Ryle



The correlator, which processes 435 antenna pairs into correlations working at 10 Teraflops



One of 30 antennas of the Giant Metrewave Radio Telescope observatory of NCRA-TIFR



Some of the people who built the correlator and keep it running 24/7

M87
HST WFC3/UVIS
F275W
F606W

The galaxy M87 (left) was long believed to harbour an energy source in its centre

One of the first galaxies associated with a strong radio source, Modeled as a black hole, accreting a rotating mass of hot gas emitting radio waves, launching a jet.

Event Horizon Telescope collaboration unveiled images of the black hole shadow in 2019

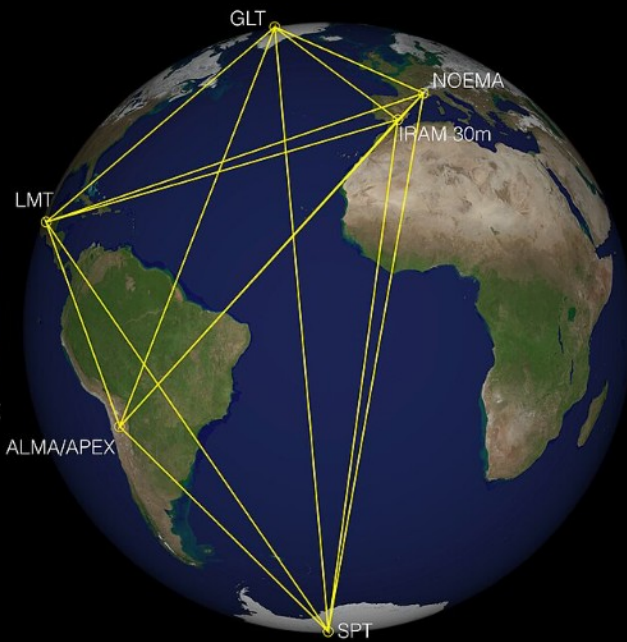
6400 light-years
25 arcseconds



2017 April 11

2018 April 21

2021 April 18



Very Very long
baseline
interferometry

4 light days / 55 million light years = angle subtended by the innermost part of the radio emitting region at earth (~ 6 Schwarzschild radii of a multibillion solar mass black hole) $\approx 2 \times 10^{-10}$ radians ≈ 40 microarcseconds $\approx$ tennis ball on the moon

1.3 mm / 10 000 km $\approx 1.3 \times 10^{-10}$ radians Doeleman, Falcke early 21 century saw and seized this opportunity Event Horizon Telescope – EHT collaboration born

Cancelling phase distortions

Instrumental and ionospheric/ atmospheric phases modify the fields and visibilities

$$E_a = \exp(i\phi_a) E_a^t, V_{ab} = \exp(i\phi_a) V_{ab}^t \exp(-i\phi_b)$$

Jennison (1958) took the phase of the product of three visibilities on baselines forming a triangle cancelling these unknown, unwanted phases

$$V_{ab} V_{bc} V_{ca} = V_{ab}^t V_{bc}^t V_{ca}^t$$

This is well before Bargmann (1964) used a similar object

$$\langle \psi_a | \psi_b \rangle \langle \psi_b | \psi_c \rangle \langle \psi_c | \psi_a \rangle$$

in a proof of Wigner's theorem. The analogy is precise. Visibilities can be regarded as inner products in a Hilbert space of signals

$$V_{ab} = \frac{1}{T} \int_0^T dt E_a(t) E_b^*(t)$$

Amplitudes can be cancelled too

Amplitude and phase errors combined into ‘Gains’ $E_a = G_a E_a^t$, $V_{ab} = G_a V_{ab}^t G_b^*$

The ‘cross ratio’ $\frac{V_{ab} V_{cd}}{V_{ad} V_{cb}}$ computed with measured visibilities

It has the same value as the cross ratio computed with the true visibilities.
Gives more information about the image. This is the “Closure amplitude” introduced by Twiss et al (1960).

We rewrite it using a “hat” to denote complex conjugate reciprocal

This brings in the idea of a closed loop of four (or any even number) of antennas, alternate visibilities hatted, $V_{ab} \hat{V}_{bc} V_{cd} \hat{V}_{da}$

Consider a triangle $V_{ab} \hat{V}_{bc} V_{ca}$. In this case we are left with a $G_a G_a^*$.
This is real, so the **phase** of this object is the same as with the true visibilities

Some numerology

COPOLAR CASE (Same polarisation at all antennas)

$$E_a = G_a E_a^t, V_{ab} = G_a V_{ab}^t G_b^*$$

$N(N-1)/2$ measured V 's , so $N(N-1)$ real parameters which are known. By a similar count, $N(N-1)$ real unknowns in the true visibilities, and $2N - 1$ real unknowns in the G ' s

