Isomorphisms among quantum Grothendieck rings and their cluster theoretical interpretation

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based on a joint work with
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Algebraic and Combinatorial Methods in Representation Theory
@ ICTS, Bengaluru
23 November 2023

Abstract

The quantum loop algebra $U_q(L\mathfrak{g})$ is a quantum affinization of simple Lie algebra $\mathfrak{g}$ originally introduced in the context of theoretical physics (lattice models, quantum integrable systems).

The category $\mathscr{C}_{\mathfrak{g}}$ of finite-dimensional modules over $U_q(L\mathfrak{g})$ is very interesting and has been studied in relation with Nakajima quiver varieties, cluster algebras, etc.

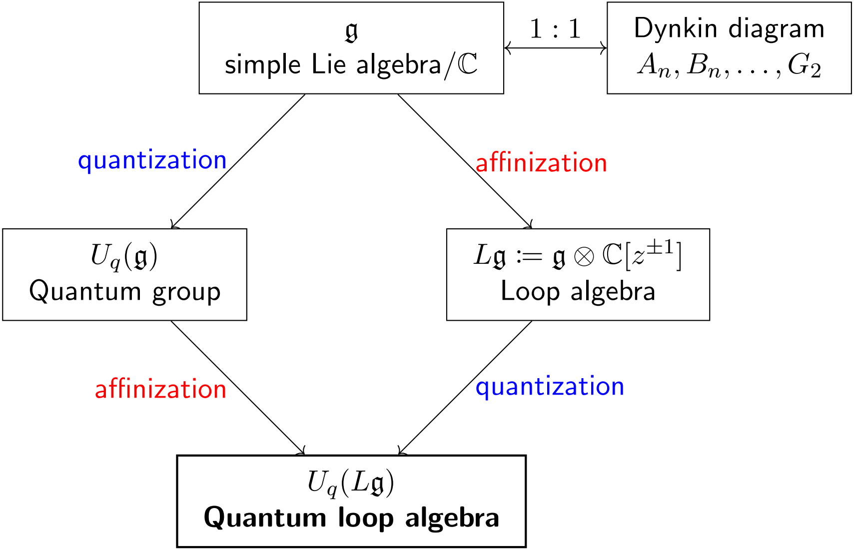
In this talk, we discuss

- quantum Grothendieck ring $K_t(\mathscr{C}_{\mathfrak{g}})$, a deformation of the Grothendieck ring $K(\mathscr{C}_{\mathfrak{g}})$, endowed with a canonical basis consisting of the so-called simple (q,t)-characters,
- isomorphisms among those of different types respecting the canonical bases, and their cluster theoretical interpretation.

Plan

(i) Representation theory of quantum loop algebras
(ii) Quantum Grothendieck rings and simple (q, t)-characters
(iii) Isomorphisms among quantum Grothendieck rings
(iv) Cluster theoretical interpretation

Quantum loop algebras



Categories of representations

In what follows, we always assume $q \in \mathbb{C}^\times$ is NOT a root of 1.

- $\mathrm{Rep}\, U :=$ category of finite-dimensional modules over U

Category	Homological prop.	Tensor prod.
$\mathrm{Rep}\, U(\mathfrak{g})$	semisimple	symmetric
$\mathrm{Rep}\, U_q(\mathfrak{g})$	semisimple	non-symmetric
$\mathrm{Rep}\, U(L\mathfrak{g})$	non-semisimple	symmetric
$\mathrm{Rep}\, U_q(L\mathfrak{g})$	**non-semisimple**	**non-symmetric**

In fact, the category $\mathrm{Rep}\, U_q(L\mathfrak{g})$ is **not even braided**. There are many pairs V_1, V_2 such that

$$V_1 \otimes V_2 \not\cong V_2 \otimes V_1 \quad \text{in } U_q(L\mathfrak{g})\text{-mod}.$$

Cluster structure of $\mathrm{Rep}\, U_q(L\mathfrak{g})$ gives a partial answer to

when does such a non-commutative phenomenon occur ?

