



Some remarks on the Macroscopic Fluctuation Theory

Giovanni Jona-Lasinio

Bengaluru, September - 2026

Abstract

The macroscopic fluctuation theory (MFT) is a consistent and self-contained description of macroscopic fluctuations using only transport coefficients. In the formulation of the Rome group an important motivation was the discovery that we could reproduce by a purely macroscopic calculation the result of Derrida, Lebowitz and Speer obtained solving the microscopic symmetric simple exclusion process. Then the MFT developed in several directions allowing a unified treatment of different physical problems. Directions for future research include: i) extension of the MFT to the quantum case ; ii) the possible use of an additional Hamiltonian structure discovered on the way; iii) a formula which recalls Clausius view of entropy; iv) developing the recent work on the integrability of the time dependent MFT.

My motivations

The term Macroscopic Fluctuation Theory appeared first in the paper by the italian team *The italian team (name due to Derrida)* in 2002.

The formal argument for the theory to be self-contained uses the notion of time reversed process which closes the theory in the sense that the trajectory creating a fluctuation is identified explicitly without solving an infinite set of coupled equations.

Definition of time reversed process

For a stationary Markov chain with a stationary distribution π and transition probabilities $P_{ij} = P(X_{n+1} = j \mid X_n = i)$, the time-reversed process $Y_n = X_{m-n}$ is also a Markov chain.

We require that also the time reversed process admits a hydrodynamic description that we call adjoint hydrodynamics non necessarily local. There are examples which can be solved explicitly

The italian team (name due to Derrida)

1. Lorenzo Bertini
2. Alberto De Sole
3. Davide Gabrielli
4. Gianni J-L
5. Claudio Landim (brasilian)

Origin of the Macroscopic Fluctuation Theory (MFT)

Technically the MFT is at the cross road of several inputs among which

1. The work of Kipnis, Olla, Varadhan on fluctuations in hydrodynamics for the the symmetric simple exclusion process
2. The work of J-L, Landim, Vares on hydrodynamical fluctuations in Glauber-Kawasaki dynamics
3. The work of Gabrielli, J-L, Landim on Onsager Symmetry
4. The work of Derrida, Lebowitz, Speer solving the non-equilibrium symmetric simple exclusion process
5. The old Einstein theory of opalescence in a fluid in equilibrium

The theory is based on variational principles determining for each model the large deviation rates (LDR) of thermodynamic variables like a density or a time average like the time averaged current flowing through the system in stationary states.

The macroscopic dynamics of diffusive systems is described by hydrodynamic equations often provided by conservation laws and constitutive equations, that is equations expressing the current in terms of the thermodynamic variables. More precisely on the basis of a local equilibrium assumption, at the macroscopic level the system is completely described by a local density $\rho(t, x)$ which may have several components and the local current $j(t, x)$. The evolution equations are of the form

$$\begin{cases} \partial_t \rho(t) + \nabla \cdot j(t) = 0, \\ j(t) = J(t, \rho(t)), \end{cases} \quad (1)$$

the space variable $x \in \Lambda \in R^d$.

For diffusive systems the constitutive equations take the form

$$J(\rho) = -D(\rho)\nabla\rho + \chi(\rho) E, \quad (2)$$

where the *diffusion coefficient* $D(\rho)$ and the *mobility* $\chi(\rho)$ are $d \times d$ symmetric and positive definite matrices, E is an external field acting on the bulk.

These equations have to be supplemented by appropriate boundary conditions on $\partial\Lambda$, the boundary of Λ . If $\lambda(t, x)$, $x \in \partial\Lambda$, is the chemical potential of the external reservoirs, the boundary conditions read

$$f'(\rho(t, x)) = \lambda(t, x), \quad x \in \partial\Lambda. \quad (3)$$

where $f(\rho)$ is the equilibrium free energy density. Non-equilibrium depends on the boundary conditions and the external field. It may happen that the two equilibrate each other in which case we have a non-homogeneous equilibrium state.

