

Solving MFT through dualities and transformations

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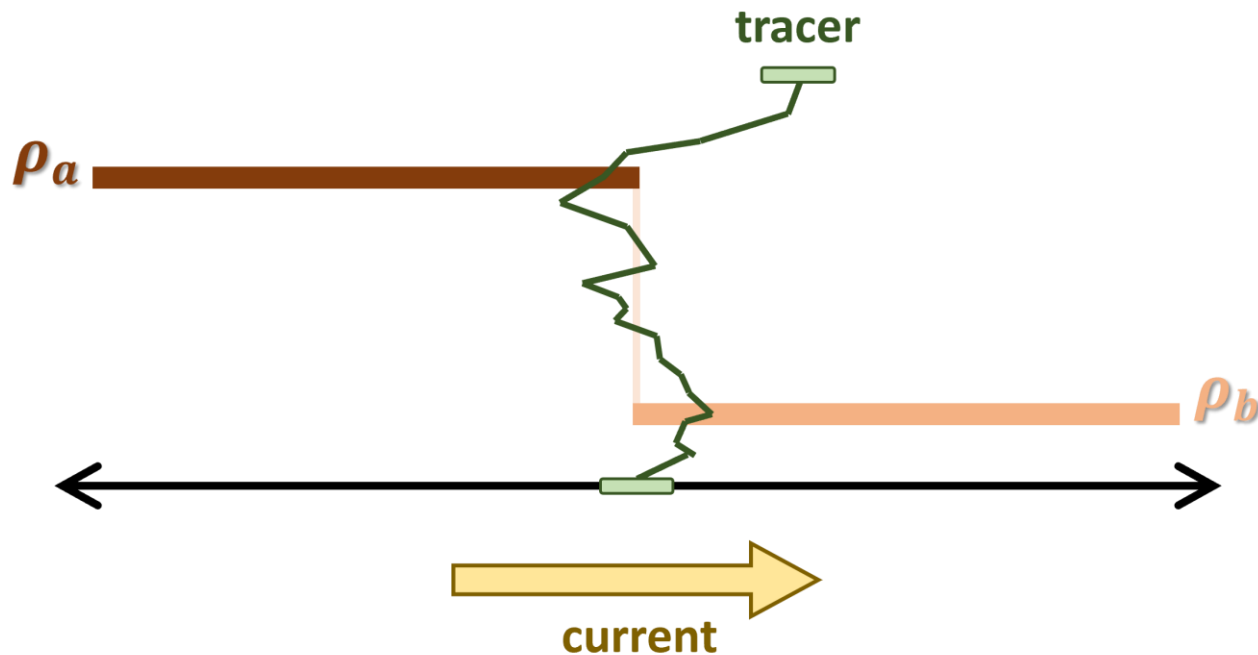
Tata Institute of Fundamental Research, Mumbai

Based on [arXiv:2607.14073](#) & [arXiv:2501.03164v3](#)

In collaboration with *Sandeep Jangid, Kapil Sharma, Jitendra Kethepalli,
Benjamin Guiselin, Jacopo De Nardis & Tridib Sadhu*

First, dynamical fluctuations

Single-file diffusion



Symmetric exclusion, hard-core points ...

Derrida, Gerschenfeld (2009)

Krapivsky, Mallick, Sadhu (2014)

Imamura, Mallick, Sasamoto (2017)