(Usual) Grothendieck rings $K(\mathscr{C})$

Let $\mathscr{C}$ be a monoidal abelian category.

$$K(\mathscr{C}) := \bigoplus_{X \in \mathscr{C}} \mathbb{Z}[X] \Big/ \left\langle \begin{array}{l} [X] = [Y] + [Z], \\ \text{if } 0 \to Y \to X \to Z \to 0 \text{ exact in } \mathscr{C} \end{array} \right\rangle$$

Ring structure :

$$[X] \cdot [Y] := [X \otimes Y] = \sum_{L \in \operatorname{irr} \mathscr{C}} [X \otimes Y : L][L]$$

If each object has finite length, $K(\mathscr{C})$ has the canonical basis

$$\{[L] \mid L \in \operatorname{irr} \mathscr{C}\}$$

whose structure constants are all non-negative.

Classical characters

Spectral decomposition w.r.t. the action of Cartan $\mathfrak{h} \subset \mathfrak{g}$ gives the character $\chi(V) = \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu) e^\mu$ of $V \in \operatorname{Rep} U(\mathfrak{g})$. This yields

$$\chi \colon K(\operatorname{Rep} U(\mathfrak{g})) \xrightarrow{\simeq} \mathbb{Z}[y_i^{\pm 1} \mid i \in I]^{W_{\mathfrak{g}}},$$

where

- I is the set of Dynkin nodes,
- $y_i = e^{\varpi_i}$ with ϖ_i being the i-th fundamental weight,
- $W_{\mathfrak{g}}$ is the Weyl group of $\mathfrak{g}$.

Recall the parametrization of irreducible representations

$$\bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i \xleftrightarrow{1:1} \operatorname{irr}(\operatorname{Rep} U(\mathfrak{g})); \quad \lambda \mapsto V(\lambda) \text{: highest weight rep}$$

and that $\chi(V(\lambda))$ is explicit by Weyl's character formula.

q-characters

In what follows, we set $\mathscr{C}_{\mathfrak{g}} := \mathrm{Rep}\, U_q(L\mathfrak{g})$.

Theorem (Chari–Pressley '94)

$$\mathcal{P}^+ := \bigoplus_{i \in I, a \in \mathbb{C}^\times} \mathbb{Z}_{\geq 0} \varpi_{i,a} \xleftrightarrow{1:1} \mathrm{irr}\, \mathscr{C}_{\mathfrak{g}}; \quad \lambda \mapsto L(\lambda): \ \ell\text{-highest wt rep}$$

Theorem (E. Frenkel–Reshetikhin '98)

Spectral decomposition w.r.t. the action of $U_q(L\mathfrak{h}) \subset U_q(L\mathfrak{g})$ yields the q-character homomorphism

$$\chi_q \colon K(\mathscr{C}_{\mathfrak{g}}) \hookrightarrow \mathcal{Y}_{\mathfrak{g}} := \mathbb{Z}[Y_{i,a}^{\pm 1} \mid i \in I, a \in \mathbb{C}^\times].$$

In particular, $K(\mathscr{C}_{\mathfrak{g}})$ is **commutative**.

q-characters (continuation)

Example ($\mathfrak{g} = \mathfrak{sl}_2$ case)

$V_a \coloneqq L(\varpi_a)$: 2-dim vector rep ($a \in \mathbb{C}^\times$), $\chi_q(V_a) = Y_a + Y_{aq^2}^{-1}$.