Trajectories which create fluctuations

“In a Stationary Non-equilibrium State (SNS) the spontaneous emergence of a macroscopic fluctuation takes place most likely following a trajectory which is the time reversal of the relaxation path according to the time reversed hydrodynamics”

The above statement follows from assuming the existence of a time reversed dynamics. In equilibrium a fluctuation emerges following with high probability the time reversal of the relaxation trajectory. This property may hold even if the microscopic dynamics does not satisfy detailed balance. Therefore Onsager symmetry and the above non-equilibrium generalization can be true under rather general conditions: this is possible because going from the microscopic to the macroscopic level there is a loss of information.

Extended Hamiltonian Structure

There is an underlying Hamiltonian structure for the joint fluctuations of density and current.

We consider the time interval $[0, T]$ and assume that the external drivings do not depend on time. Let $A_0(x)$ be a vector field related to the initial condition by $\nabla \cdot A_0(x) = \rho(x, 0)$. For example we can fix $A_0 = -\nabla h$ where h solves $\Delta h(x) = -\rho(x, 0)$. We then define the vector field

$$A(t, x) = A_0(x) - \int_0^t j(s, x) ds, \quad (4)$$

that, apart from the initial condition and a minus sign, is the time integrated current. Since ρ and j are related by the continuity equation we have $j = -\partial_t A$ and $\rho = \nabla \cdot A$.

We can write the rate functional in terms of the vector field A

$$\mathcal{I}_{[0,T]}(A) = \frac{1}{4} \int_0^T dt \int_{\Lambda} dx \left(\partial_t A + J(\nabla \cdot A) \right) \cdot \chi^{-1}(\nabla \cdot A) \left(\partial_t A + J(\nabla \cdot A) \right). \quad (5)$$

Observe that, in this form, the constraint of the continuity equation is automatically satisfied. Formula (5) has the form of an action for the Lagrangian

$$\mathbb{L}(A, \partial_t A) = \frac{1}{4} \int_{\Lambda} dx \left(\partial_t A + J(\nabla \cdot A) \right) \cdot \chi^{-1}(\nabla \cdot A) \left(\partial_t A + J(\nabla \cdot A) \right). \quad (6)$$

The corresponding Hamiltonian is

$$\begin{aligned}\mathbb{H}(A, B) &= \sup_{\xi} \left\{ \int_{\Lambda} dx B(x) \cdot \xi(x) - \mathbb{L}(A, \xi) \right\} \\ &= \int_{\Lambda} dx \left[B \cdot \chi(\nabla \cdot A) B - B \cdot J(\nabla \cdot A) \right]\end{aligned}\tag{7}$$

and the canonical equations are

$$\begin{cases} \partial_t A = 2\chi(\nabla \cdot A) B - J(\nabla \cdot A), \\ \partial_t B = -\nabla \left[\text{Tr}(D(\nabla \cdot A) \nabla^T B) \right. \\ \quad \left. + B \cdot \chi'(\nabla \cdot A) (E - B) \right], \end{cases}\tag{8}$$

Given a solution (ρ, π) of the canonical MFT equations there corresponds a solution of (8) given by

$$\begin{cases} A(t) = A_0 - \int_0^t ds [J(\rho(s)) + 2\chi(\rho(s))\nabla\pi(s)] , \\ B(t) = -\nabla\pi(t) , \end{cases} \quad (9)$$

where A_0 satisfies the condition $\nabla \cdot A_0 = \rho(0)$.

Clausius definition of entropy

I often asked myself the question: *“How did Clausius get the idea of entropy in a pure thermodynamic approach?”* . Entropy is a notion which carries some mystery with it, as was emphasized by a qualified opinion. There is a quantity called entropy in information theory which measures the content of information of a message. A story tells that such a terminology was suggested to Shannon, the inventor of the theory, by the famous mathematician John Von Neumann with the comment

“You should call it entropy, for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, no one really knows what entropy is, so in a debate you will always have the advantage.”

The fourth, sixth and ninth memoirs of Clausius by the titles respectively *“On a modified form of the second fundamental theorem of the mechanical theory of heat”*; *“On the application of the theorem of the equivalence of transformations to interior work”*; *“On several convenient forms of the fundamental equations of the mechanical theory of heat”* are very illuminating, however for some reason his arguments did not attract so much attention and are not reported in the treatises on thermodynamics I know.