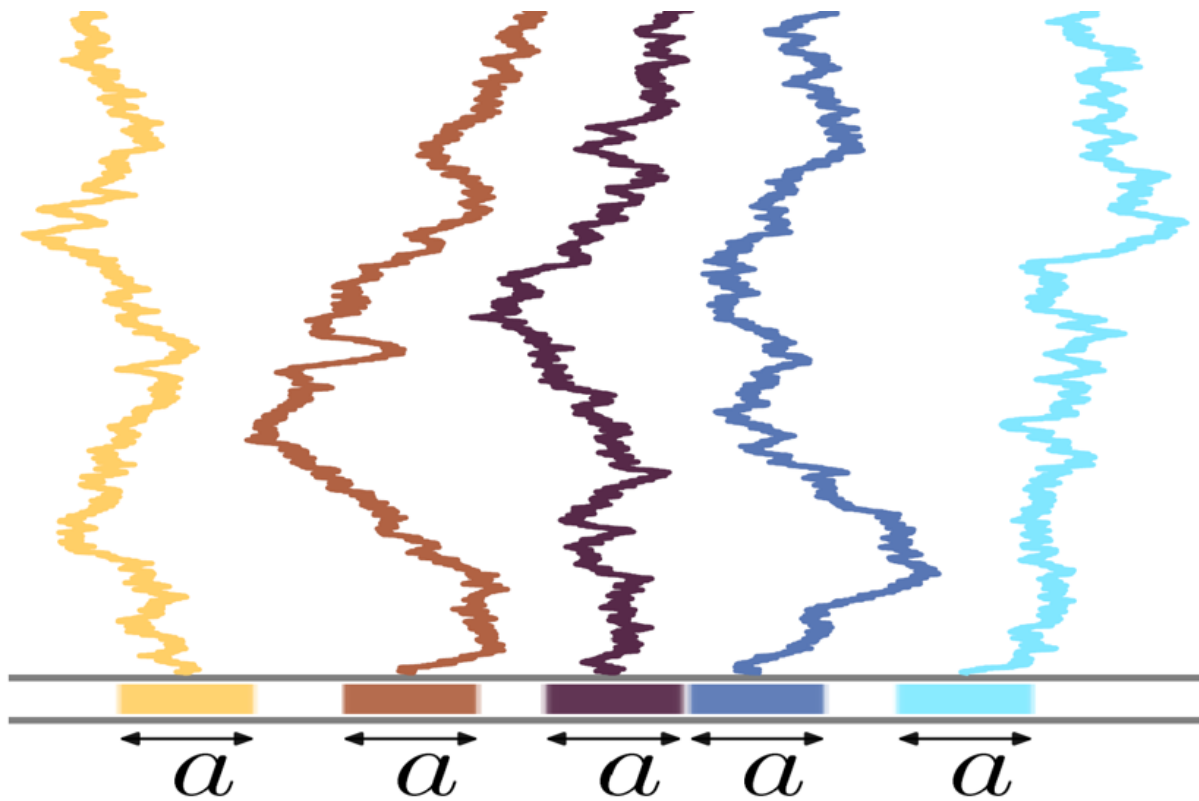
Mallick, Moriya, Sasamoto (2022)

Banerjee, Jack, Cates (2022)

Grabsch, Berlioz, Rizkallah, Illien, Benichou (2024)

- Statistics?
- Long-time memory?
 - Optimal paths?

Brownian *hard-rod* gas



$$D(\rho) = \frac{1}{(1 - a\rho)^2}$$

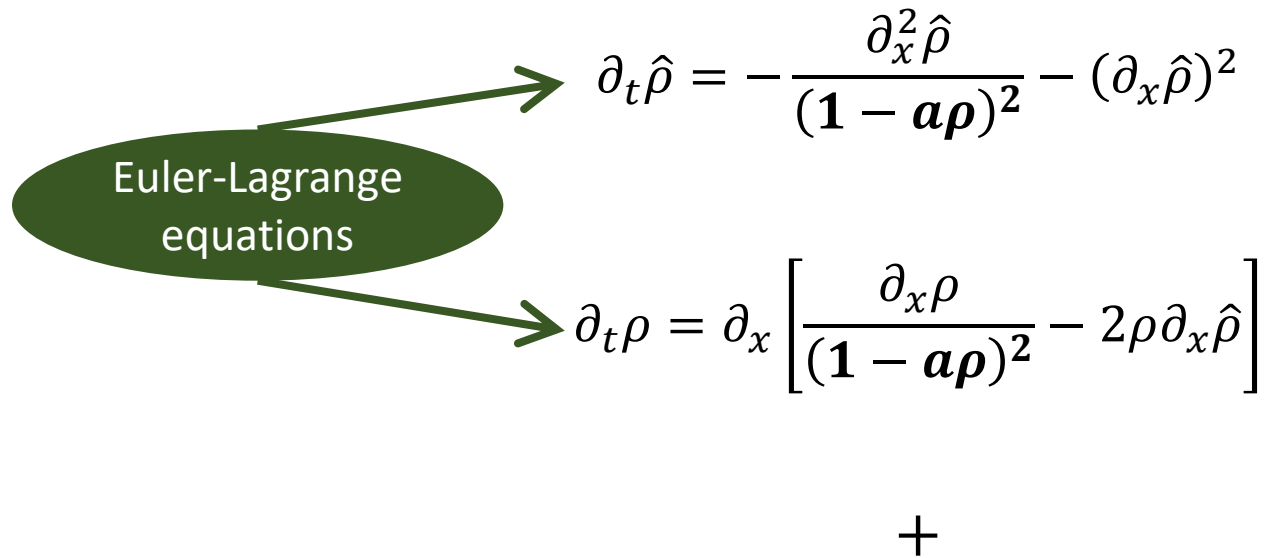
$$\sigma(\rho) = 2\rho$$

Spohn (1991)