- When $a/b \notin \{q^2, q^{-2}\}$, we have

$$V_a \otimes V_b \cong L(\varpi_a + \varpi_b) \cong V_b \otimes V_a.$$

- When $b = aq^2$, letting $L = L(\varpi_a + \varpi_{aq^2})$, we have

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{C} & \to & V_a \otimes V_{aq^2} & \to & L & \to & 0 \\ & & & & \not\cong & & & & \\ 0 & \to & L & \to & V_{aq^2} \otimes V_a & \to & \mathbb{C} & \to & 0, \end{array}$$

$$\chi_q(L) = \chi_q(V_a)\chi_q(V_{aq^2}) - 1 = Y_a Y_{aq^2} + Y_a Y_{aq^4}^{-1} + Y_{aq^2}^{-1} Y_{aq^4}^{-1}.$$

Problem: No uniform formula for $\chi_q(L(\lambda))$ is known beyond $\mathfrak{sl}_2$.
$\rightsquigarrow$ Kazhdan-Lusztig type approach

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Recall : Original Kazhdan–Lusztig theory

- $\mathcal{O}_0$: Principal block of the BGG category of $\mathfrak{g}$ (not s.s.)

Its Grothendieck group $K(\mathcal{O}_0)$ has two bases :

$\{[\text{irrep}]\} \leftarrow$ characters are difficult (want to know)

$\{[\text{Verma}]\} \leftarrow$ characters are easy

Kazhdan–Lusztig conjecture

$$K(\mathcal{O}_0) \cong \mathbb{Z}W_{\mathfrak{g}} \xleftarrow{t=1} \mathcal{H}_t(W_{\mathfrak{g}})\text{: Iwahori-Hecke alg}$$

$$\{[\text{Verma}]\} \longleftarrow\!\dashv \text{standard basis}$$

$\downarrow$ compute with an algorithm

$$\{[\text{irrep}]\} \xleftarrow{\;-\;-\;?\;-\;-\;}\!\dashv \text{canonical (KL) basis}$$

$$[\text{Verma} : \text{irrep}] \xleftarrow{?} \text{transition matrix (KL polynomials)}$$

The famous proof by Beilinson–Bernstein & Brylinski–Kashiwara requires the use of geometry of flag manifolds.

Quantum Grothendieck ring $K_t(\mathscr{C}_\mathfrak{g})$

The quantum Grothendieck ring $K_t(\mathscr{C}_\mathfrak{g})$ is a **non-commutative** t-deformation of $K(\mathscr{C}_\mathfrak{g})$ introduced by

- [Nakajima '04] and [Varagnolo-Vasserot '03] for types ADE,
- [Hernandez '04] for general $\mathfrak{g}$.

We have a commutative diagram :

$$\begin{array}{ccccccc} K(\mathscr{C}_\mathfrak{g}) & \overset{\chi_q}{\cong} & \bigcap_{i\in I} \operatorname{Ker} S_i & \subset & \mathcal{Y}_\mathfrak{g} & = & \mathbb{Z}[Y_{i,a}^{\pm 1} \mid i \in I, a \in \mathbb{C}^\times] \\ {\scriptstyle t=1}\uparrow & & & & {\scriptstyle t=1}\uparrow & & \\ K_t(\mathscr{C}_\mathfrak{g}) & \coloneqq & \bigcap_{i\in I} \operatorname{Ker} S_{i,t} & \subset & \mathcal{Y}_{\mathfrak{g},t} & = & (\mathcal{Y}_\mathfrak{g} \otimes \mathbb{Z}[t^{\pm 1/2}], *) \end{array}$$

where

- $\mathcal{Y}_{\mathfrak{g},t}$ is a quantum torus ($*$ denotes a deformed product),
- S_i is the "screening operator", and $S_{i,t}$ is its t-deformation.

Simple (q,t)-characters and KL theory in types ADE

Theorem (Nakajima (types ADE), Hernandez (general $\mathfrak{g}$))

There is a canonical basis $\{L_t(\lambda)\}_{\lambda \in \mathcal{P}^+}$ of $K_t(\mathscr{C}_\mathfrak{g})$ consisting of "simple (q,t)-characters", computed by an analog of KL algorithm from a standard basis (ordered products of $L_t(\varpi_{i,a})$'s.)