Clausius idea

Clausius idea, as developed in his fourth memoir, is that there must exist a quantity called the *the equivalence value of a transformation* expressing the strict connection of a transformation of heat into work with transfer of heat from a higher temperature to a lower one as in a Carnot cycle. The equivalence value is what he will call later the entropy.

Going one step further, all the reversible transformations connecting two given states have the same equivalence value, the variation of the entropy which is an invariant under change of the protocols connecting those states.

$$S(A) - S(B) = \int_B^A \frac{\delta Q}{T} \quad (10)$$

is independent of the protocols of the transformation and defines an equivalence class of transformations.

For example in the simple case of an ideal gas we have the exact 1-form

$$dS = C_p \frac{dT}{T} - nR \frac{dp}{p} \quad (11)$$

In Clausius words

"Carnot's theorem,....,expresses a relation between two kinds of transformations, the transformation of heat into work, and the passage of heat from a warmer to a colder body, which may be regarded as the transformation of heat at the higher, into heat at a lower temperature...."

We see, therefore, that these two transformations may be regarded as phenomena of the same nature, and we may call two transformations which can thus mutually replace one another equivalent. We have now to find the law according to which the transformations must be expressed as mathematical magnitudes, in order that the equivalence of two transformations may be evident from the equality of their values. The mathematical value of a transformation thus determined may be called its equivalence-value (Äquivalenzwerth)."

The next problem is to determine such equivalence-value. Here is Clausius argument

With respect to the magnitude of the equivalence-value, it is first of all clear that the value of a transformation from work into heat must be proportional to the quantity of heat produced; and besides this it can only depend upon the temperature. Hence the equivalence-value of the transformation of work into the quantity of heat Q , of the temperature t , may be represented generally by

$$Q f(t) \tag{12}$$

wherein $f(t)$ is a function of the temperature which is the same for all cases.....

In a similar manner the value of the passage of the quantity of heat Q , from the temperature t_1 to the temperature t_2 , must be proportional to the quantity Q , and besides this, can only depend on the two temperatures. In general therefore, it may be expressed by

$$Q F(t_1, t_2) \quad (13)$$

wherein $F(t_1, t_2)$ is a function of both temperatures, which is the same for all cases, and of which we at present only know that, without changing its numerical value, it must change its sign when the two temperatures are interchanged; so that

$$F(t_1, t_2) = -F(t_2, t_1) \quad (14)$$

Then by a general argument for which we refer to the original paper, Clausius concludes that the relationship between $f(t)$ and $F(t_1, t_2)$ is

$$F(t_1, t_2) = f(t_2) - f(t_1) \quad (15)$$

He writes $f(t) = \frac{1}{T}$ where T is provisionally a function of temperature that will be identified later with the absolute temperature

Energy balance

Consider a system in a time dependent environment, that is, E and λ depend on time. The work done by the environment on the system in the time interval $[0, T]$ is

$$W_{[0,T]} = \int_0^T dt \left\{ \int_{\Lambda} dx j(t) \cdot E(t) - \int_{\partial\Lambda} d\sigma \lambda(t) j(t) \cdot \hat{n} \right\}, \quad (16)$$

where $\hat{n}$ is the outer normal to $\partial\Lambda$ and $d\sigma$ is the surface measure on $\partial\Lambda$. The first term on the right hand side is the energy provided by the external field while the second is the energy exchanged with the reservoirs.

Fix time dependent paths $\lambda(t)$ of the chemical potential and $E(t)$ of the field. Given a density profile ρ_0 , let $\rho(t)$, $j(t)$, $t \geq 0$, be the solution of (1)–(3) with initial condition ρ_0 . By using the Einstein relation and the boundary conditions $f'(\rho(t)) = \lambda(t)$, an application of the divergence theorem yields

$$W_{[0,T]} = F(\rho(T)) - F(\rho(0)) + \int_0^T dt \int_{\Lambda} dx j(t) \cdot \chi(\rho(t))^{-1} j(t), \quad (17)$$

where F is the equilibrium free energy functional.