Lin, Meron, Cui, Rice, Diamant (2005)

Saha, Jangid, de Pirey, Klamser, Sadhu (2026)

The MFT-variational problem ...



A dark green oval containing the text "Euler-Lagrange equations" has two green arrows pointing from its right side to the two equations below.

$$\partial_t \hat{\rho} = -\frac{\partial_x^2 \hat{\rho}}{(1 - \alpha \rho)^2} - (\partial_x \hat{\rho})^2$$
$$\partial_t \rho = \partial_x \left[\frac{\partial_x \rho}{(1 - \alpha \rho)^2} - 2\rho \partial_x \hat{\rho} \right]$$
$$+$$

Krapivsky, Meerson (2012)
Krapivsky, Mallick, Sadhu (2014)
Grabsch, Venturelli, Benichou (2025)

Observable– and ensemble–dependent temporal boundary conditions

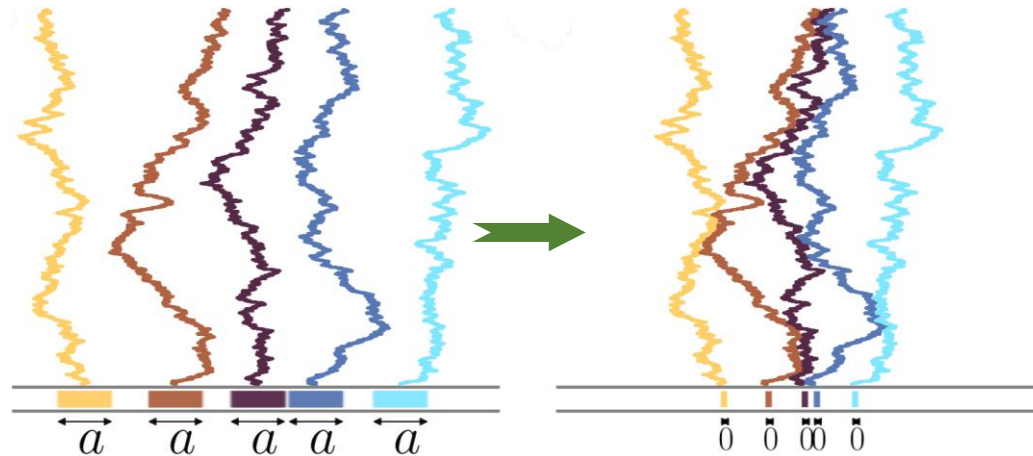
A *hard-rod-point-particle* duality

$$\hat{r}(X, t) = \hat{\rho}(x, t) - a \int_{\mathfrak{z}(t)}^x dy \rho(y, t) \partial_y \hat{\rho}(y, t)$$

$$r(X, t) = \frac{\rho(x, t)}{1 - a\rho(x, t)}$$

$$X = x - a \int_{\mathfrak{z}(t)}^x dy \rho(y, t)$$

Tracer at t



Maps the interacting problem to a non-interacting one ...

Tracer statistics

SSEP

$$\mu_{\mathcal{A}}(\lambda) = \lambda \xi + \frac{\xi}{2} \ln \frac{1 + \rho_b(e^{-B} - 1)}{1 + \rho_a(e^B - 1)} + \sum_{n \geq 1} \frac{(-\omega)^n}{2n} \left[\xi - \int_{-\xi}^{\infty} du \operatorname{erfc} \frac{\sqrt{n}u}{2} \right]$$

$$\omega = \rho_a(1 - \rho_b)(e^B - 1) + \rho_b(1 - \rho_a)(e^{-B} - 1)$$

Imamura, Mallick, Sasamoto (2017)