Theorem (Nakajima '03 + Varagnolo–Vasserot '03)

If $\mathfrak{g}$ is of type ADE, we have the following:

(KL) $L_t(\lambda)|_{t=1} = \chi_q(L(\lambda))$ for each $\lambda \in \mathcal{P}^+$;

(P1) $L_t(\lambda) \in \mathcal{Y}_{\mathfrak{g},t}$ has coefficients in $\mathbb{Z}_{\geq 0}[t^{\pm 1/2}]$ for each $\lambda \in \mathcal{P}^+$;

(P2) The basis $\{L_t(\lambda)\}_{\lambda \in \mathcal{P}^+}$ has structure constants in $\mathbb{Z}_{\geq 0}[t^{\pm 1/2}]$.

The proof uses the geometry of Nakajima quiver varieties, which is fully developped only for types ADE at this moment.

Recent progress on the other types $BCFG$

(KL) $L_t(\lambda)|_{t=1} = \chi_q(L(\lambda))$ for each $\lambda \in \mathcal{P}^+$;

(P1) $L_t(\lambda) \in \mathcal{Y}_{\mathfrak{g},t}$ has coefficients in $\mathbb{Z}_{\geq 0}[t^{\pm 1/2}]$ for each $\lambda \in \mathcal{P}^+$;

(P2) The basis $\{L_t(\lambda)\}_{\lambda \in \mathcal{P}^+}$ has structure constants in $\mathbb{Z}_{\geq 0}[t^{\pm 1/2}]$.

Conjecture (Hernandez)

The properties (KL), (P1), and (P2) hold true for general $\mathfrak{g}$.

Theorem (F.–Hernandez–Oh–Oya '22 + preprint '23)

- (P1) and (P2) hold true for general $\mathfrak{g}$.
- (KL) holds true in the following two cases:

(Case 1) when $\mathfrak{g}$ is of type B (with any λ),

(Case 2) when $L(\lambda)$ is "reachable" (with any $\mathfrak{g}$).

Our proof uses the theory of cluster algebras and categorifications.

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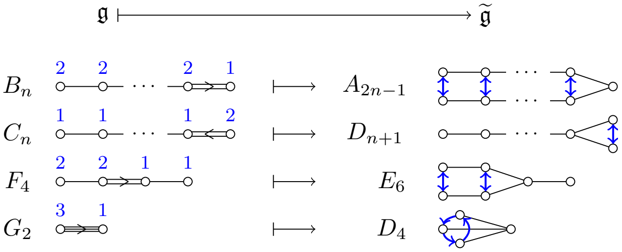
Isomorphisms among quantum Grothendieck rings

Theorem (FHOO '22)

Let $\mathfrak{g}$ be of types $BCFG$. There is an isom. of $\mathbb{Z}[t^{\pm 1/2}]$-algebras

$$\Psi\colon K_t(\mathscr{C}_{\mathfrak{g}}) \xrightarrow{\simeq} K_t(\mathscr{C}_{\widetilde{\mathfrak{g}}})$$

respecting the canonical bases, where $\widetilde{\mathfrak{g}}$ is the "unfolding" of $\mathfrak{g}$.



Here $d_i = (\alpha_i, \alpha_i)/2 \in \{1, r\}$ and r is the lacing number of $\mathfrak{g}$.

Proof of (P2)

Theorem (FHOO '22)

Let $\mathfrak{g}$ be of types $BCFG$. There is an isom. of $\mathbb{Z}[t^{\pm 1/2}]$-algebras

$$\Psi \colon K_t(\mathscr{C}_{\mathfrak{g}}) \xrightarrow{\simeq} K_t(\mathscr{C}_{\widetilde{\mathfrak{g}}})$$

respecting the canonical bases, where $\widetilde{\mathfrak{g}}$ is the "unfolding" of $\mathfrak{g}$.

Corollary ("Propagation of positivity")

The positivity of the structure constants (P2) is also true for $\mathfrak{g}$ of types $BCFG$.

Remark

Actually, we obtain a collection of such isomorphisms Ψ labelled by the so-called Q-data introduced by [F.–Oh 21].