Slow transformations and $1/\tau$ expansion

To analyze transformations over the interval $[0, \tau]$ driven by slowly changing boundary conditions and external fields, it is convenient to introduce the dimensionless variable $s = t/\tau$. The *protocol* of a transformation is defined therefore by a choice of the external drivings $E(s, x)$, $x \in \Lambda$, and $\lambda(s, x)$, $x \in \partial\Lambda$, $s \in [0, 1]$.

The transformation can then be parametrized by

$$\begin{cases} E^\tau(t) = E(t/\tau), \\ \lambda^\tau(t) = \lambda(t/\tau), \end{cases} \quad t \in [0, \tau]. \quad (18)$$

The asymptotics in which we are interested is for τ large compared to the typical relaxation time of the system. Let $\rho^\tau(t)$ and $j^\tau(t)$, $0 \leq t \leq \tau$, be the solution to the hydrodynamic equations (1), (2) and (3) with the slow external field E^τ and chemical potential λ^τ , that is

$$\begin{cases} \partial_t \rho^\tau + \nabla \cdot J(t/\tau, \rho^\tau(t)) = 0, \\ j^\tau(t) = J(t/\tau, \rho^\tau(t)) \\ f'(\rho^\tau(t))|_{\partial\Lambda} = \lambda^\tau(t) \end{cases} \quad (19)$$

where we recall that $J(t, \rho) = -D(\rho)\nabla\rho + \chi(\rho)E(t)$.

For $s \in [0, 1]$, let $\bar{\rho}(s)$ be the unique stationary solution of the hydrodynamics with external field $E(s)$ and chemical potential $\lambda(s)$. When τ is large we expand the solution (ρ^τ, j^τ)

$$\begin{aligned}\rho^\tau(\tau s) &= \bar{\rho}(s) + \frac{1}{\tau} r(s) + o\left(\frac{1}{\tau}\right), \\ j^\tau(\tau s) &= J(s, \bar{\rho}(s)) + \frac{1}{\tau} g(s) + o\left(\frac{1}{\tau}\right).\end{aligned}\tag{20}$$

By (19) we get the corresponding linear evolution equations for the first order correction (r, g) ,

$$\begin{cases} \partial_s \bar{\rho}(s) + \nabla \cdot g(s) = 0 \\ g(s) = -\nabla \cdot (D(\bar{\rho}(s))r(s)) + r(s)\chi'(\bar{\rho}(s))E(s) \\ r(s, x) = 0, \quad x \in \partial\Lambda \end{cases} \quad (21)$$

where we use the notation that, for a matrix $A = (a_{ij}(x))_{i,j=1}^n$, $\nabla \cdot A$ is the vector with i -th coordinate $\sum_j \partial_{x_j} a_{ij}(x)$.

From (16) and (17) along the transformation (ρ^τ, j^τ) we obtain

$$\begin{aligned}
 F(\rho^\tau(\tau)) - F(\rho^\tau(0)) &= \tau \int_0^1 ds \int_{\Lambda} dx j^\tau(\tau s) \cdot E(s) \\
 &- \tau \int_0^1 ds \int_{\partial\Lambda} d\sigma \lambda(s) j^\tau(\tau s) \cdot \hat{n} \\
 &- \tau \int_0^1 ds \int_{\Lambda} dx j^\tau(\tau s) \cdot \chi(\rho^\tau(\tau s))^{-1} j^\tau(\tau s).
 \end{aligned} \tag{22}$$

We can analyze the equation (22) at the different orders in $1/\tau$, obtaining an identity for each order. Direct computations yield that at order τ the right hand side of (22) vanishes.

At order τ^0 we get the first non trivial relationship,

$$\begin{aligned} F(\bar{\rho}(1)) - F(\bar{\rho}(0)) &= \int_0^1 ds \int_{\Lambda} dx E(s) \cdot g(s) \\ &\quad - \int_0^1 ds \int_{\partial\Lambda} d\sigma \lambda(s) g(s) \cdot \hat{n} \\ &\quad + \int_0^1 ds \int_{\Lambda} dx r(s) J(s, \bar{\rho}(s)) \cdot (\chi^{-1})'(\bar{\rho}(s)) J(s, \bar{\rho}(s)), \end{aligned} \tag{23}$$

where we used that $\rho^\tau(0) = \bar{\rho}(0) + O(1/\tau)$ and $\rho^\tau(\tau) = \bar{\rho}(1) + O(1/\tau)$.