Mallick, Moriya, Sasamoto (2024)

$$\mu_Q(\lambda) = ???$$

BHRG

$$\mu(\lambda) = \operatorname{extr}_{\xi, B} [\lambda \xi + \chi(\xi, B)]$$

$$\chi_{\mathcal{A}}(\xi, B) = r_a \frac{\varepsilon_B^+}{2} \int_{\xi}^{\infty} du \operatorname{erfc} \frac{u}{2} + r_b \frac{\varepsilon_B^-}{2} \int_{-\xi}^{\infty} du \operatorname{erfc} \frac{u}{2}$$

$$r_{a,b} = \frac{\rho_{a,b}}{1 - \rho_{a,b}} ; \quad \varepsilon_B^{\pm} = e^{\pm B} - 1$$

Height SCGF
of PP-gas

$$\chi_Q(\xi, B) = r_a \int_{\xi}^{\infty} du \ln \left(1 + \frac{\varepsilon_B^+}{2} \operatorname{erfc} \frac{u}{2} \right) + r_b \int_{-\xi}^{\infty} du \ln \left(1 + \frac{\varepsilon_B^-}{2} \operatorname{erfc} \frac{u}{2} \right)$$

Jangid, Saha, Sharma, Kethapalli, Guiselin, De Nardis, Sadhu [arXiv:2607.14073]

Current statistics

SSEP

$$\mu_{\mathcal{A}}(\lambda) = -\frac{1}{\sqrt{\pi}} \sum_{n \geq 1} \frac{(-\omega)^n}{n^{\frac{3}{2}}}$$

Derrida, Gerschenfeld (2009)
Mallick, Moriya, Sasamoto (2022)

$$\mu_Q(\lambda) = ???$$

BHRG

$$\mu(\lambda) = \text{extr}_{\xi, B} [\xi B + \chi(\xi, \lambda - aB)]$$

$$\chi_{\mathcal{A}}(\xi, B) = r_a \frac{\varepsilon_B^+}{2} \int_{\xi}^{\infty} du \operatorname{erfc} \frac{u}{2} + r_b \frac{\varepsilon_B^-}{2} \int_{-\xi}^{\infty} du \operatorname{erfc} \frac{u}{2}$$

$$\chi_Q(\xi, B) = r_a \int_{\xi}^{\infty} du \ln \left(1 + \frac{\varepsilon_B^+}{2} \operatorname{erfc} \frac{u}{2} \right) + r_b \int_{-\xi}^{\infty} du \ln \left(1 + \frac{\varepsilon_B^-}{2} \operatorname{erfc} \frac{u}{2} \right)$$

Jangid, Saha, Sharma, Kethepalli, Guiselin, De Nardis, Sadhu [arXiv:2607.14073]

Beyond statistics: Optimal paths

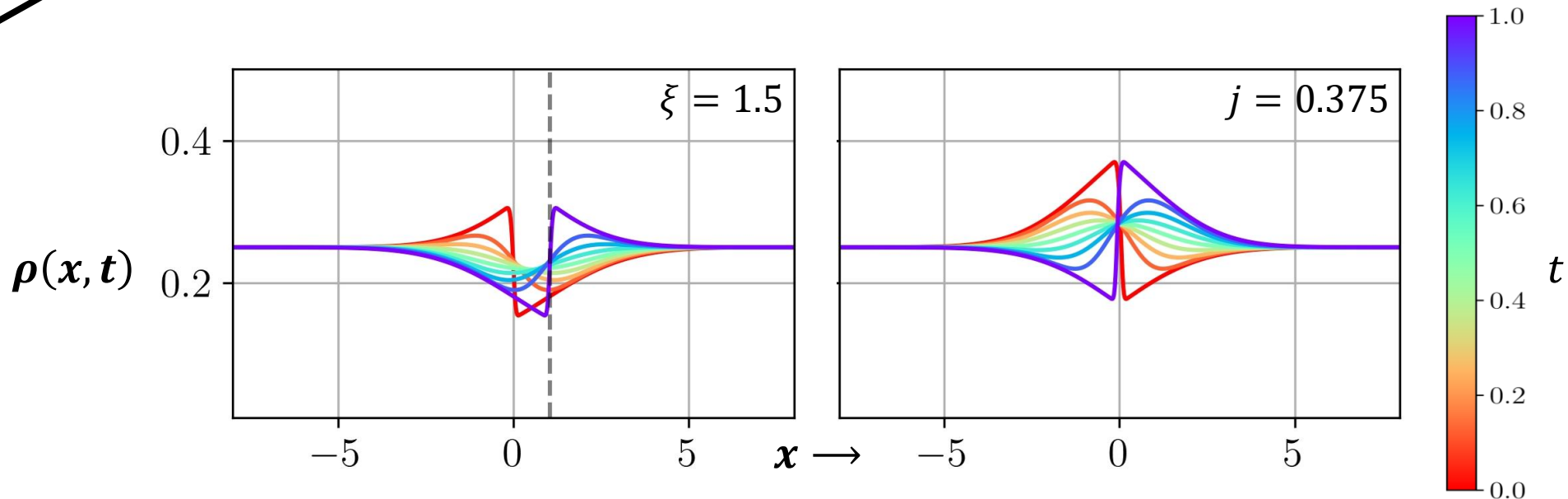
$$\rho(x, t) = \frac{r(X, t)}{1 + ar(X, t)} \quad ; \quad x = X + a \int_{\mathfrak{z}(t)}^X dY r$$