Proof of (KL) for type B = Categorification of $\Psi|_{t=1}$

Theorem (Kashiwara–Kim–Oh '19)

When $(\mathfrak{g}, \widetilde{\mathfrak{g}}) = (B_n, A_{2n-1})$, there are exact $\otimes$-functors

$$\mathscr{C}_{B_n} \leftarrow \mathscr{T} \to \mathscr{C}_{A_{2n-1}},$$

where $\mathscr{T}$ is a certain monoid. cat. of rep's of affine Hecke algebra of GL's. This induces an ring isom. $F\colon K(\mathscr{C}_{B_n}) \simeq K(\mathscr{C}_{A_{2n-1}})$ with **a bijection between the simple classes**.

Theorem (FHOO '22)

The following diagram commutes (with a special choice of Q-data).

$$\begin{array}{ccc} K_t(\mathscr{C}_{B_n}) & \xrightarrow{\Psi} & K_t(\mathscr{C}_{A_{2n-1}}) \\ {\scriptstyle t=1}\downarrow & & \downarrow{\scriptstyle t=1} \\ K(\mathscr{C}_{B_n}) & \xrightarrow{F} & K(\mathscr{C}_{A_{2n-1}}) \end{array}$$

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Notation

Fix a parity function $\varepsilon\colon I \to \{0,1\}$ such that

$$\varepsilon_i \equiv \varepsilon_j + \min(d_i, d_j) \mod 2 \quad \text{if } i \text{ and } j \text{ are adjacent.}$$

Example

- Type A_{2n-1}: 

or 

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- Type B_n: 

or 

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- Type C_{2n}: 

or 

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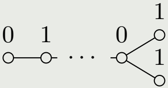
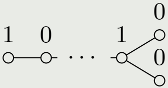
- Type D_{2n+1}: 

or 

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Hernandez–Leclerc's category $\mathscr{C}_{\mathfrak{g}}^{-}$

Definition (Hernandez–Leclerc '16)

Define the category $\mathscr{C}_{\mathfrak{g}}^{-}$ to be the Serre subcategory of $\mathscr{C}_{\mathfrak{g}}$ s.t.

$$L(\lambda) \in \mathscr{C}_{\mathfrak{g}}^{-} \iff \lambda \in \bigoplus_{i \in I, k \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \varpi_{i, q^{-\varepsilon_i - 2k}}$$

The category $\mathscr{C}_{\mathfrak{g}}^{-}$ is a monoidal subcategory which contains essential information of $\mathscr{C}_{\mathfrak{g}}$ in the following sense:

Proposition

Each **prime** simple module in $\mathscr{C}_{\mathfrak{g}}$ is contained in $\mathscr{C}_{\mathfrak{g}}^{-}$ after a suitable spectral parameter shift $L(\varpi_{i,a}) \mapsto L(\varpi_{i,ca})$.

Theorem (Kashiwara–Kim–Oh–Park, conj. by Hernandez–Leclerc)

The category $\mathscr{C}_{\mathfrak{g}}^{-}$ gives the monoidal categorification of a cluster algebra $A(\Gamma_{\mathfrak{g}}^{-})$ (to be explained).

Recall: Cluster algebra

Q: quiver (with no loops, no 2-cycles)

$$\rightsquigarrow \text{Cluster algebra } A(Q) \subset \mathbb{Q}(x_i \mid i \in Q_0)$$

- $A(Q)$ has a distinguished linearly-independent set of cluster monomials, which are grouped into overlapping multiplication-closed subsets of clusters.
- Each cluster is obtained from the initial cluster by a finite sequence of birational transformations called mutations.
- This mutation procedure involves the quiver mutation.

Recall: Quiver mutation

For $k \in Q_0$, the mutated quiver $\mu_k Q$ of Q is obtained from Q by

1. adding an arrow $i \to j$ for each subquiver $(i \to k \to j) \subset Q$;
2. reversing all the arrows adjacent to k;
3. removing all the 2-cycles (if any).

