ICTS-RRI Maths Circle
July 11, 2026

A geometer encounters a tangle of closed, non-intersecting curves in three-dimensional space.

The curves may be deformed continuously, but they may never pass through one another or break.

To understand such a configuration, the geometer studies its shadow.

What is lost in passing from space to shadow? What additional information restores the original configuration? Can different shadows represent the same object? Can a single curve be entangled with itself?

Sometimes the curves can be disentangled into separate circles lying in a plane.
Sometimes they cannot.

How can the geometer tell which situation has occurred?
As the tangle deforms, what does it remember?

How might the story change in four-dimensional space?



Fix a point x_0 in a space X, which we call the center. The space could be any shape: a plane, a plane with 7 points removed, 27 circles joined at one point, the surface of a doughnut,...

A *journey* in X is a continuous motion in X, starting at time 0, and ending at time 1.
A *loop* is a journey that begins and ends at the center.

The geometer builds a new world $\Omega_{x_0}X$ whose inhabitants (points) are loops in X.

Notice that $\Omega_{x_0}X$ has a natural center. What is it?

The algebraist invents a rule "$*$" to multiply inhabitants of $\Omega_{x_0}X$:
"$a * b$" is journey a followed by journey b. Can you make sense of that?

What are the rules of this algebra? Are $(a * b) * c$ and $a * (b * c)$ identical journeys?
Is there a journey in $\Omega_{x_0}X$ from $(a * b) * c$ to $a * (b * c)$?

Does $a * b$ equal $b * a$?

The geometer repeats the process, building a world of *loops in the space of loops*.
What algebra governs this world?

Can you distinguish a plane with two points removed and a plane with one point removed?

The geometer moves around the tangle of curves in wonder, contemplating the shape of the space of loops that do not meet any of the curves in the tangle...

There are two rules, $*$ and $\circ$, to multiply members of a certain collection X.

There are elements e_* and $e_\circ$ that behave like the number 1:
$e_* * a = a = a * e_*$ and $e_\circ \circ a = a = a \circ e_\circ$ for every member a of X.

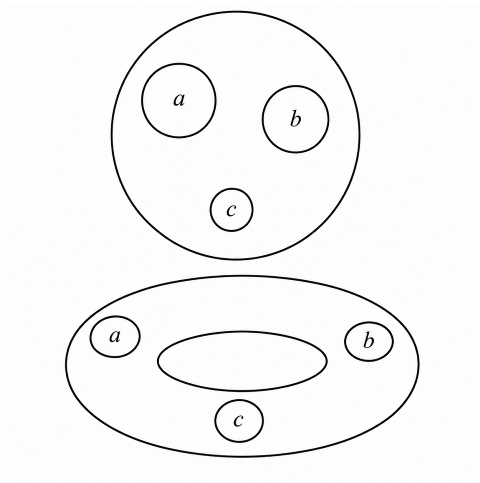
Furthermore, the two rules are compatible in the following sense:

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

for every collection of four elements a, b, c, d.

Two mathematicians are arguing about which rule of multiplication is better.

A geometer walks by, draws a two-dimensional picture, and leaves.
Immediately, the argument ends. What happened?



Ordinary algebra is written on a line. Notice how xy and yx are distinct expressions, because you can't move x past y on a line. The plane is a different story.

Nevertheless, when we substitute 'ordinary numbers' a and b for the symbols x and y respectively, we find that $ab = ba$.

Can you think of any "quantities" A, B, you can substitute for x and y so that $AB \neq BA$?

There is a world in which algebra is written on surfaces, instead of on a line. Each distinct way of arranging symbols on a surface corresponds to a different algebraic expression. Can you think of mathematical entities that follow this sort of algebra?

The three-sphere S^3 is the set of points at unit distance from the origin in four-dimensional space. In other words, it is the set of tuples of real numbers u, v, x, y such that $u^2 + v^2 + x^2 + y^2 = 1$.

We slowly rotate the (u, v)-plane and the (x, y)-plane, so that at time t each plane has rotated counterclockwise through an angle $2\pi t$.

Notice that if we start with any point $p = (u_0, v_0, x_0, y_0)$, it traces out a loop in the three-sphere as time goes from 0 to 1. Let us call curves obtained by this construction "H-loops".

Notice that every point in S^3 lies on one of these loops.
This means the three-sphere is the union of all the H-loops in it.
Do you see why?
Can you picture this?
Draw it?

Take any two distinct H-loops. They may be deformed continuously in S^3, but they may never pass through one another or break. Can they be separated, or are they entangled?

Imagine a new space X whose points are the H-loops in S^3. That is, an entire loop worth of points in S^3 corresponds to a single point in X. What is the shape of X? Can you draw it? Write equations describing it?