Tracer

$$r_{\mathcal{A}}(X, t) = \frac{1}{4\pi\sqrt{t(1-t)}} \left[\int_{-\infty}^{\infty} dY e^{-\frac{(X-Y)^2}{4(1-t)} + \frac{\lambda\theta(Y-\xi)}{r(\xi,1)}} \right] \left\{ \int_{-\infty}^{\infty} dZ [r_a\theta(-Z) + r_b\theta(Z)] e^{-\frac{(X-Z)^2}{4t} - \frac{\lambda\theta(Z)}{r(\xi,1)}} \right\}$$

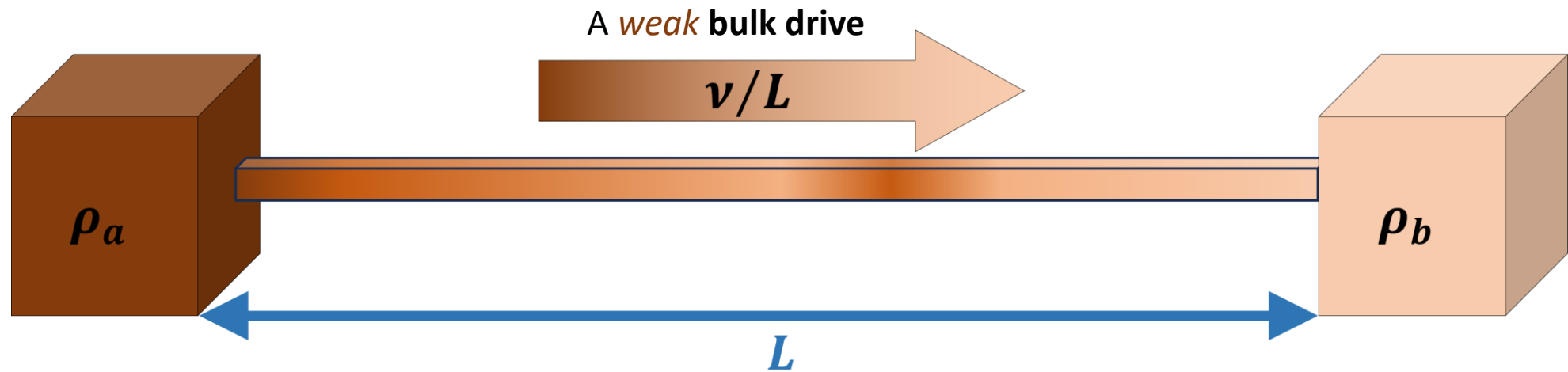
Current

$$r_{\mathcal{A}}(X, t) = \frac{1}{4\pi\sqrt{t(1-t)}} \left[\int_{-\infty}^{\infty} dY e^{-\frac{(X-Y)^2}{4(1-t)} + \frac{\lambda\theta(Y-aj)}{1+ar(aj,1)}} \right] \left\{ \int_{-\infty}^{\infty} dZ [r_a\theta(-Z) + r_b\theta(Z)] e^{-\frac{(X-Z)^2}{4t} - \frac{\lambda\theta(Z)}{1+ar(aj,1)}} \right\}$$