Then, we have

$$\begin{array}{ccc} \mathbb{Q}(x_i)_{i \in Q_0} & \xrightarrow{\text{mutation}} & \mathbb{Q}(x_i)_{i \in Q_0} \\ \cup & & \cup \\ A(\mu_k Q) & \xrightarrow{\sim} & A(Q) \\ \cup & & \cup \\ \{\text{cluster monomials}\} & \xleftrightarrow{1:1} & \{\text{cluster monomials}\}. \end{array}$$

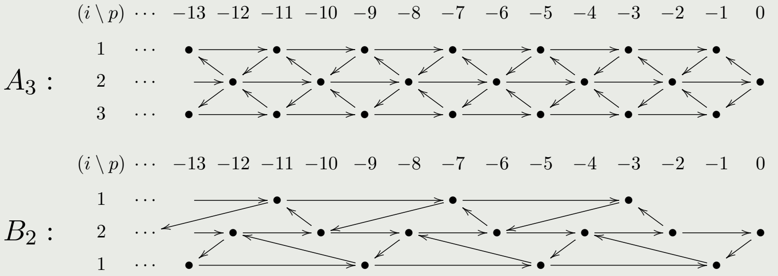
The quiver $\Gamma_{\mathfrak{g}}^{-}$

Define an infinite quiver $\Gamma_{\mathfrak{g}}^{-}$ by:

- $(\Gamma_{\mathfrak{g}}^{-})_0 = \{(i, -\varepsilon_i - 2k) \mid i \in I, k \in \mathbb{Z}_{\geq 0}\}$,
- $(i, p) \to (j, s)$ iff $c_{ij} \neq 0$ and $s - d_j = p - d_i + d_i c_{ij}$,

where $(c_{ij})_{i,j \in I}$ is the Cartan matrix of $\mathfrak{g}$. (Note $d_i c_{ij} = d_j c_{ji}$.)

Example (Types A_3 and B_2)



Monoidal categorification

Theorem (Kashiwara–Kim–Oh–Park, preprint '21)

The category $\mathscr{C}_{\mathfrak{g}}^{-}$ gives the monoidal categorification of the cluster algebra $A(\Gamma_{\mathfrak{g}}^{-})$, i.e., we have an isomorphism

$$\begin{array}{cccc} \bar{\eta}_{\mathfrak{g}}\colon & A(\Gamma_{\mathfrak{g}}^{-}) & \xrightarrow{\simeq} & K(\mathscr{C}_{\mathfrak{g}}^{-}) \\ & \cup & & \cup \\ & \{\text{cluster monomials}\} & \hookrightarrow & \operatorname{irr}\mathscr{C}_{\mathfrak{g}}^{-}. \end{array}$$

Definition

$L \in \operatorname{irr}\mathscr{C}_{\mathfrak{g}}^{-}$: reachable $\iff L \in \bar{\eta}_{\mathfrak{g}}(\{\text{cluster monomials}\})$

- For example, fundamental and KR modules are reachable.
- $\bar{\eta}_{\mathfrak{g}}(\{\text{cluster monomials}\}) \neq \operatorname{irr}\mathscr{C}_{\mathfrak{g}}^{-}$ unless $\mathfrak{g} = \mathfrak{sl}_2$.
- Expectation:

  $\bar{\eta}_{\mathfrak{g}}(\{\text{cluster monomials}\}) \overset{?}{=} \{\text{real simple modules}\}$.

A quantum analog of monoidal categorification

Theorem (FHOO preprint '23)

The cluster algebra $A(\Gamma_{\mathfrak{g}}^{-})$ can be upgraded to a **quantum** cluster algebra $A_t(\Gamma_{\mathfrak{g}}^{-})$, and we have an isomorphism

$$\begin{array}{cccc} \eta_{\mathfrak{g}}\colon & A_t(\Gamma_{\mathfrak{g}}^{-}) & \xrightarrow{\simeq} & K_t(\mathscr{C}_{\mathfrak{g}}^{-}) \\ & \cup & & \cup \\ & \{\text{quantum cluster monomials}\} & \hookrightarrow & \{L_t(\lambda)\}_{L(\lambda)\in \mathrm{irr}\,\mathscr{C}_{\mathfrak{g}}^{-}}. \end{array}$$

Moreover, we have

$$\eta_{\mathfrak{g}}|_{t=1} = \bar{\eta}_{\mathfrak{g}}.$$

Our proof uses a cluster theoretical interpretation of the isom Ψ.