Taking the pair $E(x, s), \lambda(x, s)$ as our state variables, a transformation corresponds to a path γ in an infinite dimensional space. Then the right hand side of (23) can be represented as a line integral of an exact 1- form

$$\int_{\gamma} A_1 dE + A_2 d\lambda \quad (24)$$

Taking into account that $\bar{\rho}$ depends on time only through λ, E

$$\partial_s \bar{\rho} = \frac{\delta \bar{\rho}}{\delta \lambda} \partial_s \lambda + \frac{\delta \bar{\rho}}{\delta E} \partial_s E \quad (25)$$

inserting this expression in the formulas for r, g , separating $\partial_s \lambda$ and $\partial_s E$ we can obtain explicitly A_1, A_2 .

For diffusive systems different equivalent transformations in the sense of (23) can be very different as far as energy consumption is concerned. To be concrete let us consider a conductor in a potential difference ΔV that we want to increase to a value $\Delta V'$. A quasi-static transformation increasing slowly ΔV would produce a very large quantity of heat, infinite in the limit, that we can reduce through the following procedure that we formulate in a slightly more general form.

Consider a quasi-static transformation between the stationary states corresponding to the values E_i, λ_i and E_f, λ_f . The housekeeping heat can be very costly. You may then ideally split the transformation in the following way.

Change abruptly the field from E_i to E'_i where E'_i is the value corresponding to equilibrium, i.e. $J = 0$, with chemical potential λ_i . The system will relax to equilibrium. Then perform a quasi-static transformation via equilibrium states until you reach the final value λ_f . No housekeeping heat. Finally bring abruptly the field to its final value E_f . The system will relax to its final stationary state.

Of course in every concrete case one must estimate the energy cost of the splitting and decide whether over a long but finite interval of time of a real transformation it is convenient. In 2015 we discussed a criterion of optimality based on the minimisation of the renormalized work which led to some unexpected outcome also for a system of free particles.

In conclusion it is somewhat surprising that an equivalence class of transformations could be defined for irreversible dissipative systems like purely diffusive systems. It is interesting to investigate the possibility of generalising this analysis to a more general setting like reaction diffusion macroscopic dynamics where presumably we can extend the macroscopic fluctuation theory.

References for the Macroscopic Fluctuation Theory

The first four references are reviews with an extended bibliography, the fifth contains some results not included in the reviews.

L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, J. Stat. Mech.: Theory and Experiment, P07014 (2007), arXiv:0705.1247.

B.Derrida, J. Stat. Mech.: Theory and Experiment, P07023 (2007), arXiv:0703762.

L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Rev. Mod. Phys., **87**, 593–636 (2015), arXiv:1404.6466.

P. L. Garrido, J. Stat. Mech.: Theory and Experiment (2021) 024001

L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, J. Stat. Mech.: Theory and Experiment **10**, P10018 (2015), arXiv:1506.05691.

References for Clausius

Clausius R., *On a modified form of the second fundamental theorem of the mechanical theory of heat*, Pogg. Annalen, **xciii** 481 (1854); translated in Phil. Mag. **xii** 81 (1856).

Clausius R. , *“On the application of the theorem of the equivalence of transformations to interior work”*, Pogg. Annalen, **cxvi** 73 (1862); translated in Phil. Mag. S.4. **xxiv** 81 (1862).

Clausius R.; *On several convenient forms of the fundamental equations of the mechanical theory of heat*, Pogg. Annalen **cxxv** 353 (1865); translated in Journ. de Liouville **10** 361 (1865).

Jona-Lasinio G., *On Clausius' approach to entropy and analogies in non-equilibrium*, to appear in Ensaios Matematicos special issue dedicated to Errico Presutti, arXiv: 2306.04395 (2023).

Gabrielli D., Jona-Lasinio G., *On the Equivalence and Optimality of Transformations of Diffusive Systems*, J. Stat. Phys. to appear