The -1 is because a common phase in all the G ' s disappears from the equations, there are only $N-1$ phase differences which occur

One should be able to form $N^2 - 3N + 1$ quantities by eliminating the “gains” i.e the G ' s. In the trade, these are called “closure quantities”

Putting in polarisation

$$\mathbf{E}_{lm} = \begin{bmatrix} E_{xlm} \\ E_{ylm} \end{bmatrix} \quad \mathbf{I}(l, m) = \overline{\mathbf{E}_{lm} \mathbf{E}_{lm}^+} = \begin{bmatrix} E_x E_x^* & E_x E_y^* \\ E_y E_x^* & E_y E_y^* \end{bmatrix}_{lm}$$

Stokes parameters of the source at l, m

$$\mathbf{V}(u, v) = \overline{\mathbf{E}(x, y) \mathbf{E}(x+u, y+v)^+}$$

Stokes visibility at baseline u, v

$$\mathbf{V}(u, v) = \sum_{l, m} \mathbf{I}(l, m) \exp(2\pi i(lu + mv)/\lambda)$$

$$\mathbf{V}(u, v) = \mathbf{V}(-u, -v)^+ \quad \mathbf{V}_{ab} = \mathbf{G}_a \mathbf{V}_{ab}^t \mathbf{G}_b^+$$

How to cancel $\mathbf{G}$'s ? hat defined as adjoint inverse, keeps order of matrices

$$\hat{\mathbf{V}}_{bc} = \hat{\mathbf{G}}_b \hat{\mathbf{V}}_{bc}^t \hat{\mathbf{G}}_c^+ = \mathbf{G}_b^{+-1} \hat{\mathbf{V}}_{bc}^t \mathbf{G}_c^{-1}$$

Closure quantities for full polarisation observations

Avery
Broderick,
Perimeter



Dominic
Pesce
Harvard

arXiv 2010.00612

“Closure traces: Novel calibration-insensitive quantities for radio astronomy” The Astrophysical Journal. 2020 Dec 1;904(2):126.

Investigated the 4 antenna case, used hats, looked at loops

Gauge, group, and graph

- Think of G 's as $G(L,2,c)$ gauge transformations which act on the E 's at the vertices of the graph
- The measured visibilities are associated with the edges of the graph and transform bilocally as $G_a V_{ab}^t G_b^+$
- The hat device works to cancel intermediate G 's
- Even-numbered loops give us a product of true visibilities alternately hatted and unhatted, transforming with a G on the left and a G^{-1} on the right. (called covariants). Unlike the copolar case, G and G^{-1} cannot be cancelled.
- The eigenvalues, or determinant and trace, are gauge invariants, Can be evaluated on measured visibilities but have the same values as if they were true ones

Counting invariants

- Since V and G are general complex matrices, we have $8N(N-1)/2 - (8N-1) = 4N^2 - 12N + 1$ real invariants.
- Each even numbered circuit gives four, (complex trace and determinant). A plethora - $\sim N^4/24$ of those, clearly not independent, and in any case we need an odd number!
- Triangles with one base point in common give the desired 'basis' in terms of which all other circuits, odd or even, can be constructed. They are $(N-1)(N-2)/2$ in number and the associated matrix strings are independent, they have $4N^2 - 12N + 8$ elements in all

Making invariants from triangles

$$A_{012} = V_{01} \hat{V}_{12} V_{10}$$

This matrix was christened an 'advariant' by Supurna Sinha

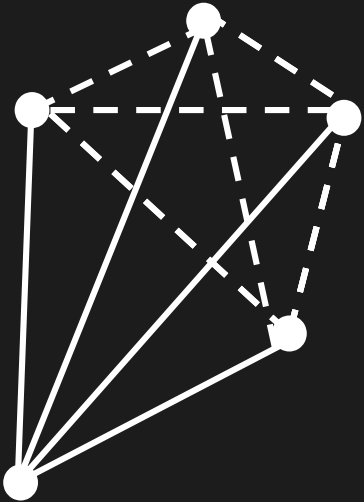
$$A = G_0 A^t G_0^+$$

Is how A transforms

Expand A^t in terms of Pauli matrices. The coefficients would transform as a complex four vector if G_0 had a determinant 1. (Silberstein, 1913, Klein and others, ca 1890)

These four vectors undergo Lorentz transformations with an additional real scale factor $\det(G G^+)$ - seven degrees of freedom. All the A 's transform according the same law

Use the first four four vectors as a basis and then express all other four vectors in that frame,. The 'scale' is fixed by normalisation
Lorentz transformations plus scale give us the extra seven d.o.f to get the right count of invariants