Second, steady-state fluctuations

Boundary-driven diffusive systems



Derrida, Lebowitz, Speer (2001, 2002)

Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim (2001, 2015)

Enaud, Derrida (2004)

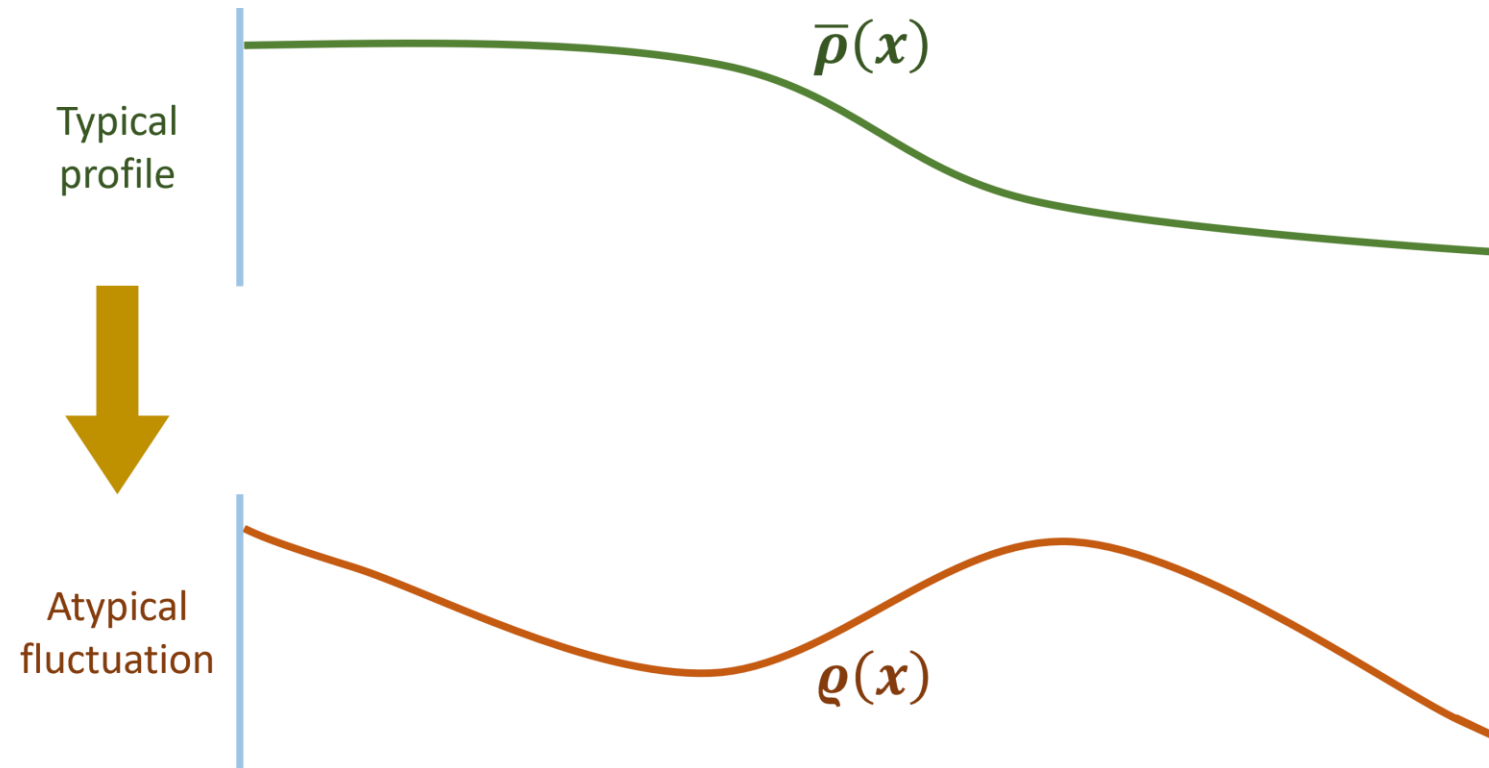
Bertini, Gabrielli, Lebowitz (2005)

Tailleur, Kurchan, Lecomte (2007)

Derrida, Hirschberg, Sadhu (2021)

Bodineau, Derrida (2025)

Density profile fluctuations



- Statistics?
- Optimal paths?