Corollary

(KL) $L_t(\lambda)|_{t=1} = \chi_q(L(\lambda))$ holds for any **reachable** modules.

Isomorphism Ψ as mutation equivalence

Let $\mathfrak{g}$ be of types $BCFG$, and $\widetilde{\mathfrak{g}}$ the unfolding of $\mathfrak{g}$.
We have $\Psi(K_t(\mathscr{C}_{\mathfrak{g}}^{-})) = K_t(\mathscr{C}_{\widetilde{\mathfrak{g}}}^{-})$ (with a special choice of Q-data).

Theorem (FHOO preprint '23)

There is an isomorphism

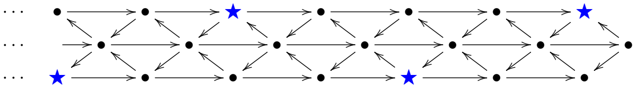
$$\Phi\colon A_t(\Gamma_{\mathfrak{g}}^{-}) \xrightarrow{\simeq} A_t(\Gamma_{\widetilde{\mathfrak{g}}}^{-})$$

arising from an infinite (but locally finite) sequence of mutations, and the following diagram commutes:

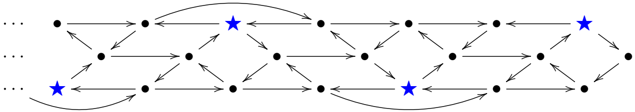
$$\begin{array}{ccc} A_t(\Gamma_{\mathfrak{g}}^{-}) & \xrightarrow{\Phi} & A_t(\Gamma_{\widetilde{\mathfrak{g}}}^{-}) \\ \eta_{\mathfrak{g}} \downarrow & & \downarrow \eta_{\widetilde{\mathfrak{g}}} \\ K_t(\mathscr{C}_{\mathfrak{g}}^{-}) & \xrightarrow{\Psi} & K_t(\mathscr{C}_{\widetilde{\mathfrak{g}}}^{-}). \end{array}$$

Example: B_2/A_3

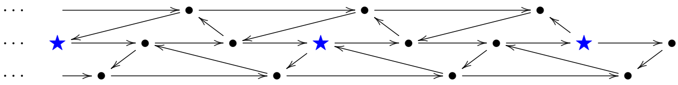
Performing the mutation at $\color{blue}\bigstar$'s in $\Gamma^-_{A_3}$



yields



which is isomorphic to $\Gamma^-_{B_2}$



Application: Substitution formula

Theorem (FHOO preprint '23)

There is an isomorphism of $\mathbb{Q}(t^{1/2})$-skew fields

$$\widetilde{\Psi} \colon \operatorname{Frac}(\mathcal{Y}_{\mathfrak{g},t}) \xrightarrow{\simeq} \operatorname{Frac}(\mathcal{Y}_{\widetilde{\mathfrak{g}},t})$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\operatorname{Frac}(\mathcal{Y}_{\mathfrak{g},t}) & \xrightarrow{\widetilde{\Psi}} & \operatorname{Frac}(\mathcal{Y}_{\widetilde{\mathfrak{g}},t}) \\
\cup & & \cup \\
\mathcal{Y}_{\mathfrak{g},t} & & \mathcal{Y}_{\widetilde{\mathfrak{g}},t} \\
\cup & & \cup \\
K_t(\mathscr{C}_{\mathfrak{g}}) & \xrightarrow{\Psi} & K_t(\mathscr{C}_{\widetilde{\mathfrak{g}}}).
\end{array}$$

In particular, $\widetilde{\Psi}$ transforms each simple (q,t)-character for $\mathfrak{g}$ into a simple (q,t)-character for $\widetilde{\mathfrak{g}}$. If $(\mathfrak{g}, \widetilde{\mathfrak{g}}) = (B_n, A_{2n-1})$, it gives a bijection between the simple q-characters.