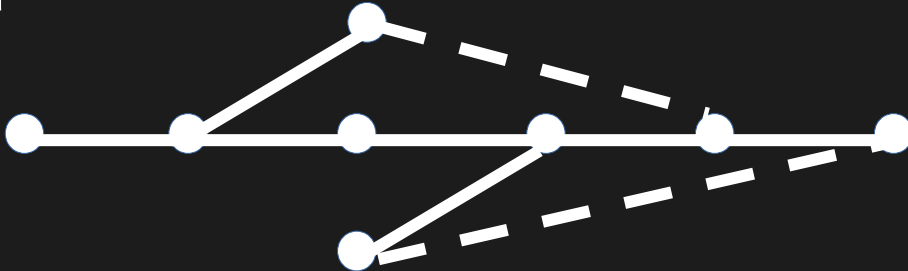


Questions

- How many invariants in the general case when not all baselines are measured? How to write them down? Multiple possibilities - which to choose?
- We are taking inverses, what about singular matrices?
- Invariants are highly nonlinear in the visibilities, what are the noise properties?
- How do you use invariants to make images?

The incomplete graph

- Form a spanning tree – no closed loops, with N vertices and $N-1$ edges, and fix a vertex as base point
- Now add the remaining edges one by one, each gives an independent loop with respect to the base point, either even or odd



- We have strings transforming with $G...$ G^{-1} and $G.....$ G^+ but can concatenate odd with even (non bipartite graph!)

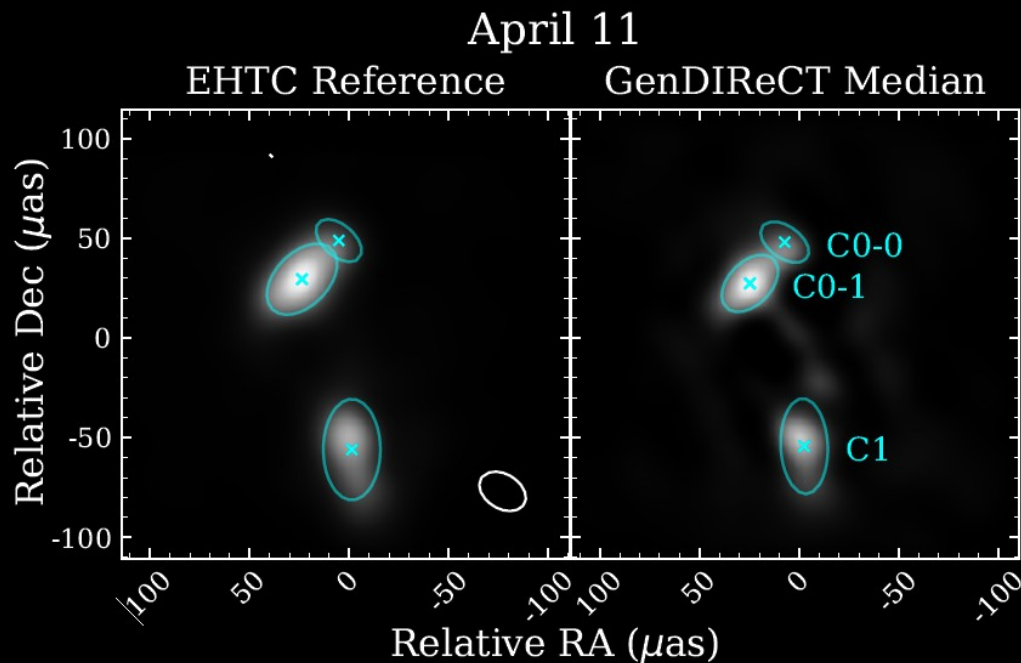
In the real world, there is noise

- The noise on the visibilities is to a good approximation, uncorrelated gaussian
- The invariants are a highly nonlinear transformation of these and have a non-gaussian distribution with correlation and asymmetric tails (VK)
- Near singular matrices probably cause the tails, but the invariants carry information about model parameters. Full polarisation, quasi realistic simulations (VK)
- NT, Samuel Lai et al (work in CSIRO, Perth WA) train an AI on images to recognise their signature in the space of invariants, and learn to go from the invariants to the image (co-polar case)
- Works!

Towards independent event horizon imaging of the supermassive black holes in M87 and the Milky Way

Nithyanandan Thyagarajan, Samuel Lai, O. Ivy Wong, and Foivos Diakogiannis

Test on
3C279 data
taken during
M87*
observations



Two for the road

- Transform parametrised models to the visibility domain. Discrepancy is

$$\chi^2 = \sum_{i < j} (V_{ij} - G_i V_{ij}^p G_j^+) (\text{c.c.}) / 2 \sigma_{ij}^2$$

- For gaussian errors, the probability is $\exp(-\chi^2)$
- Minimise χ^2 w.r.t. the parameters *and also the gains – the gauge fields* – say by Monte Carlo steps in the space of G's and parameters.
- Parisi -Wu programme?
- Invariants of CKM matrices (Chaturvedi, Gupta, Sanchez-Colon, Mukunda). Redo by graphs?