The MFT-variational problem ...

$$\partial_t \hat{\rho} = -D(\rho) \partial_x^2 \hat{\rho} - \frac{\sigma'(\rho)}{2} \partial_x \hat{\rho} (\mathbf{v} + \partial_x \hat{\rho})$$

$$\partial_t \rho = \partial_x \left[D(\rho) \partial_x \rho - \sigma(\rho) \left(\frac{\mathbf{v}}{2} + \partial_x \hat{\rho} \right) \right]$$

$$\hat{\rho}(0, t) = \hat{\rho}(1, t) = \hat{\rho}(x, -\infty) = 0$$

$$\rho = \begin{cases} \rho_a & \text{at } x = 0 \\ \rho_b & \text{at } x = 1 \\ \bar{\rho}(x) & \text{at } t \rightarrow -\infty \end{cases}$$

Derrida (2007)

Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim (2015)

A field *transformation*: $\hat{\rho} \rightarrow F$

$$\hat{\rho}(x, t) = \int_{F(x, t)}^{\rho(x, t)} \mathbf{d}w \frac{2D(w)}{\sigma(w)}$$

Saha, Sadhu [arXiv:2501.03164v3]

Recovers two exact classes ...

Derrida (2007)

Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim (2015)

$$\frac{\sigma'(\rho)}{D(\rho)} = \text{const}$$

$$D(\rho) = c \text{ and } \sigma(\rho) = b\rho + a\rho^2$$

For arbitrary D and σ ...

$$\psi(\varrho) = \int_0^1 dx [f(\varrho) - f(\bar{\rho}) - (\varrho - \bar{\rho})f'(\bar{\rho})] + \sum_{k \geq 1} (\rho_a - \rho_b)^k \Psi_k(\varrho)$$

$$(\rho_a - \rho_b)\bar{J} \geq 0$$

$$\Psi_1 = \nu \int_0^1 dx \frac{K e^{Kx}}{e^K - 1} \int_0^\infty dt \int_{\rho_a}^q dw \frac{D(w)\sigma'(\rho_a) - D(\rho_a)\sigma'(w)}{\sigma(\rho_a)}$$

$K = \nu \frac{\sigma'(\rho_a)}{2D(\rho_a)}$

Relaxation from ϱ under reversed bulk drive without boundary drive

Saha, Sadhu [arXiv:2501.03164v3]

For arbitrary D and σ ...

$$\psi(\varrho) = \int_0^1 dx [f(\varrho) - f(\bar{\rho}) - (\varrho - \bar{\rho})f'(\bar{\rho})] + \sum_{k \geq 1} (\rho_a - \rho_b)^k \Psi_k(\varrho)$$

$$\boldsymbol{\nu} = \mathbf{0} \Rightarrow \boldsymbol{\Psi}_1 = \mathbf{0}$$

Saha, Sadhu [arXiv:2501.03164v3]

$$\boldsymbol{\Psi}_2 = f''(\rho_a) \int_0^1 dx \int_0^\infty dt \int_{\rho_a}^q dw \frac{D(w)\sigma'(\rho_a) - D(\rho_a)\sigma'(w)}{\sigma(\rho_a)}$$

Bodineau, Derrida (2025)

Higher dimensions ...

Spohn (1983)

$$\psi(\varrho(\mathbf{x})) = \int d\mathbf{x} [f(\varrho) - f(\bar{\rho}) - (\varrho - \bar{\rho})f'(\bar{\rho})] + \sum_{k \geq 1} (\rho_a - \rho_b)^k \Psi_k(\varrho(\mathbf{x}))$$

*Onset of non-locality depends on **drive orientation***

Saha, Sadhu [arXiv:2501.03164v3]

The take-home message ...

Two results on non-equilibrium fluctuations from MFT

A continuum single-file diffusion through a duality

Tracer and current statistics in annealed and quenched ensembles and the optimal paths

Boundary-driven diffusive systems through a transformation

Onset of non-locality in density fluctuations with and without a weak bulk drive

“An exactly solvable macroscopic fluctuation theory of single-file diffusion”

S Jangid, S Saha, K Sharma, J Kethepalli, B Guiselin, J De Nardis and T Sadhu, arXiv:2607.14073

“Steady-state density large deviations in driven diffusive systems”

S Saha and T Sadhu, arXiv:2501.03164v3

Thank You!