Example: B_2/A_3

In our example above ($B_2 \rightsquigarrow A_3$), we have

$$\widetilde{\Psi}|_{t=1}(Y_{i,p}) = \begin{cases} Y_{1,-3-8m}Y_{1,-1-8m} & \text{if } (i,p) = (1,-3-12m), \\ Y_{1,-5-8m} & \text{if } (i,p) = (1,-7-12m), \\ Y_{1,-7-8m} & \text{if } (i,p) = (1,-11-12m), \\ Y_{2,-8m} & \text{if } (i,p) = (2,-12m), \\ Y_{2,-2-8m}Y_{1,-1-8m}^{-1} + Y_{1,-3-8m} & \text{if } (i,p) = (2,-2-12m), \\ (Y_{1,-1-8m}^{-1} + Y_{2,-2-8m}^{-1}Y_{1,-3-8m})^{-1} & \text{if } (i,p) = (2,-4-12m), \\ Y_{2,-4-8m} & \text{if } (i,p) = (2,-6-12m), \\ Y_{3,-7-8m} + Y_{2,-6-8m}Y_{3,-5-8m}^{-1} & \text{if } (i,p) = (2,-8-12m), \\ (Y_{2,-6-8m}^{-1}Y_{3,-7-8m} + Y_{3,-5-8m}^{-1})^{-1} & \text{if } (i,p) = (2,-10-12m), \\ Y_{3,-1-8m} & \text{if } (i,p) = (1,-1-12m), \\ Y_{3,-3-8m} & \text{if } (i,p) = (1,-5-12m), \\ Y_{3,-7-8m}Y_{3,-5-8m} & \text{if } (i,p) = (1,-9-12m), \end{cases}$$

where $Y_{i,p} \coloneqq Y_{i,q^p}$.

Example: B_2/A_3

For example, the simple q-character

$$\chi_q(L_{B_2}(\varpi_{1,q^{-7}})) = Y_{1,-7} + Y_{2,-6}Y_{2,-4}Y_{1,-3}^{-1} + Y_{2,-6}Y_{2,-2}^{-1} + Y_{1,-5}Y_{2,-4}^{-1}Y_{2,-2}^{-1} + Y_{1,-1}^{-1}$$

is transformed under $\widetilde{\Psi}|_{t=1}$ into

$$Y_{1,-5} + \frac{Y_{2,-4}}{(Y_{1,-1}^{-1} + Y_{2,-2}^{-1}Y_{1,-3})Y_{1,-3}Y_{1,-1}} + \frac{Y_{2,-4}}{Y_{2,-2}Y_{1,-1}^{-1} + Y_{1,-3}} + \frac{Y_{3,-3}(Y_{1,-1}^{-1} + Y_{2,-2}^{-1}Y_{1,-3})}{Y_{2,-2}Y_{1,-1}^{-1} + Y_{1,-3}} + Y_{3,-1}^{-1}$$

$$= Y_{1,-5} + Y_{2,-4}\frac{Y_{1,-3}^{-1}Y_{1,-1}^{-1} + Y_{2,-2}^{-1}}{Y_{1,-1}^{-1} + Y_{2,-2}^{-1}Y_{1,-3}} + Y_{3,-3}Y_{2,-2}^{-1} + Y_{3,-1}^{-1}$$

$$= Y_{1,-5} + Y_{2,-4}Y_{1,-3}^{-1} + Y_{3,-3}Y_{2,-2}^{-1} + Y_{3,-1}^{-1} = \chi_q(L_{A_3}(\varpi_{1,q^{-5}})).$$